PURSUING STACKS

(*À la poursuite des Champs*)

First episode:

THE MODELIZING STORY

(*histoire de modèles*)

by Alexander Grothendieck

1983

Pursuing Stacks (À la poursuite des Champs)
First episode: The modelizing story (histoire de modèles)
by Alexander Grothendieck (* March 28, 1918 in Berlin; † November 13, 2014 Lasserre, Ariège)

Original typescript written 1983

Typeset with LaTeX and the fabulous *memoir* class. Composed with Bitstream Charter for the text type and Paul Pichaureau's `mathdesign` package for the math type. All diagrams and illustrations were made with Till Tantau's Ti*k*Z (Ti*k*Z ist *kein* Zeichenprogramm) and the `tikz-cd` package.

Short contents

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. . . (to be completed)

Scrivener's preface

I wanted to read *Pursuing Stacks* (PS), and at the same time I wanted to read it carefully enough to make sure I understood as much as possible, and also make some notes and links to make it easier to navigate. That was my primary motive for creating this LaTeX-typeset version of the work. I hope it may be of some use to others, but it is not necessarily the best way to read PS.

One of the great pleasures of reading PS in the scanned typescript with the handwritten margin notes is that you get a sense of intimacy that is necessarily lost upon transcription.

However, I think it is a great shame that PS is not more widely read. This situation is likely due to the obstacles involved with navigating and parsing the typescript which only exists as `djvu`-file, accessible from Maltsiniotis' web-page:

`http://webusers.imj-prg.fr/~georges.maltsiniotis/ps.html`

I am of course aware that Maltsiniotis himself is preparing an edition of the first five chapters of PS, and with Künzer and Toën an edition of the last two chapters, both for the series *Documents Mathématiques* of the French mathematical society. This has been announced since 2010, while we now have 2015. Thus, I decided that it might not be an entirely wasted effort to publish my own edition of PS, knowing full well that it will be succeeded by presumably better ones made by more knowledgeable and capable individuals. In any event, should my files contain some small help (e.g., in the forms of diagrams) towards the goals of Maltsiniotis et al. they are more than welcome to it!

I'm also not including an index, which I don't consider myself capable of compiling. At least modern PDF-viewers are capable of search, and I provide the TeX-files which can be used for `grep`-ing.

I have tried to stay true to Grothendieck's voice, but on some matters of grammar and orthography, I have taken the liberty of correcting slightly:

- some “which”s to “that”s
- many “it's”s to “its”s
- every “standart” to “standard”

and so on. It is my hope that these tiny corrections make for a better reading experience, and I have put references in the margins to the exact pages in the original typescript for those who want to compare any particular bit with what came from the horse's mouth.

For what it's worth, my "philosophy" when TEX'ing PS was to imagine that I did it right as AG was typing (or perhaps even dictating to me! That would certainly explain the variance in punctuation.). Sometimes I would ask him whether something should be an E or an e, and thus catch little misprints. Unfortunately, I wasn't there to ask about the mathematical content, so I'll leave that as intact as I found it, for the reader to wrestle with. Among the many remarkable qualities of PS is that it is a record of mathematics *as it is being made*, and not just a polished record for a journal made after the fact.

Some people say that PS is mostly visions and loose ideas; while those are certainly present, I hope perhaps to change this impression with this edition where all the "Lemmata", "Propositions", "Corollaries", etc., are marked up in the usual way.

I hope you'll forgive me for deviating from the Computer Modern font family and choosing Bitstream Charter with Pichaureau's `mathdesign` package instead. I find it reads well on the screen and the density mimics that of the AG's typewriter. If you don't like it, you're welcome to compile your own version with different settings!

Though I swapped MathDesign's ∞-symbol for Euler Math's ∞-symbol.

I give my work to the public domain, and hope merely that others may find it useful. Some people might ask me to identify myself, to which I can only reply: "I'd prefer not to."

the scrivener

Part I

The Take-off (a letter to Daniel Quillen)

Les Aumettes 19.2.1983 [p. 1]

Dear Daniel,

The importance of innocence.

1 Last year Ronnie Brown from Bangor sent me a heap of reprints and preprints by him and a group of friends, on various foundational matters of homotopical algebra. I did not really dig through any of this, as I kind of lost contact with the technicalities of this kind (I was never too familiar with the homotopy techniques anyhow, I confess) – but this reminded me of a few letters I had exchanged with Larry Breen in 1975, where I had developed an outline of a program for a kind of "topological algebra", viewed as a synthesis of homotopical and homological algebra, with special emphasis on topoi – most of the basic intuitions in this program arising from various backgrounds in algebraic geometry. Some of those intuitions we discussed, I believe, at IHES eight or nine years before, at a time when you had just written up your nice ideas on axiomatic homotopical algebra, published since in Springer's Lecture Notes. I write you under the assumption that you have not entirely lost interest for those foundational questions you were looking at more than fifteen years ago. One thing which strikes me, is that (as far as I know) there has not been any substantial progress since – it looks to me that an understanding of the basic structures underlying homotopy theory, or even homological algebra only, is still lacking – probably because the few people who have a wide enough background and perspective enabling them to feel the main questions, are devoting their energies to things which seem more directly rewarding. Maybe even a wind of disrepute for any foundational matters whatever is blowing nowadays! In this respect, what seems to me even more striking than the lack of proper foundations for homological and homotopical algebra, is the absence I daresay of proper foundations for topology itself! I am thinking here mainly of the development of a context of "tame" topology, which (I am convinced) would have on the everyday technique of geometric topology (I use this expression in contrast to the topology of use for analysts) a comparable

[cf. Künzer (2015)]

[Grothendieck (1975)]

[Quillen (1967)]

impact or even a greater one, than the introduction of the point of view of schemes had on algebraic geometry. The psychological drawback here I believe is not anything like messiness, as for homological and homotopical algebra (as for schemes), but merely the inrooted inertia which prevents us so stubbornly from looking innocently, with fresh eyes, upon things, without being dulled and imprisoned by standing habits of thought, going with a familiar context – *too* familiar a context! The task of working out the foundations of tame topology, and a corresponding structure theory for "stratified (tame) spaces", seems to me a lot more [p. 1'] urgent and exciting still than any program of homological, homotopical or topological algebra.

A short look into purgatory.

2 The motivation for this letter was the latter topic however. Ronnie Brown and his friends are competent algebraists and apparently strongly motivated for investing energy in foundational work, on the other hand they visibly are lacking the necessary scope of vision which geometry alone provides. They seem to me kind of isolated, partly due I guess to the disrepute I mentioned before – I suggested to try and have contact with people such as yourself, Larry Breen, Illusie and others, who have the geometric insight and who moreover, may not think themselves too good for indulging in occasional reflection on foundational matters and in the process help others do the work which should be done.

At first sight it has seemed to me that the Bangor group had indeed come to work out (quite independently) one basic intuition of the program I had envisioned in those letters to Larry Breen – namely that the study of n-truncated homotopy types (of semisimplicial sets, or of topological spaces) was essentially equivalent to the study of so-called n-groupoids (where n is any natural integer). This is expected to be achieved by associating to any space (say) X its "fundamental n-groupoid" $\Pi_n(X)$, generalizing the familiar Poincaré fundamental groupoid for $n = 1$. The obvious idea is that 0-objects of $\Pi_n(X)$ should be the points of X, 1-objects should be "homotopies" or paths between points, 2-objects should be homotopies between 1-objects, etc. This $\Pi_n(X)$ should embody the n-truncated homotopy type of X, in much the same way as for $n = 1$ the usual fundamental groupoid embodies the 1-truncated homotopy type. For two spaces X, Y, the set of homotopy-classes of maps $X \to Y$ (more correctly, for general X, Y, the maps of X into Y in the homotopy category) should correspond to n-equivalence classes of n-functors from $\Pi_n(X)$ to $\Pi_n(Y)$ – etc. There are very strong suggestions for a nice formalism including a notion of geometric realization of an n-groupoid, which should imply that any n-groupoid (or more generally of an n-category) is relativized over an arbitrary topos to the notion of an n-gerbe (or more generally, an n-stack), these become the natural "coefficients" for a formalism of non-commutative cohomological algebra, in the spirit of Giraud's thesis.

But all this kind of thing for the time being is pure heuristics – I never so far sat down to try to make explicit at least a definition of n-categories and n-groupoids, of n-functors between these etc. When I [p. 2] got the Bangor reprints I at once had the feeling that this kind of work had been done and the homotopy category expressed in terms of ∞-groupoids. But finally it appears this is not so, they have been working throughout with a notion of ∞-groupoid too restrictive for the purposes I had in mind (probably because they insist I guess on strict associativity of compositions, rather than associativity up to a (given) isomorphism, or rather, homotopy) – to the effect that the simply connected homotopy types they obtain are merely products of Eilenberg-MacLane spaces, too bad! They do not seem to have realized yet that this makes their set-up wholly inadequate to a sweeping foundational set-up for homotopy. This brings to the fore again to work out the suitable definitions for n-groupoids – if this is not done yet anywhere. I spent the afternoon today trying to figure out a reasonable definition, to get a feeling at least of where the difficulties are, if any. I am guided mainly of course by the topological interpretation. It will be short enough to say how far I got. The main part of the structure it seems is expressed by the sets F_i ($i \in \mathbb{N}$) of i-objects, the source, target and identity maps

$$\begin{aligned} \mathrm{s}_1^i, \mathrm{t}_1^i &: F_i \to F_{i-1} \quad (i \geq 1) \\ \mathrm{k}_1^i &: F_i \to F_{i+1} \quad (i \in \mathbb{N}) \end{aligned}$$

and the symmetry map (passage to the inverse)

$$\mathrm{inv}_i : F_i \to F_i \quad (i \geq 1),$$

satisfying some obvious relations: k_1^i is right inverse to the source and target maps $\mathrm{s}_1^{i+1}, \mathrm{t}_1^{i+1}$, inv_i is an involution and “exchanges” source and target, and moreover for $i \geq 2$

notation:
$d_a = \mathrm{k}_1^i(a)$,
$\check{u} = \mathrm{inv}_i(u)$

$$\begin{aligned} \mathrm{s}_1^{i-1}\mathrm{s}_1^i &= \mathrm{s}_1^{i-1}\mathrm{t}_1^i \left(\overset{\text{def}}{=} \mathrm{s}_2^i : F_i \to F_{i-2}\right) \\ \mathrm{t}_1^{i-1}\mathrm{s}_1^i &= \mathrm{t}_1^{i-1}\mathrm{t}_1^i \left(\overset{\text{def}}{=} \mathrm{t}_2^i : F_i \to F_{i-2}\right); \end{aligned}$$

thus the composition of the source and target maps yields, for $0 \leq j \leq i$, just *two* maps

$$\mathrm{s}_\ell^i, \mathrm{t}_\ell^i : F_i \to F_{i-\ell} = F_j \quad (\ell = i - j).$$

The next basic structure is the composition structure, where the usual compoisition of arrows, more specifically of i-objects ($i \geq 1$) $v \circ u$ (defined when $\mathrm{t}_1(u) = \mathrm{s}_1(v)$) must be supplemented by the Godement-type operations $\mu * \lambda$ when μ and λ are “arrows between arrows”, etc. Following this line of thought, one gets the composition maps

$$(u, v) \mapsto (v *_\ell u) : (F_i, \mathrm{s}_\ell^i) \times_{F_{i-\ell}} (F_i, \mathrm{s}_\ell^i) \to F_i,$$

the composition of i-objects for $1 \leq \ell \leq i$, being defined when the ℓ-target of u is equal to the ℓ-source of v, and then we have

$$\left.\begin{aligned} \mathrm{s}_1^i(v *_\ell u) &= \mathrm{s}_1^i(v) *_{\ell-1} \mathrm{s}_1^i(u) \\ \mathrm{t}_1^i(v *_\ell u) &= \mathrm{t}_1^i(v) *_{\ell-1} \mathrm{s}_1^i(u) \end{aligned}\right\} \quad \ell \geq 2 \text{ i.e. } \ell - 1 \geq 1$$

and for $\ell = 1$ [p. 2’]

$$\mathrm{s}_1(v *_1 u) = s_1(u)$$
$$\mathrm{t}_1(v *_1 u) = t_1(v)$$

(NB the operation $v *_1 u$ is just the usual composition $v \circ u$).

One may be tempted to think that the preceding data exhaust the structure of ∞-groupoids, and that they will have to be supplemented only by a handful of suitable *axioms*, one being *associativity* for the operation $\overset{i}{\underset{\ell}{*}}$, which can be expressed essentially by saying that that composition operation turns F_i into the set of arrows of a category having $F_{i-\ell}$ as a set of objects (with the source and target maps s^i_ℓ and t^i_ℓ, and with identity map $\mathrm{k}^{i-\ell}_\ell : F_{i-\ell} \to F_i$ the composition of the identity maps $F_{i-\ell} \to F_{i-\ell+1} \to \cdots \to F_{i-1} \to F_i$), and another being the Godement relation

$$(v' *_\alpha v) *_\nu (u' *_\alpha u) = (v' *_\nu u') *_\alpha (v *_\nu u)$$

(with the assumptions $1 \le \alpha \le \nu$, and $u, u', v, v'' in F_i$ and

$$\begin{cases} \mathrm{t}_\alpha(u) = \mathrm{s}_\alpha(u') \\ \mathrm{t}_\alpha(v) = \mathrm{s}_\alpha(v') \end{cases} \quad \mathrm{t}_\nu(u) = \mathrm{s}_\nu(v) = \mathrm{s}_\nu(v') = \mathrm{t}_\nu(u')$$

implying that both members are defined), plus the two relations concerning the inversion of i-objects ($i \ge 1$) $u \mapsto \check{u}$,

$$u *_1 \check{u} = \mathrm{id}_{\mathrm{t}_1(u)}, \qquad \check{u} *_1 u = \mathrm{id}_{\mathrm{s}_1(u)}, \qquad (\check{v} *_\ell \check{u}) = \ ? \quad (\ell \ge 2)$$

It just occurs to me, by the way, that the previous description of basic (or "primary") data for an ∞-groupoid is already incomplete in some rather obvious respect, namely that the symmetry-operation $\mathrm{inv}_i : u \mapsto \check{u}$ on F_i must be complemented by $i-1$ similar involutions on F_i, which corresponds algebraically to the intuition that when we have an $(i+1)$-arrow λ say between two i-arrows u and v, then we must be able to deduce from it another arrow from $\check{u}$ to $\check{v}$ (namely $u \mapsto \check{u}$ has a "functorial character" for variable u)? This seems a rather anodine modification of the previous set-up, and is irrelevant for the main point I want to make here, namely: that for the notion of ∞-groupoids we are after, all the equalities just envisioned in this paragraph (and those I guess which will ensure naturality by the necessary extension of the basic involution on F_i) should be replaced by "homotopies", namely by $(i+1)$-arrows between the two members. These arrows should be viewed, I believe, as being part of the data, they appear here as a kind of "secondary" structure. The difficulty which appears now is to work out the natural
[p. 3] coherence properties concerning this secondary structure. The first thing I could think of is the "pentagon axiom" for the associativity data, which occurs when looking at associativities for the compositum (for $\overset{i}{\underset{\ell}{*}}$ say) of four factors. Here again the first reflex would be to write down, as usual, an *equality* for two compositions of associativity isomorphisms, exhibited in the pentagon diagram. One suspects however that such

equality should, again, be replaced by a “homotopy”-arrow, which now appears as a kind of “ternary” structure – before even having exhausted the list of coherence “relations” one could think of with the respect to the secondary structure! Here one seems caught at first sight in an infinite chain of ever “higher”, and presumably, messier structures, where one is going to get hopelessly lost, unless one discovers some simple guiding principle for shedding some clarity in the mess.

“Fundamental ∞-groupoids” as objects of a “model category”?

3 I thought of writing you mainly because I believe that, if anybody, you should know if the kind of structure I am looking for has been worked out – maybe even *you* did? In this respect, I vaguely remember that you had a description of n-categories in terms of n-semisimplicial sets, satisfying certain exactness conditions, in much the same way as an ordinary category can be interpreted, via its “nerve”, as a particular type of semisimplicial set. But I have no idea if your definition applied only for describing n-categories with strict associativities, or not.

Still some contents in the spirit of your axiomatics of homotopical algebra – in order to make the question I am proposing more seducing maybe to you! One comment is that presumably, the category of ∞-groupoids (which is still to be defined) is a “model category” for the usual homotopy category; this would be at any rate one plausible way to make explicit the intuition referred to before, that a homotopy type is “essentially the same” as an ∞-groupoid up to ∞-equivalence. The other comment: the construction of the fundamental ∞-groupoid of a space, disregarding for the time being the question of working out in full the pertinent structure on this messy object, can be paraphrased in any model category in your sense, and yields a functor from this category to the category of ∞-groupoids, and hence (by geometric realization, or by localization) also to the usual homotopy category. Was this functor obvious beforehand? It is of a non-trivial nature only when the model category is *not* pointed – as a matter of fact the whole construction can be carried out canonically, in terms of a “cylinder object” I for the final object e of the model category, playing the role of the unit argument. [p. 3’]
It’s high time to stop this letter – please excuse me if it should come ten or fifteen years too late, or maybe one year too early. If you are not interested for the time being in such general nonsense, maybe you know someone who is …

Very cordially yours

20.2. [p. 4]

I finally went on pondering about a definition of ∞-groupoids, and it seems to me that, after all, the topological motivation does furnish the “simple guiding principle” which yesterday seemed to me to be still to be discovered, in order not to get lost in the messiness of ever higher order structures. Let me try to put it down roughly.

A bit of ordering in the mess of "higher order structures".

4 First I would like to correct somewhat the rather indiscriminate description I gave yesterday of what I thought of viewing as "primary", secondary, ternary etc. structures for an ∞-groupoid. More careful reflection conduces to view as the most primitive, starting structure on the set of sets F_i ($i \in \mathbb{N}$), as a skeleton on which progressively organs and flesh will be added, the mere data of the source and target maps

$$\mathrm{s}_i, \mathrm{t}_i : F_i \rightrightarrows F_{i-1} \quad (i \geq 1),$$

which it will be convenient to supplement formally by corresponding maps $\mathrm{s}_0, \mathrm{t}_0$ for $i = 1$, from F_0 to $F_{-1} \overset{\text{def}}{=}$ one-element set. In a moment we will pass to a universal situation, when the F_i are replaced by the corresponding "universal" objects $\boldsymbol{F}_i$ in a suitable category stable under finite products, where F_{-1} will be the final element. For several reasons, it is not proper to view the inversion maps $\mathrm{inv}_i : F_i \to F_i$, and still less the other $i-1$ involutions on F_i which I at first overlooked, as being part of the primitive or "skeletal" structure. One main reason is that already for the most usual 2-groupoids, such as the 2-groupoid whose 0-objects are ordinary (1-)groupoids, the 1-objects being equivalences between these (namely functors which are fully faithful and essentially surjective), and the 2-objects morphisms (or "natural transformations") between such, there is *not*, for an i-object $f : C \to C'$, a natural choice of an "inverse" namely of a quasi-inverse in the usual sense. And even assuming that such quasi-inverse is chosen for every f, it is by no means clear that such choice can be made involutive, namely such that $(f^\vee)^\vee = f$ for every f (and not merely $(f^\vee)^\vee$ isomorphic to f). The maps inv_i will appear rather, quite naturally, as "primary structure", and they will not be involutions, but "pseudo-involutions" (namely involutions "up to homotopy"). It turns out that among the various functors that we will construct, from the category of topological spaces to the category of ∞-groupoids (the construction depending on arbitrary choices and yielding a large bunch of mutually non-isomorphic functors, which however are "equivalent" in a sense which will have to be made precise) – there are choices neater than others, and some of these will yield in the primary structure maps inv_i which are actual involutions and similarly for the other pseudo-involutions, appearing in succession as higher order structure. The possibility of such neat and fairly natural
[p. 4'] choices had somewhat misled me yesterday.

What may look less convincing though at first sight, is my choice to view as non-primitive even the "degeneration maps" $\mathrm{k}_1^i : F_i \to F_{i+1}$, associating to every i-objects the $i+1$-object acting as an identity on the former. In all cases I have met so far, these maps are either given beforehand with the structure (of a 1-category or 2-category say), or they can be uniquely deduced from the axioms. In the present set-up however, they seem to me to appear more naturally as "primary" (not as primitive) structure, much in the same way as the inv_i. Different choices for associating an ∞-groupoid to a topological space, while yielding the same base-sets F_i, will however (according to this point of view) give rise to different maps k_1^i. The main motivation for this point of view comes from the fact that the mechanism for a uniform construction of the chain of ever higher order structures makes a basic use of the source and target maps only and of the "transposes" (see below), and (it seems to me) not at all of the degeneration maps, which in this respect rather are confusing the real picture, if viewed as "primitive". The degeneration maps rather appear as typical cases of primary structure, probably of special significance in the practical handling of ∞-groupoids, but not at all in the conceptual machinery leading up to the construction of the structure species of "∞-groupoids".

Much in the same way, the composition operations $\overset{i}{\underset{\ell}{*}}$ are viewed as primary, not as primitive or skeletal structure. Their description for the fundamental ∞-groupoid of a space – for instance the description of composition of paths – depends on arbitrary choices, such as the choice of an isomorphism (say) between $(I,1) \amalg_e (I,0)$ and I, where I is the unit interval, much in the same way as the notion of an inverse of a path depends on the choice of an isomorphism of I with itself, exchanging the two end-points 0 and 1. The operations $\overset{i}{\underset{2}{*}}$ of Godement take sense only once the composition operations $*_1$ are defined – they are "secondary structure", and successively the operations $*_3, \dots, *_i$ appear as ternary etc. structure. This is correctly suggested by the notations which I chose yesterday, where however I hastily threw all the operations into a same pot baptized "primary structure"!

5 It is about time though to come to a tentative precise definition of description of the process of stepwise introduction of an increasing chain of higher order structure. This will be done by introducing a canonical sequence of categories and functors *Jumping over the abyss!*

$$C_0 \to C_1 \to C_2 \to \cdots \to C_n \to C_{n+1} \to \cdots,$$

where C_n denotes the category harbouring the "universal" partial structure of a would-be ∞-groupoid, endowed only with its "structure of order $\leq n$". The idea is to give a direct inductive construction of this sequence, by describing C_0, and the passage from C_n to C_{n+1} ($n \geq 0$), namely from an n-ary to $(n+1)$-ary structure. As for the meaning of "universal structure", once a given structure species is at hand, it depends on the type of categories (described by the exactness properties one is assuming for these) one wants to take as carriers for the considered structure, and the type of exactness properties one assumes for the functor one allows between these. The choice depends partly on the particular species; if it is an algebraic structure which can be described say by a handful of composition laws between a bunch of base sets (or base-objects, when looking at "realizations" of the structure not only in the category (Sets)), one natural choice is to take categories with finite products, and functors which commute to these. For more sophisticated algebraic structures (including the structure of category, groupoid or the like), which requires for the description of data or axioms not only finite products, but also some fiber products, one other [p. 5] familiar choice is to take categories with finite inverse limits, and left exact functors. Still more sophisticated structures, when the description of the structure in terms of base objects requires not only some kind of inverse limits, but also more or less arbitrary direct limits (such as the structure of a comodule over an algebra, which requires consideration of tensor products over a ring object), still more stringent conditions will have to be imposed upon categories and functors between these, for the structure to make a sense in these categories, and the functors to transform a structure of this type in one category into one of same type in another. In most examples I have looked up, everything is OK taking categories which are topoi, and functors between these which are inverse image functors for morphisms of topoi, namely which are left exact and commute with arbitrary direct limits. There is a general theorem for the existence of universal structures, covering all these cases – for instance there is a "classifying topos" for most algebro-geometric structures, whose cohomology say should be viewed as the "classifying cohomology" of the structure species considered. In the case we are interested in here, it is convenient however to work with the smallest categories C_n feasible – which amounts to being as generous as possible for the categories one is allowing as carriers for the structure of an ∞-groupoid, and for the functors between these which are expected to carry an ∞-groupoid into an ∞-groupoid. What we will do is define ultimately a structure of an ∞-groupoid in a category C, as a sequence of objects F_i ($i \in \mathbb{N}$), endowed with some structure to be defined, assuming merely that in C finite products of the F_i exist, plus certain finite inverse limits built up with the F_i's and the maps s^i_ℓ, t^i_ℓ between them (the iterated source and target maps). It should be noted that the type of $\varprojlim$ we allow, which will have to be made precise below, is fixed beforehand in terms of the "skeletal" or "primitive" structure alone, embodied by the family of couples $(s_i, t_i)_{i\in\mathbb{N}}$. This implies that the categories $\mathbb{C}_i$ can be viewed as having *the same set of objects*, namely the objects $\boldsymbol{F}_i$ (written in bold now to indicate their universal nature, and including as was said before $\boldsymbol{F}_{-1}$ = final object), plus the finite products and iterated fiber products of so-called "standard" type. While I am writing, it appears to me even that the finite products here are of no use (so we just drop them both in the condition on categories which are accepted for harbouring ∞-groupoids, and in the set of objects of the categories C_i). Finally, the common set of objects of the categories $\mathbb{C}_i$ is the set of "standard" iterated fiber products of the $\boldsymbol{F}_i$, built up using only the primitive structure embodied by the maps s_i and t_i (which I renounce to underline!). This at the same time gives, in principle, a precise definition of $\mathbb{C}_0$, at least up to equivalence – it should not be hard anyhow to give a wholly explicit description of $\mathbb{C}_0$ as a small category, having a countable set of objects, once the basic notion of the standard iterated fiber-products has been explained.

or torsors under a group…

[p. 5']

[was: *underlined*]

Once $\mathbb{C}_0$ is constructed, we will get the higher order categories $\mathbb{C}_1$ (primary), $\mathbb{C}_2$ etc. by an inductive process of *successively adding arrows*. The category $\mathbb{C}_\infty$ will then be defined as the direct limit of the categories

$\mathbb{C}_n$, having the same objects therefore as $\mathbb{C}_0$, with

$$\mathrm{Hom}_\infty(X,Y) = \varinjlim_n \mathrm{Hom}_n(X,Y)$$

for any two objects. This being done, giving a structure of ∞-groupoid in any category C, will amount to giving a functor

$$\mathbb{C}_\infty \to C$$

commuting with the standard iterated fiber-products. This can be reexpressed, as amounting to the same as to give a sequence of objects (F_i) in C, and maps s^i_ℓ, t^i_ℓ between these, satisfying the two relations I wrote down yesterday (page 2) (and which of course have to be taken into account when defining C_0 to start with, I forgot to say before), and such that "standard" fiber-products defined in terms of these data should exist in C, plus a bunch of maps between these fiber-products (in fact, it will suffice to give such maps with target among the F_i's), satisfying certain relations embodied in the structure of the category $\mathbb{C}_\infty$. I am convinced that this bunch of maps (namely the maps stemming from arrows in $\mathbb{C}_\infty$) not only is infinite, but cannot either be generated in the obvious sense by a finite number, nor even by a finite number of infinite series of maps such as k^i_ℓ, inv_i, $\overset{i}{\underset{\ell}{*}}$, the compatibility arrows in the pentagon, and the like. More precisely still, I am convinced that none of the functors $\mathbb{C}_n \to \mathbb{C}_{n+1}$ is an equivalence, which amounts to saying that the structures of increasing order form a strictly increasing sequence – at every step, there is actual extra structure added. This is perhaps evident beforehand to topologists in the know, but I confess that for the time being it isn't to me, in terms uniquely of the somewhat formal description I will make of the passage of $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$. This theoretically is all that remains to be done, in order to achieve an explicit construction of the structure species of an ∞-category (besides the definition of standard fiber-products) – without having to get involved, still less lost, in the technical intricacies of ever messier diagrams to write down, with increasing order of the structures to be added...

I'll drop the qualification "iterated" henceforth!

[p. 6]

The topological model: hemispheres building up the (tentative) "universal ∞-(co)groupoid".

6 In the outline of a method of construction for the structure species, there has not been any explicit mention so far of the topological motivation behind the whole approach, which could wrongly give the impression of being a purely algebraic one. However, topological considerations alone are giving me the clue both for the description of the so-called standard fiber products, and of the inductive step allowing to wind up from $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$? The heuristics indeed of the present approach is simple enough, and suggested by the starting task, to define pertinent structure on the system of sets $F_i(X)$ of "homotopies" of arbitrary order, associated to an arbitrary topological space. In effect, the functors

$$X \mapsto F_i(X)$$

are representable by spaces D_i, which are easily seen to be i-disks. The source and target maps $s^i_\ell, t^i_\ell : F_i(X) \rightrightarrows F_{i-1}(X)$ are transposed to maps, which I may denote by the same letters,

$$s_\ell^i, t_\ell^i : D_{i-1} \rightrightarrows D_i.$$

Handling around a little, one easily convinces oneself that all the main structural items on $F_*(X)$ which one is figuring out in succession, such as the degeneracy maps k_ℓ^i, the inversion maps inv_i, the composition $v \cdot u = v *_i u$ for i-objects, etc., are all transposed of similar maps which are defined between the cells D_i, or which go from such cells to certain spaces, deduced from these by gluing them together – the most evident example in this respect being the composition of paths, which is transposed of a map from the unit segment I into $(I,1) \amalg_e (I,0)$, having preassigned values on the endpoints of I (which correspond in [p. 6'] fact to the images of the two maps $s_1^1, t_1^1 : D_0 = \text{one point} \rightrightarrows D_1 = I$). In a more suggestive way, we could say from this experiment that the family of discs $(D_i)_{i \in \mathbb{N}}$, together with the maps s, t and a lot of extra structure which enters into the picture step by step, is what we would like to call a *co-∞-groupoid in the category* (Top) of topological spaces (namely an ∞-groupoid in the dual category $(\mathrm{Top})^{\mathrm{op}}$), and that the structure of ∞-category on $F_*(X)$ we want to describe is the transform of this co-structure into an ∞-groupoid, by the contravariant functor from (Top) to (Sets) defined by X. The (iterated) amalgamated sums in (Top) which allow to glue together the various D_i's using the s and t maps between them, namely the corresponding fibered products in $(\mathrm{Top})^{\mathrm{op}}$, are indeed transformed by the functor h_X into fibered products of (Sets). The suggestion is, moreover, that if we view our co-structure in (Top) as a co-structure in the subcategory of Top, say B_∞, whose objects are the cells D_i and the amalgamated sums built up with these which step-wise enter into play, and whose arrows are all those arrows which are introduced step-wise to define the co-structure, and all compositions of these – that this should be *the universal structure* in the sense dual to the one we have been contemplating before; or what amount to the same, that the corresponding ∞-groupoid structure in the dual category B_∞^{op} is "universal" – which means essentially that it is none other than the universal structure in the category $\mathbb{C}_\infty$ we are after. Whether or not this expectation will turn out to be correct (I believe it is), we should be aware that, while the successive introduction of maps between the cells D_i and their "standard" amalgamated sums (which we will define precisely below) depends at every stage on arbitrary choices, the categories $\mathbb{C}_n$ and their limit $\mathbb{C}_\infty$ do not depend on any of these choices; assuming the expectation is correct, this means that up to (unique) isomorphism, the category B_∞ (and each of the categories B_n of which it appears as the direct limit) is independent of those choices – the isomorphism between two such categories transforming any one choice made for the first, into the corresponding choice made for the second. Also, while this expectation was of course the crucial motivation leading to the explicit description of $\mathbb{C}_0$ and of the inductive step from $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$, this description seems to me a reasonable one and in any case it makes a formal sense, quite independently of whether the expectation proves valid or not.

25.2. But no longer now and I do not really care! Cf. p. 10.

7 Before pursuing, it is time to give a more complete description of the primitive structure on (D_i), as embodied by the maps s, t, which I will now denote by

Gluing hemispheres: the "standard amalgamations".

NB it is more natural to consider φ^+ as "target" and φ^- as "source".

[p. 7]

$$\varphi_i^+, \varphi_i^- : D_i \rightrightarrows D_{i+1}.$$

It appears that these maps are injective, that their images make up the boundary $S_i = \dot{D}_{i+1}$ of D_{i+1}, more specifically these images are just two "complementary" hemispheres in S_i, which I will denote by S_i^+ and S_i^-. The kernel of the pair $(\varphi_i^+, \varphi_i^-)$ is just $S_{i-1} = \dot{D}_i$, and the common restriction of the maps φ_i^+, φ_i^- to S_{i-1} is an isomorphism

$$S_{i-1} \simeq S_i^+ \cap S_i^-.$$

This S_{i-1} in turn decomposes into the two hemispheres S_{i-1}^+, S_{i-1}^-, images of D_{i-1}. Replacing D_{i+1} by D_i, we see that the i-cell D_i is decomposed into a union of $2i+1$ closed cells, one being D_i itself, the others being canonically isomorphic to the cells S_j^+, S_j^- $(0 \leq j \leq i)$, images of $D_j \to D_n$ by the iterated morphisms

$$\varphi_{n,j}^+, \varphi_{n,j}^- : D_j \rightrightarrows D_n.$$

This is a cellular decomposition, corresponding to a partition of D_n into $2n+1$ open cells D_n, $S_j^+ = \varphi_{n,j}^+(D_j)$, $S_j^- = \varphi_{n,j}^-(D_j)$. For any cell in this decomposition, the incident cells are exactly those of strictly smaller dimension.

When introducing the operation $\overset{n}{\underset{\ell}{*}}$ with $\ell = n - j$, it is seen that this corresponds to choosing a map

$$D_n \to (D_n, S_j^+) \amalg_{D_j} (D_n, S_j^-),$$

satisfying a certain condition *, expressing the formulas I wrote down yesterday for s_1 and t_1 of $u *_\ell v$ – the formulas translate into demanding that the restriction of the looked-for map of D_n to its boundary S_{n-1} should be a given map (given at any rate, for $\ell \geq 2$, in terms of the operation $*_{\ell-1}$, which explains the point I made that the $*_\ell$-structure is of order just above the $*_{\ell-1}$-structure, namely (inductively) is of order ℓ ...). That the extension of this map of S_{n-1} to D_n does indeed exist, comes from the fact that the amalgamated sum on the right hand side is contractible for obvious reasons.

This gives a clue of what we should call "standard" amalgamated sums of the cells D_i. The first idea that comes to mind is that we should insist that the space considered should be contractible, excluding amalgamated sums therefore such as

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

which are circles. This formulation however has the inconvenience of not being directly expressed in combinatorial terms. The following formulation, which has the advantage of being of combinatorial nature, is presumably equivalent to the former, and gives (I expect) a large enough notion of "standardness" to yield for the corresponding notion of ∞-category enough structure for whatever one will ever need. In any [p. 7'] case, it is understood that the "amalgamated sum" (rather, finite $\varinjlim$) we are considering are of the most common type, when X is the finite union of closed subsets X_i, with given isomorphisms

$$X \simeq D_{n(i)},$$

the intersection of any two of these $X_i \cap X_j$ being a union of closed cells both in $D_{n(i)}$ and in $D_{n(j)}$. (This implies in fact that it is either a closed cell in both, or the union of two closed cells of same dimension m and hence isomorphic to S_m, a case which will be ruled out anyhow by the triviality condition which follows.†) The triviality or "standardness" condition is now expressed by demanding that the set of indices I can be totally ordered, i.e., numbered in such a way that we get X by successively "attaching" cells $D_{n(i)}$ to the space already constructed, $X(i-1)$, by a map from a sub-cell of $D_{n(i)}$, $S_j^\xi \to X(i-1)$ ($\xi \in \{\pm 1\}$), this map of course inducing an isomorphism, more precisely *the* standard isomorphism, $\varphi_{n(i)}^\xi : S_j^\xi \simeq D_j$ with S_j^ξ one of the two corresponding cells S_j^+, S_j^- in some $X_{i'} \simeq D_{n(i')}$. The dual translation of this, in terms of fiber products in a category C endowed with objects F_i ($i \in \mathbb{N}$) and maps s_1^i, t_1^i between these, is clear: for a given set of indices I and map $i \mapsto n(i) : I \to \mathbb{N}$, we consider a subobject of $\prod_{i \in I} D_{n(i)}$, which can be described by equality relations between iterated sources and targets of various components of $u = (u_i)_{i \in I}$ in P, the structure of the set of relations being such that I can be numbers, from 1 to N say, in such a way that we get in succession $N-1$ relations on the N components u_i respectively ($2 \le i \le N$), every relation being of the type $f(u_i) = g(u'_i)$, with f and g being iterated source of target maps, and $i' < i$. (Whether source or target depending in obvious way on the two signs ξ, ξ'.)

†False, [picture of disk divided in three]

[p. 8] 21.2.

8 Returning to the amalgamated sum $X = \bigcup_i X_i$, the cellular decompositions of the components $X_i \simeq D_{n(i)}$ define a cellular decomposition of X, whose set of cells with incidence relation forms a finite ordered set K, finite union of a family of subsets $(K_i)_{i \in I}$, with given isomorphisms

Description of the universal primitive *structure.*

$$f_i : K_i \simeq \mathrm{J}_{n(i)} \quad (i \in I),$$

where for every index $n \in \mathbb{N}$, J_n denotes the ordered set of the $2n+1$ cells S_j^ξ ($0 \le j \le n-1$, $\xi \in \{\pm 1\}$), D_n of the pertinent cellular decomposition of D_n. We may without loss of generality assume there is no inclusion relation between the K_i, moreover the standardness condition described above readily translates into a condition on this structure K, $(K_i)_{i \in I}$, $(f_i)_{i \in I}$, and implies that for $i, i' \in I$, $K_i \to K_{i'}$, is a "closed" subset in the two ordered sets $K_i, K_{i'}$ (namely contains with any element x the elements smaller than x), and moreover isomorphic (for the induced order) to some J_n. Thus the category B_0 can be viewed as the category of such "standard ordered sets" (with the extra structure on these just said), and the category C_0 can be defined most simply as the dual category B_0^{op}. (NB the definition of morphisms in B_0 is clear I guess ...) I believe the category B_0 is stable under amalgamated sums $X \amalg_Z Y$, provided however we insist that the empty structure K is *not* allowed – otherwise we have to restrict to amalgamated sums with $Z \neq \emptyset$. It seems finally more convenient to exclude the empty structure in B_0, i.e. to exclude the final element from $\mathbb{C}_0$, for the benefit of being able to state that $\mathbb{C}_0$ (and all categories $\mathbb{C}_n$) are stable under amalgamated sums, and that the functors $\mathbb{C}_n \to C$ (and ultimately $\mathbb{C}_\infty \to C$) we are interested in are just those commuting to arbitrary binary amalgamated sums (without awkward reference to the objects $\boldsymbol{F}_i$ and the iterated source and target maps between them...).

this is obviously false, see P.S. p.12.

9 The category $\mathbb{C}_0$ being fairly well understood, it remains to complete the construction by the inductive step, passing from $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$. The main properties I have in mind therefore, for the sequence of categories $\mathbb{C}_n$ and their limit $\mathbb{C}_\infty$, are the following two.

The main inductive step: just add coherence arrows! The abridged story of an (inescapable and irrelevant) ambiguity

(A) For any $K \in \mathrm{Ob}(\mathbb{C}_\infty)$ $(= \mathrm{Ob}(\mathbb{C}_0))$, and any two arrows in $\mathbb{C}_\infty$

$$f, g : K \rightrightarrows F_i,$$

with $i \in \mathbb{N}$, and such that either $i = 0$, or the equalities

$$s_1^i f = s_1^i g, \quad t_1^i f = t_1^i g \tag{1}$$

hold (case $i \geq 1$), there exists $h : K \to F_{i+1}$ such that

$$s_1^{i+1} h = f, \quad t_1^{i+1} h = g. \tag{2}$$

(B) For any $n \in \mathbb{N}$, the category $\mathbb{C}_{n+1}$ is deduced from $\mathbb{C}_n$ by keeping the same objects, and just adding new arrows h as in (A), with f, g arrows in $\mathbb{C}_n$.

The expression "deduces from" in (B) means that we are adding arrows $h : K \to F_i$ (each with preassigned source and target in $\mathbb{C}_n$), with as "new axioms" on the bunch of these uniquely the two relations (2) of (A), the category $\mathbb{C}_{n+1}$ being deduced from $\mathbb{C}_n$ in an obvious way, as the solution of a universal problem within the category of all categories where binary amalgamated products exist, and "maps" between these being functors which commute to those fibered products. In practical terms, the arrows of $\mathbb{C}_{n+1}$ are those deduced from the arrows in $\mathbb{C}_n$ and the "new" arrows h, by combining formal operations of composing arrows by $v \circ u$, and taking (binary) amalgamated products of arrows.

[p. 8']

This has to be corrected – amalgamated sums exist in $\mathbb{C}$, only – and *those* should be respected.

NB Of course the condition (1) in (A) is necessary for the existence of an h satisfying (2). That it is sufficient too can be viewed as an extremely strong, "universal" version of coherence conditions, concerning the various structures introduced on an ∞-groupoid. Intuitively, it means that whenever we have two ways of associating to a finite family $(u_i)_{i \in I}$

of objects of an ∞-groupoid, $u_i \in F_{n(i)}$, subjected to a standard set of relations on the u_i's, an element of some F_n, in terms of the ∞-groupoid structure only, then we have automatically a "homotopy" between these built in in the very structure of the ∞-groupoid, provided it makes at all sense to ask for one (namely provided condition (1) holds if $n \geq 1$). I have the feeling moreover that conditions (A) and (B) (plus the relation $\mathbb{C}_\infty = \varinjlim \mathbb{C}_n$) is all what will be ever needed, when using the definition of the structure species, – plus of course the description of $\mathbb{C}_0$, and the implicit fact that the categories $\mathbb{C}_n$ are stable under binary fiber products and the inclusion functors commute to these.† Of course, the category which really interests us is $\mathbb{C}_\infty$, the description of the intermediate $\mathbb{C}_n$'s is merely technical – the main point is that there should exist an increasing sequence $(\mathbb{C}_n)$ of subcategories of $\mathbb{C}_\infty$, having the same objects (and the "same" fiber-products), such that $\mathbb{C}_\infty$ should be the limit (i.e., every arrow in $\mathbb{C}_\infty$ should belong to some $\mathbb{C}_n$), and such that the passage from $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$ should satisfy (B). It is fairly obvious that these conditions alone do by no means characterize $\mathbb{C}_\infty$ up to equivalence, and still less the sequence of its subcategories $\mathbb{C}_n$. The point I wish to make though, before pursuing with a proposal of an explicit description, is that *this ambiguity is in the nature of things*. Roughly saying, two different mathematicians, working independently on the conceptual problem I had in mind, assming they both wind up with some explicit definition, will almost certainly get non-equivalent definitions – namely with non-equivalent categories of (set-valued, say) ∞-groupoids! And, [p. 9] secondly and as importantly, that *this ambiguity however is an irrelevant one*. To make this point a little clearer, I could say that a third mathematician, informed of the work of both, will readily think out a functor or rather a pair of functors, associating to any structure of Mr. X one of Mr. Y and conversely, in such a way that by composition of the two, we will associate to an X-structure (T say) another T', which will not be isomorphic to T of course, but endowed with a canonical ∞-equivalence (in the sense of Mr. X) $T \underset{\infty}{\simeq} T'$, and the same on the Mr. Y side. Most probably, a fourth mathematician, faced with the same situation as the third, will get his own pair of functors to reconcile Mr. X and Mr. Y, which very probably won't be equivalent (I mean isomorphic) to the previous one. Here however, a fifth mathematician, informed about this new perplexity, will probably show that the two Y-structures U and U', associated by his two colleagues to an X-structure T, while not isomorphic alas, admit however a canonical ∞-equivalence between U, and U' (in the sense of the Y-theory). I could go on with a sixth mathematician, confronted with the same perplexity as the previous one, who winds up with another ∞-equivalence between U and U' (without being informed of the work of the fifth), and a seventh reconciling them by discovering an ∞-equivalence between these equivalences. The story of course is infinite, I better stop with seven mathematicians, a fair number nowadays to allow themselves getting involved with foundational matters ... There should be a mathematical statement though resuming in finite terms this infinite story, but in order to write it down I guess a

†Inaccurate; see above

minimum amount of conceptual work, in the context of a given notion of ∞-groupoids satisfying the desiderata (A) and (B) should be done, and I am by no means sure I will go through this, not in this letter anyhow.

10 Now in the long last the explicit description I promised of $\mathbb{C}_{n+1}$ in terms of $\mathbb{C}_n$. As a matter of fact, I have a handful to propose! One choice, about the widest I would think of, is: for every pair (f, g) in $\mathbb{C}_n$ satisfying condition (1) of (A), add one new arrow h. To avoid set-theoretic difficulties though, we better first modify the definition of $\mathbb{C}_0$ so that the set of its objects should be in the universe we are working in, preferably even it be countable. Or else, and more reasonably, we will pick one h for every isomorphism class of situations (f, g) in $\mathbb{C}_n$. Another restriction to avoid too much redundancy – this was the first definition actually that flipped to my mind the day before yesterday – is to add a *new* h only when there is no "old" one, namely in $\mathbb{C}_n$, serving the same purpose. Then it came to my mind that there is a lot of redundancy still, thus there would be already infinitely many operations standing for the single operation $v \overset{i}{\circ} u$ say, which could be viewed in effect in terms of an arbitrary n-sequence ($n \geq 2$) of "composable" i-objects $u_1 = u, u_2 = v, u_3, \ldots, u_n$. The natural way to meet this "objection" would be to restrict to pairs (f, g) which cannot be factored non-trivially through another objects K' as

Cutting down redundancies – or: "l'embarras du choix".

[p. 9']

$$K \longrightarrow K' \underset{g'}{\overset{f'}{\rightrightarrows}} F_i \ .$$

But even with such restrictions, there remain a lot of redundancies – and this again seems to me in the nature of things, namely that there is no really natural, "most economic" way for achieving condition (A), by a stepwise construction meeting condition (B). For instance, in $\mathbb{C}_1$ already we will have not merely the compositions $v \overset{i}{\circ} u$, but at the same time simultaneous compositions

$$u_n \circ u_{n-1} \circ \cdots \circ u_1 \qquad (*)$$

for "composable" sequences of i-objects ($i \geq 1$), without reducing this (as is customary) to iteration of the binary composition $v \circ u$. Of course using the binary composition, and more generally iteration of $n' - ary$ compositions with $n' < n$ (when $n \geq 3$), we get an impressive bunch of operations in the n variables $u_1, \ldots, u_n$, serving the same purpose as (*). All these will be tied up by homotopies in the next step $\mathbb{C}_2$. We would like to think of this set of homotopies in $\mathbb{C}_2$ as a kind of "transitive system of isomorphisms" (of associativity), now the transitivity relations one is looking for will be replaced by homotopies again between compositions of homotopies, which will enter in the picture with $\mathbb{C}_3$, etc. Here the infinite story is exemplified by the more familiar situation of the two ways in which one could define a "$\otimes$-composition with associativity" in a category, starting either in terms of a binary operation, or with a bunch of n-ary operations – with, in both cases the associativity isomorphisms being an essential part of the structure. Here again, while it is generally (and quite validly) felt that the two points of view are equivalent; and both have their advantages and their drawbacks, still it is not true, I believe, that the two categories of algebraic structures “category with associative ⊗-operation”, using one or the other definition, are equivalent. Here the story though of the relation between the two notions is a finite one, due to the fact that it is related to the notion of 2-categories or 2-groupoids, instead of ∞-groupoids as before…

even not in the compoid??? context.

Thus I don't feel really like spending much energy in cutting down redundancies, but prefer working with a notion of ∞-groupoid which remains partly indeterminate, the main features being embodied in the conditions (A) and (B) and in the description of $\mathbb{C}_0$, without other specification.

[p. 10]

11 *Returning to the topological model* (*the canonical functor from spaces to “∞-groupoids”*).

One convenient way for constructing a category $\mathbb{C}_\infty$ would be to define for every $K, L \in \mathrm{Ob}(\mathbb{C}_0) = \mathrm{Ob}(B_0)$ the set $\mathrm{Hom}_\infty(K, L)$ as a subset of the set $\mathrm{Hom}(|L|, |K|)$ of continuous maps between the geometric realizations of L and K in terms of gluing together cells D_i, the composition of arrows in $\mathbb{C}_\infty$ being just composition of maps. This amounts to defining $\mathbb{C}_\infty$ as the dual of a category $\mathbb{B}_\infty$ of topological descriptions. It will be sufficient to define for every cell D_n and every subset $\mathrm{Hom}_\infty(D_n, |K|)$ of $\mathrm{Hom}(D_n, |K|)$, satisfying the two conditions:

(a) stability by compositions $D_n \to |K| \to |K'|$, where $K \to K'$ is an “allowable” continuous map, namely subjected only to the condition that its restriction to any standard subcell $D_{n'} \subset |K|$ is again “allowable”, i.e., in Hom_∞.

(b) Any “allowable” map $S_n \to |K|$ (i.e., whose restrictions to S_n^+ and S_n^- are allowable) extends to an allowable map $D_{n+1} \to |K|$.

Condition (a) merely ensures stability of allowable maps under composition, and the fact that $\mathbb{B}_\infty$ (endowed with the allowable maps as morphisms) has the correct binary amalgamated sums, whereas (b) expresses condition (a) on $\mathbb{C}_\infty$. These conditions are satisfied when we take as Hom_∞ subsets defined by tameness conditions (such as piecewise linear for suitable piecewise linear structure on the D_n's, or differentiable, etc.). The condition (b) however is of a subtler nature in the topological interpretation and surely not met by such sweeping tameness requirements only! Finally, the question as to whether we can actually in this way describe an “acceptable” category $\mathbb{C}_\infty$, by defining sets Hom_∞, namely describing $\mathbb{C}_\infty$ in terms of $\mathbb{B}_\infty$, seems rather subsidiary after all. We may think of course of constructing stepwise $\mathbb{B}_\infty$ via subcategories $\mathbb{B}_n$, by adding stepwise new arrows in order to meet condition (b), thus paraphrasing condition (B) for passage from $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$ – but it is by no means clear that when passing to the category $\mathbb{B}_{n+1}$ by composing maps of $\mathbb{B}_n$ and “new” ones, and using amalgamated sums too, there might not be some undesirable extra relations in $\mathbb{B}_{n+1}$, coming from the topological interpretation of the arrows in $\mathbb{C}_{n+1}$ as maps. To say it differently, universal algebra furnishes us readily with an acceptable sequence of categories $\mathbb{C}_n$ and hence $\mathbb{C}_\infty$, and by the universal properties of the $\mathbb{C}_n$ in terms of $\mathbb{C}_{n+1}$, we readily get (using arbitrary choices) a contravariant functor $K \mapsto |K|$ from $\mathbb{C}_\infty$ to the category of topological spaces (i.e., a co-∞-groupoid in (Top)), but it is by no means clear that this functor is faithful – and it doesn't really matter after all! [p. 10']

About replacing spaces by objects of a "model category".

12 I think I really better stop now, except for one last comment. The construction of a co-∞-groupoid in (Top), giving rise to the fundamental functor

$$(\mathrm{Top}) \longrightarrow (\infty\text{-groupoids}),$$

generalizes, as I already alluded to earlier, to the case when (Top) is replaced by an arbitrary "model category" M in your sense. Here however the choices occur not only stepwise for the primary, secondary, ternary etc. structures, but already for the primitive structures, namely by choice of objects D_i ($i \in \mathbb{N}$) in M, and source and target maps $D_i \rightrightarrows D_{i+1}$. These choices can be made inductively, by choosing first for D_0 the final object, or more generally any object which is fibrant and trivial (over the final objects), D_{-1} being the initial object, and defining further $S_0 = D_0 \amalg_{D_{-1}} D_0 = D_0 \amalg D_0$ with obvious maps $\psi_0^+, \psi_0^- : D_0 \to S_0$, and then, if everything is constructed up to D_n and $S_n = (D_n, \varphi_{n-1}^+) \amalg_{D_{n-1}} (D_n, \varphi_{n-1}^-)$, defining D_{n+1} as any fibrant and trivial object together with a cofibrant map

$$S_n \to D_{n+1},$$

and φ_n^+, φ_n^- as the compositions of the latter with $\psi_n^+, \psi_n^- : D_n \rightrightarrows S_n$. Using this and amalgamated sums in M, we get our functor

$$\mathbb{B}_0 = \mathbb{C}_0^{\mathrm{op}} \to M, \quad K \mapsto |K|_M,$$

commuting with amalgamated sums, which we can extend stepwise through the $\mathbb{C}_n^{\mathrm{op}}$'s to a functor $\mathbb{B}_\infty = \mathbb{C}_\infty^{\mathrm{op}} \to M$, provided we know that the objects $|K|_M$ of M ($K \in \mathrm{Ob}\,\mathbb{C}_0$) obtained by "standard" gluing of the D_n's in M, are again fibrant and trivial – and I hope indeed that your axioms imply that, via, say, that if $Z \to X$ and $Z \to Y$ are cofibrant and X, Y, Z are fibrant and trivial, then $X \amalg_Z Y$ is fibrant and trivial...

Among the things to be checked is of course that when we localize the category of ∞-groupoids with respect to morphisms which are "weak equivalences" in a rather obvious sense (NB the definition of the Π_i's of an ∞-groupoid is practically trivial!), we get a category equivalent to the usual homotopy category (Hot). Thus we get a composed functor

$$M \to (\infty\text{-groupoids}) \to (\mathrm{Hot}),$$

as announced. I have some intuitive feeling of what this functor stands for, at least when M is say the category of semisimplicial sheaves, or (more or less equivalently) of n-gerbes or ∞-gerbes on a given topos: namely it should correspond to the operation of "integration" or "sections" for n-gerbes (more generally for n-stacks) over a topos – which is indeed *the* basic operation (embodying non-commutative cohomology objects of the topos) in "non-commutative homological algebra".

I guess that's about it for today. It's getting late and time to go to bed! Good night.

[p. 11]

22.2.1983

13 *An urgent reflection on proper names: "Stacks" and "coherators".*

It seems I can't help pursuing further the reflection I started with this letter! First I would like to come back upon terminology. Maybe to give the name of n-groupoids and ∞-groupoids to the objects I was after is not proper, for two reasons: a) it conflicts with a standing terminology, applying to structure species which are frequently met and deserve names of their own, even if they turn out to be too restrictive kind of objects for the use I am having in mind – so why not keep the terminology already in use, especially for two-groupoids, which is pretty well suited after all; b) the structure species I have in mind is not really a very well determined one, it depends as we saw on choices, without any one choice looking convincingly better than the others – so it would be a mess to give an unqualified name to such structure, depending on the choice of a certain category $\mathbb{C}_\infty = \mathbb{C}$. I have been thinking of the terminology n-stack and ∞-stack (stack = "champ" in French), a name introduced in Giraud's book (he was restricting to champs = 1-champs), which over a topos reduced to a one-point space reduces in his case to the usual notion of a category, i.e., 1-category. Here of course we are thinking of "stacks of groupoids" rather than arbitrary stacks, which I would like to call (for arbitrary order $n \in \mathbb{N}$ or $n = \infty$) n-Gr-stack – suggesting evident ties with the notion of Gr-categories, we should say Gr-1-categories, of Mme Hoang Xuan Sinh. One advantage of the name "stack" is that the use it had so far spontaneously suggests the extension of these notions to the corresponding notions over an arbitrary topos, which of course is what I am after ultimately. Of course, when an ambiguity is possible, we should speak of n-$\mathbb{C}$-stacks – the reference to $\mathbb{C}$ should make superfluous the "Gr" specification. Thus n-$\mathbb{C}$-stacks are essentially the same as n-$\mathbb{C}$-stacks over the final topos, i.e., over a one point space. When both $\mathbb{C}$ and "Gr" are understood in a given context, we will use the terminology n-stack simply, or even "stack" when n is fixed throughout. Thus it will occur that in certain contexts "stack" will just mean a usual groupoid, in others it will mean just a category, but when $n = 2$ it will not mean a usual 2-groupoid, but something more general, defined in terms of $\mathbb{C}$.

The categories $\mathbb{C} = \mathbb{C}_\infty$ described before merit a name too – I would like to call them "coherators" ("cohéreurs" in French). This name is meant to suggest that $\mathbb{C}$ embodies a hierarchy of coherence relations, more accurately of coherence "homotopies". When dealing with stacks, the term *i-homotopies* (rather than i-objects or i-arrows) for the elements of the i^{th} component F_i of a stack seems to me the most suggestive – they [p. 11'] will of course be denoted graphically by arrows, such as $h : f \to g$ in the formulation of (A) yesterday. More specifically, I will call coherator any category *equivalent* to a category $\mathbb{C}_\infty$ as constructed before. Thus a coherator is stable under binary fiber products, moreover the F_i are recovered up to isomorphism as the indecomposable elements of $\mathbb{C}$ with respect to amalgamation. However, in a category $\mathbb{C}_\infty$, the objects $\boldsymbol{F}_i$ have non-trivial automorphisms – namely the "duality involutions" and their compositions (the group of automorphisms of $\boldsymbol{F}_i$ should turn out to be canonically isomorphic to $(\pm 1)^i$),† in other words by the mere category structure of a coherator we will not be able to recover the objects $\boldsymbol{F}_i$ in $\mathbb{C}$ up to unique isomorphism. Therefore, in the structure of a coherator should be included, too, the choice of the basic indecomposable objects $\boldsymbol{F}_i$ (one in each isomorphism class), and moreover the arrows $\mathrm{s}_1^i, \mathrm{t}_1^i : \boldsymbol{F}_i \rightrightarrows \boldsymbol{F}_{i-1}$ for $i \geq 1$ (a priori, only the *pair* $(\mathrm{s}_1^i, \mathrm{t}_1^i)$ can be described intrinsically in terms of the category structure of $\mathbb{C}$, once $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i-1}$ are chosen...). But it now occurs to me that we don't have to put in this extra structure after all – while the $\boldsymbol{F}_i$'s separately do have automorphisms, the system of objects $(\boldsymbol{F}_i)_{i\in\mathbb{N}}$ and of the maps $(\mathrm{s}_1^i)_{i\geq 1}$ and $(\mathrm{t}_1^i)_{i\geq 1}$ has only the trivial automorphism (all this of course is heuristics, I didn't really prove anything – but the structure of the full subcategory of a $\mathbb{C}_\infty$ formed by the objects $\boldsymbol{F}_i$ seems pretty obvious...). To finish getting convinced that the mere category structure of a coherator includes already all other relevant structure, we should still describe a suitable intrinsic filtration by subcategories $\mathbb{C}_n$. We define the $\mathbb{C}_n$ inductively, $\mathbb{C}_0$ being the "primitive structure" (the arrows are those deducible from the source and target arrows by composition and fiber products), and $\mathbb{C}_{n+1}$ being defined in terms of $\mathbb{C}_n$ as follows: add to $\mathrm{Fl}\,\mathbb{C}_n$ all arrows of $\mathbb{C}$ of the type $h : K \to \boldsymbol{F}_i$ $(i \geq 1)$ such that $\mathrm{s}_1^i h$ and $\mathrm{t}_1^i h$ are in $\mathbb{C}_n$, and the arrows deduced from the bunch obtained by composition and fibered products.‡ In view of these constructions, it would be an easy exercise to give an intrinsic characterization of a coherator, as a category satisfying certain internal properties.

I was a little rash right now when making assertions about the structure of the group of automorphisms of $\boldsymbol{F}_i$ – I forgot that two days ago I pointed out to myself that even the basic operation inv_i upon $\boldsymbol{F}_i$ need not even be involutions!§ However, I just checked that if in the inductive construction of coherators $\mathbb{C}_\infty$ given yesterday, we insist on the most trivial irredundancy condition (namely that we don't add a "new" homotopy $h : f \to g$ when there is already an old one), then any morphism $h : \boldsymbol{F}_i \to \boldsymbol{F}_i$ such that $\mathrm{s}f = \mathrm{s}$ and $\mathrm{t}f = \mathrm{t}$, is the identity – and that implies inductively that an automorphism of the system of $\boldsymbol{F}_i$'s related by the source and target maps $\mathrm{s}_1^i, \mathrm{t}_1^i$ is the identity. Thus it is correct after all, it seems, that the category structure of a coherator implies all other structure relevant to us.¶ [p. 12]

I do believe that the description given so far of what I mean by a coherator, namely something acting like a kind of pattern in order to define a corresponding notion of "stacks" (which in turn should be the basic coefficient objects in non-commutative homological algebra, as well as a convenient description of homotopy types) embodies some of the essential features of the theory still in embryo that wants to be developed. It is quite possible of course that some features are lacking still, for instance that some extra conditions have to be imposed

false, see PS

†false, see below

‡This may give however *too large* a category $\mathbb{C}_{n+1}$.

§Nor even automorphisms

¶Maybe false; it is safer to give moreover the subcategory $\mathbb{C}_0$.

upon $\mathbb{C}$, possibly of a very different nature from mere irredundancy conditions (which, I feel, are kind of irrelevant in this set-up). Only by pushing ahead and working out at least in outline the main aspects of the formalism of stacks, will it become clear whether or not extra conditions on $\mathbb{C}$ are needed. I would like at least to make a commented list of these main aspects, and possibly do some heuristic pondering on some of these in the stride, or afterwards. For today it seems a little late though – I have been pretty busy with non-mathematical work most part of the day, and the next two days I'll be busy at the university. Thus I guess I'll send off this letter tomorrow, and send you later an elaboration (presumably much in the style of this unending letter) if you are interested. In any case I would appreciate any comments you make – that's why I have been writing you after all! I will probably send copies to Ronnie Brown, Luc Illusie and Jean Giraud, in case they should be interested (I guess at least Ronnie Brown is). Maybe the theory is going to take off after all, in the long last!

Very cordially yours

PS (25.2.) I noticed a rather silly mistake in the notes of two days ago, when stating that the categories $\mathbb{C}_n$ admit fiber products: what is true is that the category $\mathbb{C}_0$ has fiber products (by construction, practically), and that these are fiber products also in the categories $\mathbb{C}_n$ (by construction equally), i.e., that the inclusion functors $\mathbb{C}_0 \to \mathbb{C}_n$ commute to fiber products. Stacks in a category C correspond to functors $\mathbb{C}_\infty \to C$ whose restriction to $\mathbb{C}_0$ commutes with fiber products. I carried the mistake along in the yesterday notes – it doesn't really change anything substantially. I will have to come back anyhow upon the basic notion of a coherator…

Part II

Test categories and test functors

Reflections on homotopical algebra [p. 1]

27.2.83.

The following notes are the continuation of the reflection started in my letter to Daniel Quillen written previous week (19.2 – 23.2), which I will cite by (L) ("letter"). I begin with some corrections and comments to this letter.

The unnoticed failure (of the present foundations of topology for expressing topological intuition).

14 Homotopical algebra can be viewed as being concerned mainly with the study of spaces of continuous maps between spaces and the algebraic analogons of these, with a special emphasis on homotopies between such homotopies between homotopies etc. The kind of restrictive properties imposed on the maps under consideration is exemplified by the typical example when demanding that the maps should be extensions, or liftings, of a given map. Homotopical algebra is not directly suited for the study of spaces of homeomorphisms, spaces of immersions, of embeddings, of fibrations, etc. – and it seems that the study of such spaces has not really yet taken off the ground. Maybe the main obstacle here lies in the wildness phenomena, which however, one feels, makes a wholly artificial difficulty, stemming from the particular way by which topological intuition has been mathematically formalized, in terms of the basic notion of topological spaces and continuous maps between them. This transcription, while adequate for the homotopical point of view, and partly adequate too for the use of analysts, is rather coarsely inadequate, it seems to me, in most other geometrico-topological contexts, and particularly so when it comes to studying spaces of homeomorphisms, immersions, etc. (in all those questions when "isotopy" is replacing the rather coarse homotopy relation), as well as for a study of stratified structures, when it becomes indispensable to give intrinsic and precise meaning to such notions as tubular neighborhoods, etc. For a structure theory of stratifications, it turns out (somewhat surprisingly maybe) that even the somewhat cumbersome context of topoi and pretopoi is better suited than topological spaces, and moreover directly applicable to unconventional contexts such as étale topology of schemes, where the conventional transcription of topological intuition in terms of topological spaces is quite evidently breaking down. To emphasize one point I was making in (L, p. 1), it seems to me that this breakdown is almost as evident in isotopy questions or for the needs of a structure theory of stratified "spaces" (whatever we mean by "spaces" . . .). It is a matter of amazement to me that this breakdown has not been clearly noticed, and still less overcome by working out a more suitable transcription of [p. 2] topological objects and topological intuition, by the people primarily concerned, namely the topologists. The need of eliminating wildness phenomena has of course been felt repeatedly, and (by lack of anything better maybe, or rather by lack of any attempt of a systematic reflection on what *was* needed) it was supposed to be met by the notion of piecewise linear structures. This however was falling from one extreme into another – from a structure species with vastly too many maps between "spaces", like a coat vastly too wide and floating around in a million wild wrinkles, to one with so few (not even a quadratic map from $\mathbb{R}$ to $\mathbb{R}$ is allowed!) that it feels like too narrow a coat, bursting apart on all edges and ends. The main defect here, technically speaking, seems to me the fact that numerical piece-wise linear functions are not stable under multiplication, and as a geometric consequence of this, that when contracting a compact p.l. subspace of a (compact, say) piecewise linear space into one point, we do not get on the quotient space a natural p.l. structure. This alone should have sufficed, one might think, to eliminate the piecewise linear structure species as a reasonable candidate for "doing topology" without wildness impediments – but strangely enough, it seems to be hanging around till this very day!

15 *Overall review of standard descriptions of homotopy types.*

But my aim here is not to give an outline of foundations on "tame topology", but rather to fill some foundational gaps in homotopy theory, more specifically in homotopical algebra. The relative success in the homotopical approach to topology is probably closely tied to the well known fact (Brouwer's starting point as a matter of fact, when he introduced the systematics of triangulations) that every continuous map between triangulated spaces can be approximated by simplicial maps. This gave rise, rather naturally, to the hope expressed in the "Hauptvermutung" – that two homeomorphic triangulated spaces admit isomorphic subdivisions, a hope that finally proved a delusion. With a distance of two or three generations, I would comment on this by saying that this negative result was the one to be expected, once it has become clear that neither of the two structure species one was comparing, namely topological spaces and triangulated spaces, was adequate for expressing what one is really after – namely an accurate mathematical transcription, in terms of "spaces" of some kind or other, of some vast and deep and misty and ever transforming mass of intuitions in our psyche, which we are referring to as "topological" intuition. There *is* something positive though, definitely, which can be viewed as an extremely weakened version of the Hauptvermutung, namely the fact that topological spaces on the one hand, and semi-simplicial sets [p. 3] on the other, give rise, by a suitable "localization" process (formally

analogous to the passage from categories of chain complexes to the corresponding derived categories), to eventually[?] "the same" (up to equivalence) "homotopy category". One way of describing it is via topological spaces which are not "too wild" as objects (the CW-spaces), morphisms being homotopy classes of continuous maps. The other is via semi-simplicial sets, taking for instance Kan complexes as objects, and again homotopy classes of "maps" as morphisms. The first description is the one most adapted to direct topological intuition, as long as least as no more adequate notion than "topological spaces" is at hand. The second has the advantage of being a purely algebraic description, with rather amazing conceptual simplicity moreover. In terms of the two basic sets of algebraic invariants of a space which has turned up so far, namely cohomology (or homology) on the one hand, and homotopy groups on the other, it can be said that the description via topological spaces is adequate for direct description of neither cohomology nor homotopy groups, whereas the description via semi-simplicial sets is fairly adequate for description of cohomology groups (taking simply the abelianization of the semi-simplicial set, which turns out to be a chain complex, and taking its homology and cohomology groups). The same can be said for the alternative algebraic description of homotopy types, using cubical complexes instead of semi-simplicial ones, which were introduced by Serre as they were better suited, it seems, for the study of fibrations and of the homology and cohomology spectral sequences relative to these. One somewhat surprising common feature of those two standard algebraic descriptions of homotopy types, is that neither is any better adapted for a direct description of homotopy groups than the objects we started with, namely topological spaces. This is all the more remarkable as it is the homotopy groups really, rather than the cohomology groups, which are commonly viewed as *the* basic invariants in the homotopy point of view, sufficient, e.g., for test whether a given map is a "weak equivalence", namely gives rise to an isomorphism in the homotopy category. It is here of course that the point of view of "stacks" ("champs" in French) of (L) (previously called "∞-groupoids" in the beginning of the reflections of (L)) sets in. These presumably give rise to a "category of models" and [?] there, to the usual homotopy category, in much the same way as topological spaces or simplicial (or cubical) complexes, thus yielding a third [?] description of homotopy types, and corresponding wealth of algebraico-geometric intuitions. Moreover, stacks are ideally suited for expressing the homotopy groups, in an even more direct way than simplicial complexes allow description of [p. 4]
homology and cohomology groups. As a matter of fact, the description is formally analogous, and nearly identical, to the description of the homology groups of a chain complex – and it would seem therefore that that stacks (more specifically, Gr-stacks) are in a sense the closest possible non-commutative generalization of chain complexes, the homology groups of the chain complex becoming the homotopy groups of the "non-commutative chain complex" or stack.

It is well understood, since Dold-Puppe, that chain complexes form a category equivalent to the category of abelian group objects in the category of semi-simplicial sets, or equivalently, to the category of semi-simplicial abelian groups. By this equivalence, the homology groups of the chain complex are identified with the *homotopy* groups of the corresponding homotopy type. As for the homology and cohomology groups of this homotopy type, their description in terms of the chain complex we started with is kind of delicate (I forgot all about it I am afraid!). A fortiori, when a homotopy type is described in terms of a stack, i.e., a "non-commutative chain complex", there is no immediate way for describing its homology or cohomology groups in terms of this structure. What probably should be done, is to define first a *nerve* functor from stacks to semi-simplicial sets (generalizing the familiar nerve functor defined on the category of categories), and define homology invariants of a stack via those of the associated semi-simplicial set (directly suited for calculating these).

3.5. This is an error – it appeared to me after that the cohomology can be expressed much in the same way as for simplicial or "cubical complexes, using a [?] the "source" and "target" structure of the stack (i.e., part of the primitive structure).

Stacks over topoi as unifying concept for homotopical and cohomological algebra.

16 These reflections on the proper place of the notion of a stack which in standard homotopy algebra are largely a posteriori – the clues they give are surely not so strong as to give an imperative feeling for the need of developing this new approach to the homotopy category. Rather, the imperative feeling comes from the intuitions tied up with non-commutative cohomological algebra over topological spaces, and more generally over topoi, in the spirit of Giraud's thesis, where a suitable formalism for non-commutative K^i's for $i = 0$, 1 or 2 is developed. He develops in extenso the notion of stacks, we should rather say now 1-stacks, over a topos, constantly alluding (and for very understandable reasons!) to the notion of a 2-stack, appearing closely on the heels of the 1-stacks. Keeping in mind that 0-stacks are just ordinary sheaves of sets, on the space or the topos considered, the hierarchy of increasingly higher and more sophisticated notions of 0-stacks, 1-stacks, 2-stacks, etc., which will have to be developed over an arbitrary topos, just parallels the hierarchy of corresponding notions over the one-point space, namely sets (= 0-stacks), categories (or 1-stacks), 2-categories, etc. Among these structures, those generalizing groupoids among categories, namely Gr-stacks of various orders n, play a significant role, especially for the description of homotopy types, but equally for a non-commutative "geometric" interpretation of the cohomology groups $\mathrm{H}^i(X,F)$ of arbitrary dimension (or "order"), of a topos X with coefficients in an abelian sheaf F. The reflections in (L) therefore were directly aimed at getting a grasp on a definition of such Gr-stacks, and whereas it seems to me to have come to a concrete starting point for such a definition, a similar reflection for defining just stacks rather than Gr-stacks is still lacking. This is one among the manifold things I have in mind while sitting down on the present reflections.

[Giraud 1971]

[p. 5]

Thus n-stacks, relativized over a topos to "n-stacks over X", are viewed primarily as the natural "coefficients" in order to do (co)homological algebra of dimension $\leq n$ over X. The "integration" of such coefficients, in much the same spirit as taking objects $\mathrm{R}\Gamma_*$ (with $\mathrm{R}\Gamma$ the derived functor of the sections functor Γ) for complexes of abelian sheaves F_j on X, is here merely the trivial operation of taking sections, namely the "value" of the n-stack on the final object of X (or of a representative site of X, if X is described in terms of a site). The result of integration is again an n-stack, whose homotopy groups (with a dimension shift of n) should be viewed as the cohomology invariants $\mathrm{H}^i(X, F_*)$, where F_* now stands for the n-stack rather than for a complex of abelian sheaves. In my letters (two or three) to Larry Breen in 1975, I develop some heuristics along this point of view, with constant reference to various geometric situations (mainly from algebraic geometry), providing the motivations. The one motivation maybe which was the strongest, was the realization that the classical Lefschetz theorem about comparison of homology and homotopy invariants of a projective variety, and a hyperplane section – once it was reformulated suitably so as to get rid of non-singularity assumptions, replaced by suitable assumptions on cohomological "depth" – could be viewed as comparison statements of "cohomology" with coefficients in more or less arbitrary stacks. This is carries through completely, within the then existing conceptual framework restricted to 1-stacks, in the thesis of Mme Raynaud, a beautiful piece of work. There seems to me to be overwhelming evidence that her results (maybe her method of proof too?) should generalize in the context of non-commutative cohomological algebra of arbitrary dimension, with a suitable property of ind-finiteness as the unique restriction on the coefficient stacks under consideration. [Grothendieck (1975)] [Raynaud (1975)] [p. 6]

Technically speaking, ∞-stacks are the common denominator of n-stacks for arbitrary n, in much the same way as n-stacks appear both as the next-step generalization of $(n-1)$-stacks (the former forming a category which admits the category of $(n-1)$-stacks as a full subcategory), and as the most natural "higher order structure" appearing on the category of all $(n-1)$-stacks (and on various analogous categories whose objects are $(n-1)$-stacks subject to some restrictions or endowed with some extra structure). I'll have to make this explicit in due course. For the time being, when speaking of "stacks" or "Gr-stacks", it will be understood (unless otherwise specified) that we are dealing with the infinite order notions, which encompass the finite ones.

Working out a theory of stacks over topoi, as the natural foundation of non-commutative cohomological algebra, would amount among others to write Giraud's book within this considerably wider framework. Of course, this mere prospect wouldn't be particularly exciting by itself, if it did not appear as something more than grinding through an unending exercise of rephrasing and reproving known things, replacing everywhere $n = 0$, 1 or 2 by arbitrary n. I am convinced however that there is a lot more to it – namely the fascination of gradually discovering and naming and getting acquainted with presently still unknown, unnamed, mysterious structures. As is the case so often when making a big step backwards for gaining new perspective, there is not merely a quantitative change (from $n \leq 2$ to arbitrary n say), but a qualitative change in scope and depth of vision. One such step was already taken I feel by Daniel Quillen and others, when realizing that homotopy constructions make sense not only in the usual homotopy category, or in one or the other categories of models which give rise to it, but in more or less

arbitrary categories, by working with semi-simplicial objects in these. The step I am proposing is of a somewhat different type. The notion of a stack here appears as the unifying concept for a synthesis of homotopical algebra and non-commutative cohomological algebra. This (rather than merely furnishing us with still another description of homotopy types, more convenient for expression of the homotopy groups) seems to me the real "raison d'être" of the notion of a stack, and the main motivation
[p. 7] for pushing ahead a theory of stacks.

16bis *Categories as models for homotopy types. First glimpse upon an "impressive bunch" (of modelizers).*

One last comment before diving into more technical matters. Without even climbing up the ladder of increasing sophistication, leading up to [stacks?], there is on the very first step, namely with just usual categories, the possibility of describing [?]. Namely, there are two natural, well-known ways to associate to a category C (I mean here a "small" category, belonging to the given universe we are working in) some kind of topological object, and hence a homotopy type. One is by associating to C the topos $C^\wedge$ or [?], namely arbitrary contravariant functors from C to (Sets). The other is through the [nerve?] functor, associating to C a semi-simplicial set – and hence, if this suits us better, a topological space, by taking the geometrical realization. By a construction of Verdier, any topos and therefore $\mathrm{Top}(C)$ gives rise canonically to a [pro-object?] in the category of semi-simplicial sets, and hence by "localization" to a pro-object in the homotopy category (namely a "pro-homotopy type" in the terminology of Artin-Mazur). In the same way, the nerve $\mathrm{N}(C)$ gives rise to a homotopy type – and of course [?] and may be called *the* homotopy type of C. When C is a groupoid, we get merely a 1-truncated homotopy type, namely with homotopy groups π_n which vanish for $n \geq 2$, or equivalently, with connected components (corresponding of course to connected components of C) $K(\pi, 1)$ spaces. This had led me at one moment in the late sixties to hastily surmise that even for arbitrary C, we got merely such sums of $K(\pi, 1)$ spaces (namely, that the homotopy type of C does not change when replacing C by the universal enveloping groupoid, deduced from C by making formally invertible all its arrows). As Quillen pointed out to me, this is definitely not so – indeed, using categories, we get (up to isomorphism) arbitrary homotopy types. This is achieved, I guess, using the left adjoint functor N' from the inclusion functor

$$(\mathrm{Cat}) \overset{\mathrm{N}}{\hookrightarrow} (\mathrm{Ss\ sets})$$

which is fully faithful (the adjoint functor being therefore a localization functor), and showing that for a semisimplicial set K, the natural map

$$K \to \mathrm{N}\,\mathrm{N}'(K)$$

is a weak equivalence; or what amounts to the same, that the set of arrows in (Ss sets) by which we localize in order to get (Cat) (namely those transformed into invertible arrows by N') is made up with weak equivalences only. This would imply that we may reconstruct the usual homotopy category, up to equivalence, in terms of (Cat), by just localiz-
[p. 8]

5.3. This is false, see §24 below (p. 21–23).

ing (Cat) with respect to weak isomorphism, namely functors $C \to C'$ inducing an isomorphism between the corresponding homotopy types. Pushing a little further in this direction, one may conjecture that (Cat) is a "closed model category", whose weak equivalences are the functors just specified, and whose cofibrations are just functors which are injective on objects and injective on arrows.

The fact that [?] and one moreover it seems which has not found its way still into the minds of topologists or homotopists, with only few exceptions I guess. These objects are extremely simply and familiar to most mathematicians; what is somewhat more sophisticated is the process of localization towards homotopy types, or equivalently, the explicit description of weak equivalences, within the framework of usual category theory. This would amount more or less to the same as describing the homotopy groups of a category, which does not seem any simpler than the same task for its nerve. As for the cohomology invariants, which can be interpreted as the left derived functors of the $\varprojlim_C$ functor, or rather its values on particular presheaves (for instance constant presheaves), they are of course known to be significant, independently of any particular topological interpretation, but they are not expressible in direct terms. (The most common computation for these is again via the nerve of C.)

This situation suggests that for any natural integer $n \geq 1$, the category of n-stacks can be used as a category of models for the usual homotopy category, in particular any n-stack gives rise to a homotopy type, and up to equivalence we should get any homotopy type in this way (for instance, through the n-category canonically associated to any 1-category giving rise to this homotopy type). The homotopy types coming from n-Gr-stacks, however, should be merely the n-truncated ones, namely those whose homotopy groups in dimension $> n$ are zero. Moreover, n-Gr-stacks appear as the most adequate algebraic structures for expressing n-truncated homotopy types, the latter being deduced from the former, presumably, by the same process of localization by weak equivalences. Moreover, in the context of n-Gr-stacks, the notions of homotopy groups and of weak equivalences are described in a particularly obvious way. Thus, passing to the limit case $n = \infty$, it is [∞-Gr-stacks?] rather than general ∞-stacks which appear as the neatest [model for homotopy types?].

These reflections suggest that there should be a rather impressive bunch of algebraic structures, each giving rise to a model category for the usual homotopy category, or in any case yielding this category by localization with respect to a suitable notion of "weak equivalences". [p. 9]
The "bunch" is all the more impressive, if we remember that the notion of stack (dropping now the qualification n, namely assuming $n = \infty$) is not really a uniquely defined one, but depends on the choice of a "coherator", namely (mainly) a category C satisfying certain requirements, which can be met in a vast variety of ways, presumably. The construction of coherators is achieved in terms of universal algebra, which seems here the indispensable Ariadne's thread not to get lost in overwhelming

messiness. The natural question which arises here (and which do not feel though like pursuing) is to give in terms of universal algebra some kind of characterization, among all algebraic structures, of those which give rise in some specified way (including the known cases) to a category of models say for usual homotopy theory.

17 *The Artin-Mazur cohomological criterion for weak equivalence.*

When referring (p. 5) to the notion of a stack as a unifying concept for homotopical algebra and non-commutative cohomological algebra, I forgot to mention one significant observation of Artin-Mazur along those lines (messy unification), namely that (for ordinary homotopy types) weak equivalences (namely maps inducing isomorphisms for all homotopy groups) can be characterized as being those which induce isomorphisms on cohomology groups of the spaces considered not only for constant coefficients, but also for arbitrary twisted coefficients on the target space, including also the non-commutative H^0 and H^1 for twisted (non-commutative) group coefficients. This is indeed the basic technical result enabling them, from known results on étale cohomology of schemes (including non-commutative H^1's) to deduce corresponding information on homotopy types. Maybe however that the observation has acted rather as a dissuasion for developing higher non-commutative cohomological algebra, as it seemingly says that the non-commutative H^1, plus the commutative H^i's, was all that was needed to recover stringent information about homotopy types. In other words, there wasn't too little in Giraud's book, but rather, too much!

[p. 10] 28.2.

18 *Corrections and contents to letter. Bénabou's lonely approach.*

I still have to correct a number of "étourderies" of (L). The most persistent one, ever since page 8 of that letter, is about fiber products in the coherator $\mathbb{C}_\infty$, or, equivalently, amalgamated sum in the dual category $\mathbb{B}_\infty$. The "correction" I added in the last PS (p. 12) is still incorrect, namely it is not true even in the subcategory $\mathbb{B}_0$ of $\mathbb{B}$ that arbitrary amalgamated sums exist. I was mislead by the interpretation of elements of $\mathbb{B}_0$ in terms of (contractible) spaces, obtained inductively by gluing together discs D_n ($n \in \mathbb{N}$) via subdiscs, corresponding to the cellular subdivisions of the discs D_n considered p. 7. I was implicitly thinking of amalgamated sums of the type

$$K \amalg_L M,$$

where $L \to K$ and $M \to K$ are *monomorphisms*, corresponding to the geometric vision of embeddings – in which case the usual amalgamated sum in the category of topological spaces is indeed contractible, which was enough to make me happy. But I overlooked the existence of morphisms in $\mathbb{B}_0$ which are visibly no monomorphisms, such as $K \amalg_L K \to K$ the codiagonal map, when $L \to K$ is a strict inclusion of discs. In any case, I will have to come back upon the description of the categories $\mathbb{B}_0$ and its dual $\mathbb{C}_0$ and upon the definition of coherators, after the heuristic introduction (L). It will be time then too to correct the mistaken description of $\mathbb{C}_{n+1}$ in terms of $\mathbb{C}_n$, which I propose on p. 11', yielding probably much too big a subcategory of $\mathbb{C}_\infty$ – the correct definition should make explicit use of the total set A of "new" arrows, by which $\mathbb{C}_\infty$ is described in terms of universal algebra via $\mathbb{C}_0$. Another étourdie the day before, p. 7', is the statement that in a standard amalgamated sum, any intersection of two maximal subcells is a subcell – which is seen to be false in the standard example.

Drawing of two hemispheres intersecting in a sphere.

Of lesser import is the misstatement on p. 2', stating that the associativity relation for the operation $*_\ell$ should by replaced by a homotopy arrow $(\lambda * \mu) * \nu \to \lambda * (\mu * \nu)$. This is OK for $\ell = 1$ (primary composition), but already for $\ell = 2$ does not make sense as stated, because [highlighted, maybe: because the two sides do not have the same source and target?]. Here the statement should be replaced by one making sense, with a homotopy "making commutative" a certain square, and accordingly for cubes, etc. for higher order compositions $\lambda *_\ell \mu$, to give reasonable meaning to associativity. Anyhow, such painstaking explicitations of particular coherence properties (rather, coherence homotopies) is kind of ruled out by the sweeping axiomatic description of the kind of structure species we want for a "stack", at least, I guess, in a large part of the development of the theory of Gr-stacks. A systematic study of particular sets of homotopies is closely connected of course to an investigation into irredundancy conditions which can be figured out for a coherator $\mathbb{C}$. This is indeed an interesting topic, but I decided not to get involved in this, unless I am really forced to! [p. 11]

The basic notion which has been peeling out in the reflection (L) is of course the notion of a coherator. Concerning terminology, it occurred to me that the dual category $\mathbb{B}_\infty$ to $\mathbb{C}_\infty$ is more suggestive in some cases, for instance because of the topological interpretation attached to its objects, and (more technically) because of formal analogy of the role of this category, for developing homotopical algebra, with the category of the standard (ordered) simplices. Both mutually dual objects $\mathbb{B}_\infty$, $\mathbb{C}_\infty$ seem to me to merit a name, I suggest to call them respectively *left* and *right coherators*, or simply *coherators* of course when for a while it is understood on which side of the mirror we are playing the game.

One last comment still before taking off for a heuristic voyage of discovery of stacks! I just had a glance at Bénabou's exposé in 1967 of what he calls "bicategories" (Springer Lecture Notes nº 47, p. 1–77). [Bénabou (1967)] These are none else, it appears, than *non-associative* 2-categories, namely 2-stacks in the terminology I am proposing (but not 2-Gr-stacks – namely it is a particular case of a general notion of ∞-stack which has still to be developed). The most interesting feature of this exposé, it seems to me, is the systematic reference to topological intuition, notably of the structure of various diagrams. His terminology, referring to elements of F_0, F_1, F_2 respectively as 0-cells, 1-cells and 2-cells, is quite suggestive of an idea of topological realization of a 2-stack – it is not clear from this exposé whether Bénabou has worked out this idea, nor whether he has made a connection with Quillen's ideas on axiomatics of homotopy theory, which appeared the same year in the same series. In any case, in the last section of his exposé, he deals with his bicategories formally as with topological spaces, much in the same spirit as the one I was contemplating since around 1975, and which is now motivating the present notes. While there is no mention of Bénabou's ideas in my letters to Larry Breen, it is quite possible that on the unconscious level, the little I had heard of his approach on one or two casual occasions, had entered into reaction with my own intuitions, coming mainly from geometry and cohomological algebra, and finally resulted in the program outlined in those letters.

Beginning of a provisional itinerary (through stacks).

19 I would like now to write down a provision itinerary of the voyage ahead – namely to make a list of those main features of a theory of stacks which are in my mind these days. I will write them down in the order in which they occur to me – which will be no obligation upon me to follow this order, when coming back on those features separately to [p. 12] elaborate somewhat on them. This I expect to do, mainly as a way to check whether the main notions and intuitions introduced are sound indeed, and otherwise, to see how to correct them.

1º) Definition of the categories $\mathbb{B}_0$ and its dual $\mathbb{C}_0$, and formal definition of [?]. This definition will still be a provisional one, and will presumably have to be adjusted somewhat to allow for the various structures we are looking for in the corresponding category of Gr-stacks.

2º) Relation between the category of Gr-stacks and the category of topological spaces, via two adjoint functors, the "topological realization functor" $F_* \mapsto |F_*|$, and the "singular stack functor" $X \mapsto \mathbb{F}_*(X)$. The situation should be formally analogous to the corresponding situation for semi-simplicial sets versus topological spaces, the role of the category S_* of standard ordered simplices being taken by the left coherator $\mathbb{B}$ we are working with. The main technical difference here is that the category of Gr-stacks is not just the category of presheaves on $\mathbb{B}$, but the full subcategory defined by the requirement that the presheaves considered should transform "standard" amalgamated sums into fibered products. As a matter of fact, the topological realization functor $F_* \mapsto |F_*|$ could be defined in a standard way on the whole of $\mathbb{B}\hat{}$, in terms of any functor

$$(*) \qquad \mathbb{B} \mapsto (\text{Spaces})$$

(by the requirement that the extension of this functor to $B\hat{}$ commutes with arbitrary direct limits). A second difference with the simplicial situation lies in the fact that the only really compelling choice for the functor (*) is it's restriction to $\mathbb{B}_0$, in terms of the cells D_n and the standard "half-hemisphere maps" between these (L, p. 6–7). The extension of this functor to $\mathbb{B}$ is always possible, due to the inductive construction of $\mathbb{B}$ and to the interpretation of elements of $\mathbb{B}_0$ as *contractible* spaces, via the functor $(*)_0$; but it depends on a bunch of arbitrary choices. To give precise meaning to the intuition that these choices don't really make a difference,

> and that the choice of coherator we are starting with doesn't make much of a difference either, will need some elaboration on the notion of equivalences between Gr-stacks, which will have to be developed at a later stage. [p. 13]

The idea just comes to my mind whether the exactness condition implying standard amalgamated sums, defining the subcategory of stacks within $\mathbb{B}^{\wedge}$, cannot be interpreted in terms of some more or less obvious topology on $\mathbb{B}$ turning $\mathbb{B}$ into a site, as the subcategory of corresponding *sheaves*. This would mean that the category (Gr-stacks) is in fact a topos, with the host of categorical information and topological intuition that goes with such a situation. In this connection, it is timely to recall that the related categories (Cat) of "all" categories, and (Groupoids) of "all" groupoids, are definitely *not* topoi (if my recollection is correct – it isn't immediately clear to me why they are not). This seems to suggest that, granting that (Gr-stacks) is indeed a topos, that this would be a rather special feature of the structure species of infinite order we are working with (as one ward so to say, among a heap of others, for conceptual sophistication!), in contrast to the categories (Gr-n-stacks) with finite n, which presumably are not topoi. (For a definition of Gr-n-stacks in terms of Gr-stacks, namely Gr-∞-stacks, see below.)

A related question is whether the category (Gr-stacks) is a model category for the usual homotopy category, the pair of adjoint functors considered before satisfying moreover the conditions of Quillen's comparison theorem. The obvious idea that comes to mind here, in order to define the model structure on (Gr-stacks), is to take as "weak equivalences" the maps which are transformed into weak equivalences by the topological realization functor (which should be readily expressible in algebraic terms), for cofibrations the monomorphisms, and defining fibrations by the Serre-Quillen lifting property with respect to cofibrations which are weak equivalences (with the expectation that we even get a "closed model category" in the sense of Quillen). Here it doesn't look too unreasonable to expect the same constructions to work in each of the categories (Gr-n-stacks), $n \geq 1$, as well as in the categories (n-stacks) without Gr, which are still to be defined though.

Coming back upon the question of a suitable topology on $\mathbb{B}$, the idea that comes to mind immediately of course is to define covering families of an object K of $\mathbb{B}$, i.e., of $\mathbb{B}_0$, in terms of the components which occur in the description of K as iterated amalgamated sum of cells D_n. A quick glance (too quick a glance?) seems to show this is indeed a topology, and that the sheaves for this topology are what we expect.

[p. 14] 5.3.

Are model categories sites?

20 I had barely stopped writing last Tuesday, when it became clear that the "quick glance" had been too quick indeed. As a matter of fact, the "topology" I was contemplating on $\mathbb{B}$ in terms of covering families does not satisfy the conditions for a "site" – that something was fishy first occurred to me through the heuristic consequence, that the functors "n-th component"

$$F_* \mapsto F_n$$

from stacks to sets are fiber functors, or equivalently, that direct limits in the category of Gr-stacks can be computed componentwise – now this is definitely false, for much the same reasons why it is false already in the category (Cat) of categories. However, it occurred to me that the latter category (Cat), although not a topos (for instance because, according to Giraud's paper on descent theory of 1965, the implications for epimorphisms

[Giraud 1964]

$$\begin{tikzcd} & \text{effective} \arrow[dr, Rightarrow] & \\ \text{effective universal} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{just epi} \\ & \text{universal} \arrow[ur, Rightarrow] & \end{tikzcd}$$

are strict), there *is* a very natural topology, turning it into a site, namely the one where a family of morphisms namely functors

$$A_i \to A$$

is covering if and only if the corresponding family in (Ss sets)

$$\mathrm{Nerve}(A_i) \to \mathrm{Nerve}(A)$$

is covering, i.e., iff every sequence of composable arrows in A

$$a_0 \to a_1 \to a_2 \to a_3 \to \cdots \to a_n$$

can be lifted to one among the A_i's. As a matter of fact, this condition (where it suffices to take $n = 2$, visibly) is equivalent (according to Giraud) to the condition that the family be "universally effectively epimorphic", i.e., covering with respect to the "canonical topology" of (Cat). This suggests a third ward for associating to a category A a topology-like structure, namely the topos of all sheaves over $(\mathrm{Cat})_{/A}$, the site of all categories over A, endowed with the topology induced by the canonical topology of (Cat) (which is indeed the canonical topology of $(\mathrm{Cat})_{/A}$). It should be an easy exercise in terms of nerves to check that the homotopy type (a priori, a pro-homotopy type) associated to this site is just "the" homotopy type of A, defined either as the homotopy type of Nerve(A), or as the (pro)homotopy type of the topos $A^\wedge$ of all presheaves over A. This of course parallels the similar familiar fact in the category (Ss sets) of all semisimplicial sets, namely that for such a ss set K, the homotopy

5.3. This is definitely false, see §24 below.

type of K can be viewed also as the homotopy type of the induced topos $(\text{Ss sets})_{/K}$ of all ss sets over K. The difference in the two cases is that in the second case the category (Ss sets) (and hence any induced category) is already a topos, namely equivalent to the category of sheaves on the same for the canonical topology, whereas on the category (Cat) this is not so. [p. 15]

These reflections suggest that in most if not all categories of models encountered for describing usual homotopy types, there is a natural structure of a site on the model category M (*presumably*[†] the one corresponding to the canonical topology), with the property that the homotopy type of any object X in M is canonically isomorphic to the (pro)homotopy type of the induced site $M_{/X}$, or what amounts to the same of the corresponding topos

[†]5.3. this assumption is false actually.

$$(M_{/X})^{\sim} = M^{\sim}_{/X}.$$

I would expect definitely this to be the case for each of the categories (Gr-stacks), (n-Gr-stacks) (although this is not really a model category in the sense of Quillen, as it gives rise only to n-truncated homotopy types . . .), (stacks) and (n-stacks), corresponding to an arbitrary choice of coherator, defining the notion of a (Gr-stack), or of a stack (for the latter and relations between the two, see below).

Further glimpse upon the "bunch" of possible model categories and a relation between n-complexes and n-stacks.

21 I would like to digress a little more, to emphasize still about the vast variety of algebraic structures giving rise to model categories for the usual homotopy category, or at any rate suitable for expressing more or less arbitrary homotopy types. Apart from stacks, where everything is still heuristics for the time being, we have noticed so far three examples of such structures, namely [cubical? and semisimplicial complexes? and topological spaces?]. There are a few familiar variants of the two former, such as the "simplicial complexes" (in contract to semi-simplicial ones), namely presheaves on the category of non-empty finite sets, which can be interpreted as ss complexes enriched with symmetry operations on each component – and there is the corresponding variant in the cubical case. It would be rather surprising that there were not just as good model categories, as the more habitual ones – all the more as the singular complex (simplicial or cubical) is naturally endowed with this extra structure, which one generally chooses to forget. More interesting variants are the n-multicomplexes (simplicial or cubical, with or without symmetries), defined by contravariant functors to sets in n arguments rather than in just one, where $n \geq 1$. These complexes are familiar mainly, it seems, because of their connections with product spaces and the Künneth-Eilenberg-Zilber type relations. It is generally understood that to such a multicomplex is associated the corresponding "diagonal" complex, which is just a usual complex and adequately describes the "homotopy type" of the multicomplex. So why bother with relatively messy kinds of models, when just usual complexes suffice! Here however the point is not to get the handiest possible model categories (whatever our criteria of "handyness"), but rather to get an idea of the variety of [p. 16] algebraic structures suitable for defining homotopy types, and perhaps to come to a clue of what is common to all these. Moreover, I feel the relation between ordinary complexes and n-multicomplexes, is of much the same nature as the relation between just ordinary categories, perfectly sufficient for describing homotopy types, and n-categories or n-stacks. This reminds of course of Quillen's idea of [?], in much the same way as categories can be defined (via the Nerve functor) in terms of usual ss complexes. I hit again upon multisimplices (without symmetry), when trying to reduce to a minimum the category $\mathbb{B}_0$ of what should be called "standard" amalgamated sums of the cells D_n, where my tendency initially (L, p. 7) had rather been to be as generous as possible, in order to be as stringent as imaginable for the completion condition (A) of (L, p. 8). Now it turns out that the coherence relations which seem to have been written down so far (and the like of which presumably will suffice to imply full completeness of coherence relations, in the sense of (A)) make use only of very restricted types of such amalgamated sums, expressible precisely in terms of multisimplices. This I check for instance on the full list of data and axioms for Bénabou's "bicategories" namely 2-stacks, in his 1967 Midwest Category Seminar exposé (already referred to). I'll have to come back upon this point with some care, which gives also a pretty natural way for getting Quillen's functor from n-stacks to n-ss complexes.

22 *Oriented sets as models for homotopy types.*

These examples of possible models for homotopy types can be viewed as generalizations of usual complexes, or of usual categories; I would like to give a few others which go in the opposite direction – they may be viewed as particular cases of categories. One is the (pre)order structure, which may be viewed as a category structure when the map $\mathrm{Fl} \to \mathrm{Ob} \times \mathrm{Ob}$ defined by the source and target maps is injective. Such category is equivalent (and hence homotopic) to the category associated with the corresponding *ordered* set (when $x \geq y$ and $y \geq x$ imply $x = y$). Ordered sets are more familiar I guess as model objects for describing combinatorially a topological space, in terms of a "cellular [p. 17] subdivision" by compact subsets or "cells" ("strata" would be a more appropriate term), which actually need not be topological cells in the strict sense, but rather conical (and hence contractible) spaces, each being homeomorphic to the cone over the union of all strictly smaller strata (this union is compact). The ordered set associated to such a (conically) stratified space X is just the set of strata, with the inclusion relation, and it can be shown that there is a perfect dictionary between the topological objects (at least in the case of finite or locally finite stratifications), and the corresponding (finite or locally finite) ordered sets, via a "topological realization functor"

$$X \mapsto |X|$$

from ordered sets to (conically stratified) topological spaces. As a matter of fact, when K is finite, K is endowed with a canonical triangulation (the so-called barycentric subdivision), the combinatorial model of which (of "maquette") is as follows: the vertices are in one-to-one correspondence with the elements of K (they correspond to the vertices of the corresponding cones), and the combinatorial simplices are the "flags", or subsets of K which are totally ordered for the induced order. This is still OK when K is only locally finite (restricting of course to subsets of K which are finite, when describing simplices), but in any case we can define (via infinite direct limits in the category (Spaces)of all topological spaces) the geometric realization of K, together with an interpretation of K as the ordered set of strata of $|K|$. As a matter of fact, we get a canonical isomorphism (nearly tautological)

$$|K| \simeq |\mathrm{Nerve}(K)|$$

where in the second member, we have written shortly K for the category defined by K. I did not reflect whether it was reasonable to expect that the categories (Preord) and (Ord) of preordered and ordered sets are model categories, or even closed model categories, in Quillen's precise sense – but it is clear though that using ordered sets we'll get practically any homotopy type, in any case any homotopy type which can be described in terms of locally finite triangulations. However it should be noted that the inclusion functors into (Cat) or (Ss sets)

$$(\mathrm{Preord}) \hookrightarrow (\mathrm{Ord}) \hookrightarrow (\mathrm{Cat}) \hookrightarrow (\mathrm{Ss\ sets}),$$

while giving the correct results on homotopy types, do not satisfy Quillen's general conditions on pairs of adjoint functors between model categories – namely the adjoint functor say

$$(\mathrm{Cat}) \to (\mathrm{Preord}),$$

associating to a category A the set $\mathrm{Ob}A$ with the obvious relation, does *not* commute to formation of homotopy types, as we see in the trivial case when $\mathrm{Ob}A$ is reduced to one point ...

There is another amusing interpretation of the homotopy type associated to any preordered set K, via the topological space whose underlying set is K itself, and where the closed sets are the subsets J of K such that $x \in J$, $y \subseteq x$ implies $y \in J$. This is a highly non-separated topology τ (except when the preorder relation is the discrete one), where an arbitrary union of closed subsets is again closed. I doubt its singular homotopy type to make much sense, however its homotopy type as a topos does, and (possibly under mild local finiteness restrictions) it should be the same as the homotopy type of K just envisioned. Thus, a sheaf on the topological space K can be interpreted via its fibers as being just a *covariant* functor [p. 18]

$$K \to (\mathrm{Sets})$$

(NB the open sets of K are just the closed sets of K^{op}, namely for the opposite order relation, and thus every $x \in K$ has a smallest neighborhood, namely the set $K_{\geq x}$ of all $y \in K$ such that $y \geq x$), or what amounts to the same, a sheaf on the topos $(K^{\mathrm{op}})\hat{}$ defined by the opposite order.

Hence the derived functors of the "sections" functor, when working with abelian sheaves on (K, τ), i.e., the cohomology of the topological space (K, τ), can be interpreted in terms of the topos associated to K^{op}. This suggests that the definition I gave of the topology of K was awkward and maybe it is indeed (although it is the more natural one in terms of incidence relations between open strata of $|K|$), and that we should have called "open" the sets I called "closed" and vice-versa, or equivalently, replace K by K^{op} in the definition I gave of a topology on the set K. But as far as homotopy types are concerned, it doesn't make a difference, namely the homotopy types associated to K and K^{op} are canonically isomorphic. This can be seen most simply on the topological realizations, via a homeomorphism (not only a homotopism)

$$|K| \simeq |K^{\mathrm{op}}|,$$

coming from the fact that the maquettes (by which I mean the combinatorial model for a triangulation, which Cartan time ago called "schéma simplicial" ...) of the two spaces are canonically isomorphic, because the "flags" of K are K^{op} are the same. A similar argument due to Quillen using the nerves shows that for any category K (not necessarily ordered), K and K^{op} are homotopic, although $K^\wedge$ and $(K^{\mathrm{op}})^\wedge$ are definitely not equivalent, i.e., not "homeomorphic".

To come back to the decreasing cascade of algebraic structure suitable for describing homotopy types, we could go down one more step still, to the category of "maquettes" (Maq), namely sets S together with a family
[p. 19] K of finite subsets (the simplices of the set of vertices S), such that one-point subsets are simplices and a subset of a simplex is a simplex. This category, via the functor $(S, K) \mapsto K$, is equivalent to the full subcategory of (Ord), whose objects are those ordered sets K, such that for every $x \in K$, the set $K_{\leq x}$ be isomorphic to the ordered set of non-empty subsets of some finite set (or "simplex"). Here the question whether this category is a model category in the technical sense doesn't really arise, because this category doesn't even admit finite products – rien à faire!

5.3. indeed, the notion of a maquette is *not* an algebraic structure species!

23

Getting a basic functor $M \to$ (Hot) from a site structure M (altering beginning of a systematic reflection).

It may be about time to get back to stacks, still I can't help going on pondering about algebraic structures as models for homotopy types. If we have any algebraic structure species, giving rise to a category M of set-theoretic realizations, the basic question here doesn't seem so much whether M is a model category for a suitable choice of the three sets of arrows (fibrations, cofibrations, weak equivalences), but rather how to define a natural functor

$$M \to (\mathrm{Hot}), \tag{*}$$

where (Hot) is the category of usual homotopy types, and see whether via this functor (Hot) can be interpreted as a category of fractions (or "localization") of M – namely, of course, by the operation of making invertible those arrows in M which are transformed into isomorphisms in (Hot). In any case, if we have such a natural functor, the natural thing to do is to call those arrows "weak equivalences". If we want M

to be a category of models, various examples suggest that the natural thing again is to take as cofibrations the monomorphisms, and then (expecting that the model categories we are going to meet will be closed model categories) to define fibrations by the Serre-Quillen lifting property with respect to cofibrations (=monomorphisms) which are weak equivalences. This being done, it becomes meaningful to ask if indeed M is a category of models.

Now the reflections of the beginning of today's notes (p. 14–15) suggest a rather natural way for describing a functor (*), which makes sense in fact, in principle, for any category M, namely: endow M with its canonical topology (unless a still more natural one appears at hand – I am not sure there is any better one in the present context), and assume that for every $X \in \operatorname{Ob} M$, the pro-homotopy type of the induced site $M_{/X}$ is essentially constant, i.e., can be identified with an object in (Hot) itself. We then get the functor (*) in an obvious way. It then becomes meaningful to ask whether this functor is a localization functor.

5.3. the canonical topology is *not* always suitable, see §24 below.

[p. 20]

When M is defined in terms of an algebraic structure species, it admits both types of limits, without finiteness requirement – and we certainly would expect indeed at least existence of finite *and* infinite direct sums in M, if objects of M were to describe arbitrary homotopy types. However, in view of the special exactness properties of (Hot), which are by no means autodual, we will expect moreover direct sums in M to be "universal and disjoint", in Giraud's sense. This condition, which characterizes to a certain extent categories which at least mildly resemble or parallel categories such as (Sets), (Spaces) and similar categories, whose objects more or less express "shapes" – this condition at once rules out the majority of the most common algebraic structures, such as rings, groups, modules over a ring or anything which yields for M an abelian category, etc. If we describe an algebraic structure species in terms of its universal realization in a category stable under finite inverse limits, then such a structure species can be viewed as being defined by such a category $\mathbb{C}$, and its realizations in any other such C as the left-exact functors $\mathbb{C} \to C$ (the universal realization of the structure within $\mathbb{C}$ corresponding to the identity functor $\mathbb{C} \to \mathbb{C}$). In terms of the dual category $\mathbb{B}$, associating to every element in $\operatorname{Ob} \mathbb{B} = \operatorname{Ob} \mathbb{C}$ the covariant functor $\mathbb{C} \to (\mathrm{Sets})$ it represents, we get a fully-faithful embedding

$$\mathbb{B} \hookrightarrow M,$$

by which $\mathbb{B}$ can be interpreted as the category of the (set-theoretic) realizations of the given structure which are of "finite presentation" in a suitable sense (in terms of a given family of generators of $\mathbb{B}$ namely co-generators of $\mathbb{C}$, considered as corresponding to the choice of "base-sets" for the given structure species – such choice however being considered as a convenient way merely to describe the species in concrete terms. . .). If I remember correctly, M can be deduced from $\mathbb{B}$, up to equivalence, as being merely the category of Ind-objects of $\mathbb{B}$, i.e., the inclusion functor above yields an equivalence of categories

$$\operatorname{Ind}(\mathbb{B}) \xrightarrow{\approx} M,$$

which implies, I guess, that the exactness properties of M mainly reflect those of $\mathbb{B}$. Thus I would expect the condition we want on direct sums in M to correspond to the same condition for finite direct sums in $\mathbb{B}$, not more not less. Thus the algebraic structure species satisfying this condition should correspond exactly to small categories $\mathbb{B}$, stable under finite direct limits, and such that finite sums in $\mathbb{B}$ are disjoint and universal. This condition is presumably necessary, if we want the functor (*) from M to (Hot) to be defined and to be a localization functor – a [p. 21] condition which it would be nice to understand directly in terms of $\mathbb{B}$, and (presumably) in terms of the canonical topology of $\mathbb{B}$, which should give rise to a localization functor

$$\mathbb{B} \to (\mathrm{Hot})_{\mathrm{ft}},$$

where the subscript ft means "finite type" – granting that the notion of homotopy types of finite type (presumably the same as homotopy types of finite triangulations, or of finite CW space) is a well-defined notion. As usual, it is in $\mathbb{B}$, not in $\mathbb{C}$, that geometrical constructions take place which make sense for topological intuition. More specifically, it seems that in the cases met so far, there *are* indeed privileged base-sets for the structure species considered (such as the "components" or a semisimplicial or cubical complex, or of a stack, etc.), indexed by the natural integers or n-tuples of such integers, and which "correspond" to topological cells of various dimensions. Moreover, some of the basic structural monomorphic maps between these objects of $\mathbb{B}$ define cellular decompositions of the topological spheres building these topological cells. These objects and "boundary maps" between them define a (non-full, in general) subcategory of $\mathbb{B}$, say $\mathbb{B}_\infty$, which looks like the core of the category $\mathbb{B}$, from which the topological significance of $\mathbb{B}$ is springing. It is in terms of $\mathbb{B}_\infty$ that the "correspondence" (vaguely referred to above) with topological cells and spheres takes a precise meaning. Namely, associating to any object of $\mathbb{B}_\infty$ the ordered set of its subobjects (within $\mathbb{B}_\infty$ of course, not $\mathbb{B}$), the (stratified) topological realization of this ordered set is a cell, the family of subcells of smaller dimension (namely different from the given one) is a cellular subdivision of the sphere bounding this cell.

7.3.

A bunch of topologies on (Cat).

24 Yesterday there occurred to me a big "étourderie" again of the day before, in connection with a reflection on a suitable "natural" site structure on the category (Cat) – namely when asserting that for a family of morphisms, i.e., functors in (Cat)

$$A_i \to A, \tag{*}$$

in order for the corresponding family to be "covering" namely epimorphic in the category (a topos, as a matter of fact) (Ss sets), namely for the corresponding families of mappings of sets

$$\mathrm{Fl}_n(A_i) \to \mathrm{Fl}_n(A) \tag{**}$$

to be epimorphic, it was sufficient that his condition be satisfied for $n = 2$ (which, according to Giraud, just means that the family (*) is covering for the canonical topology of (Cat)). This is obviously false – morally, it would mean that an n-simplex is "covered" in a reasonable sense by its sub-2-simplices, which is pretty absurd. To give specific [p. 22] positive statements along these lines, let's for any $N \in \mathbb{N}$, denote by T_N the topology on (Cat) for which a family (*) is covering iff the families (**) are for $n \leq N$ – which means also that the corresponding family of N-truncated nerves

$$\mathrm{Nerve}(A_i)[N] \to \mathrm{Nerve}(A)[N]$$

is epimorphic. For $N = 1$, this means that the family is "universally epimorphic", for $N = 2$, that it is "universally effectively epimorphic", i.e., covering for the canonical topology of (Cat) (Giraud, loc. cit. page 28). It turns out that this decreasing sequence of topologies on (Cat) is strictly decreasing – as a matter of fact, denoting by $\Delta[N]$ the category of standard, ordered simplices of dimension $\leq N$, and using the inclusions

$$\Delta[N] \hookrightarrow (\mathrm{Cat}) \hookrightarrow \Delta[N]^\wedge$$

(for $N \geq 2$ say), an immediate application of the "comparison lemma" for sites shows that we have an equivalence

$$((\mathrm{Cat}), T_N)^\sim \simeq \Delta[N]^\wedge,$$

and hence, for any object A in (Cat), namely a category A, we get an equivalence

$$((\mathrm{Cat}), T_N)^\sim_{/A} \simeq (\Delta[N]_{/A})^\wedge,$$

and hence the homotopy type of the first hand member is *not* described by and equivalent to the homotopy type of the whole Nerve(A) object, but rather by its N-skeleton, which has the same homotopy and cohomology invariant in dimension $\leq N$, but by no means for higher dimensions. This shows that among the topologies T_N, none is suitable for recovering the homotopy type of objects of (Cat) in the way contemplated two days ago (page 15); the one topology which *is* suitable is the one which may be denoted by T_∞, and which is the one indeed which I first contemplated (page 14), before the mistaken idea occurred to me that it was the same as the canonical topology.

A very similar mistake occurred earlier, when I surmised that the left adjoint functor N' to the inclusion or Nerve functor N from (Cat) to (Ss sets) $= \mathbb{\Delta}^\wedge$ had the property that for any object K in (Ss sets), K and $N'(K)$ had the same homotopy type. Looking up yesterday the description in Gabriel-Zisman of this functor, this recalled to my mind that it factors (via the natural restriction functor) through the category $\Delta[2]^\wedge$, hence any morphism $K \to K'$ inducing an isomorphism on the 2-skeletons (which by no means implies that it is a homotopy equivalence) induces an isomorphism $N'(K) \xrightarrow{\sim} N'(K')$, and a fortiori a homotopy equivalence. This shows that, even if it should be true that (Cat) is [p. 23] a model category in Quillen's sense, the situation with the inclusion functor

$$N : (\mathrm{Cat}) \to (\mathrm{Ss\ sets})$$

and the left adjoint functor N' is by no means the one of Quillen's comparison theorem, where the two functors play mutually dual roles and both induce equivalences on the corresponding localized homotopical categories. Here only N, not N', induces such equivalence. This is analogous to the situation, already noted before, of the inclusion functors

$$(\mathrm{Preord}) \hookrightarrow (\mathrm{Cat}) \quad \text{or} \quad (\mathrm{Ord}) \hookrightarrow (\mathrm{Cat}),$$

which by localization induce equivalences on the associate homotopy categories, but the left adjoint functors do not share this property.

The topology T_∞ just considered on (Cat) as a "suitable" topology for describing homotopy types of objects of (Cat), was of course directly inspired by the semi-simplicial approach to homotopy types, via simplicial complexes, namely via the two associated inclusion functors

$$\mathbb{\Delta} \hookrightarrow (\mathrm{Cat}) \hookrightarrow \mathbb{\Delta}^\wedge,$$

the second functor associating to every category A the "complex" of its "simplicial diagrams" $a_0 \to a_1 \to \cdots \to a_n$. If we had been working with cubical complexes rather than ss ones for describing homotopy types, this would give rise similarly to two functors

$$\square \hookrightarrow (\mathrm{Cat}) \hookrightarrow \square^\wedge,$$

where $\square$ is the category of "standard cubes" and face and degeneracy maps between them, and where the second inclusion associates to every category A the cubical complex of its "cubical diagrams", namely commutative diagrams in A modelled after the diagram types $\square_n[1]$, the 1-skeleton of the standard n-cubes $\square_n$, with suitable orientations on its edges (indicative of the direction of corresponding arrows in A). The natural idea would be to endow (Cat) with the topology, T'_∞ say, induced by $\square^\wedge$, namely call a family (*) covering iff for every $n \in \mathbb{N}$, the corresponding family of maps of sets

$$\mathrm{Cub}_n(A_i) \to \mathrm{Cub}_n(A)$$

is epimorphic. This topology appears to be coarser than T_∞ (i.e., there are fewer covering families), and the comparison lemma gives now the equivalence

$$((\mathrm{Cat}), T'_\infty)^\sim \simeq \square^\wedge,$$

which shows that definitely the topology is *strictly* coarser than T_∞, as it gives rise to a non-equivalent topos, $\square^\wedge$ instead of $\mathbb{\Delta}^\wedge$.

These reflections convince me 1) there are indeed topologies on (Cat), suitable for describing the natural homotopy types of objects of
[p. 24] (Cat) namely of categories, and 2) that there is definitely no privileged choice for such a topology. We just described two such, but using multicomplexes (cubical or semisimplicial) rather than simple complexes

should give us infinitely many others just as suitable, and I suspect now that there must be a big lot more of them still!

A tentative equivalence relation for topologists.

25 A fortiori, coming back to the intriguing question of characterizing the algebraic structure species suitable for describing homotopy types in the usual homotopy category (Hot), and for recovering (Hot) as a category of fractions of the category M of all set-theoretic realizations of this species, it becomes clear now that we cannot hope reasonably for a "natural" topology on M, distinguished among all others, giving rise to the wished-for functor

$$M \to (\mathrm{Hot}) \tag{*}$$

in the way contemplated earlier (page 19 ff.). Here it occurs to me that anyhow, if concerned mainly with defining the functor (*), we should consider that two topologies T, T' on M such that $T \geq T'$ and hence giving rise to a morphism of topoi

$$M^{\sim}_{T} \to M^{\sim}_{T'} \tag{**}$$

(the direct image functor associated to this morphism being the natural inclusion functor, when considering T-sheaves as particular cases of T'-sheaves) are "*equivalent*" – maybe we should rather say "Hot-equivalent" – if for any object A in M, the induced morphism

$$(M^{\sim}_{T})_{/A} \to (M^{\sim}_{T'})_{/A}$$

is a homotopy equivalence. This can be viewed as an intrinsic property of the morphism of topoi (**), of a type rather familiar I guess to people used to the dialectics of étale cohomology, where a very similar notion was met and given the name of a "globally acyclic morphism". The suitable name here would be "globally aspheric morphism" which is a reinforcement of the former, in the sense of being expressible in terms of isomorphism relations in cohomology with arbitrary coefficient sheaves on the base, *including non-commutative* coefficient sheaves. The relation just introduced between two topologies T, T' on a category M makes sense for any M (irrespective of the particular way M was introduced here), it is not yet an equivalence relation though – so why not introduce the equivalence relation it generates, and call this "Hot-equivalence" – unless we find a coarser, and cleverer notion of equivalence, deserving this name. The point which, one feels now, should be developed, is that this notion of equivalence should be the coarsest we can find out, and which still implies that to any Hot-equivalence class of topologies on M there should be canonically associated a functor (*), which should essentially be "the" common value of all the similar functors, associated to the topologies T within this class. Maybe even it could be shown that this equivalence class can be recovered in terms of the corresponding functor (*), in the same way as (according to Giraud) a "topology" on M can be recovered from the associated subtopos of $M^{\wedge}$, namely the associated category of sheaves on M (in such a way that the set of

For a proper map of paracompact spaces, this condition just means that the fibers are "aspheric", namely "contractible" (in Čech' sense)

[p. 25]

"topologies" on M can be identified with the set of "closed subtopoi" of $M^{\wedge}$). The very best one could possibly hope for along these lines would be a one-to-one correspondence between isomorphism classes of functors (*) (satisfying certain properties?), and the so-called "Hot-equivalence classes" of topologies on M.

Whether or not this tentative hope is excessive, when it comes to the (still somewhat vague) question of understanding "which algebraic structure species are suitable for expressing homotopy types", it might not be excessive though to expect that in all such cases, M should be equipped with just *one* "natural" Hot-equivalence class of topologies on M, which moreover (one hopes, or wonders) should be expressible directly in terms of the intrinsic structure of M, or, what amounts to the same, in terms of the full subcategory $\mathbb{B}$ of objects of "finite presentation", giving rise to M via the equivalence

$$M \simeq \mathrm{Ind}(\mathbb{B}).$$

As we just saw in the case $M = (\mathrm{Cat})$, the so-called "canonical topology" on M need not be within the natural Hot-equivalence class – and I am at a loss for the moment to give a plausible intrinsic characterization of the latter, in terms of the category $M = (\mathrm{Cat})$.

26 *The dawn of test categories and test functors...*

What comes to mind though is that the categories such as $\triangle$, $\square$ and their analogons (corresponding to multicomplexes rather than monocomplexes, for instance) can be viewed as (generally not full) subcategories of M (in fact, even of the smaller category (Ord) of all ordered sets). The topologies we found on M were in fact associated in an evident way to the choice of such subcategories. As was already felt by the end of the reflection two days ago (p. 21), these subcategories (denoted there by $\mathbb{B}_\infty$) have rather special features – they are associated to simultaneous cellular decompositions of spheres of all dimensions – and it is this feature, presumably, that makes the associated "trivial" algebraic structure species, giving rise to the category of set-theoretic realizations $\mathbb{B}_\infty^{\wedge}$, eligible for "describing homotopy types". In the typical example $M = (\mathrm{Cat})$ though, contrarily to what was suggested on page 21 (when thinking mainly of the rather special although important case when [p. 26] M is expressible as a category $B^{\wedge}$, for some category B such as $\triangle$, $\square$ etc.), there is no really privileged choice of such subcategory $\mathbb{B}_\infty$ – we found indeed a big bunch of such, the ones just as good as the others. The point of course is that the corresponding topologies on M, namely induced from the canonical topology on $\mathbb{B}_\infty^{\wedge}$ by the canonical functor

$$M \to \mathbb{B}_\infty^{\wedge},$$

are Hot-equivalent for some reason or other, which should be understood. The plausible fact that emerges here, is that the "natural" Hot-equivalence classes of topologies on M is associated, in the way just described, to a class (presumably an equivalence class in a suitable set for suitable equivalence relation...) of subcategories $\mathbb{B}_\infty$ in M. The question of giving an intrinsic description of the former, is apparently reduced to the (possibly more concrete) one, of giving an intrinsic description of a bunch of subcategories $B = \mathbb{B}_\infty$ of M. This description, one feels, should both a) insist on intrinsic properties of B, independent of M, namely of the structure species one is working with, and b) be concerned with the particular way in which B is embedded in M, which should by no means be an arbitrary one.

The properties a) should be, I guess, no more no less than those which express that the "trivial" algebraic structure species defined by B, giving a category of set-theoretic realizations $M_B = B^\wedge$, should be "suitable for describing homotopy types". The examples at hand so far suggest that in this case, the canonical topology on the topos M_B is within the natural Hot-equivalence class, which gives a meaning to the functor

$$M_B = B^\wedge \to (\mathrm{Hot}),$$

indeed it associates to any $a \in \mathrm{Ob}\, B^\wedge$ the homotopy type of the induced category $B_{/a}$ of all objects of B "over a". Thus a first condition on B is that this functor should be a "localization functor", identifying (Hot) with a category of fractions of $M_B = B^\wedge$. This does look indeed as an extremely stringent condition on B, and I wonder if the features we noticed in the special cases dealt with so far, connected with cellular decompositions of spheres, have any more compulsive significance than just giving some handy *sufficient* conditions (which deserve to be made explicit sooner or later!) for "eligibility" of B for recovering (Hot). Beyond this, one would of course like to have a better understanding of what it really means, in terms of the internal structure of B, that the functor above from $B^\wedge$ to (Hot) is a localization functor.

because $B^\wedge_{/a} \simeq (B_{/a})^\wedge$.

Once this internal condition on B is understood, step b) then would amount to describing, in terms of an arbitrary "eligible" algebraic structure species expressed by the category M, of what we should mean by "eligible functors"[†] [p. 27]

$$B \to M,$$

[†] we will rather say "test functors", see below. . .

giving rise in the usual way to a functor

$$M \to B^\wedge.$$

In any case, the latter functor defines upon M an induced topology, T say, and the comparison lemma tells us that if either $B \to M$ or $M \to B^\wedge$ is fully faithful, then the topos associated to M is canonically equivalent to $B^\wedge$ (using this comparison lemma for the functor which happens to be fully faithful). From this follows that the functor (*) $M^\sim \to (\mathrm{Hot})$ defined by T is nothing but the compositum

$$M^\sim \xrightarrow{\simeq} B^\wedge \to (\mathrm{Hot}),$$

and hence a localization functor. In other words, when B is a category satisfying the condition seen above,[‡] then *any* functor $B \to M$ satisfying one of the two fullness conditions above yields a corresponding description of (Hot) as a localization of $M^\sim$. What is still lacking though

[‡] a "test category", as we will say

is a grasp on when two such functors $B \to M$, $B' \to M$ define essentially "the same" functor $M \to$ (Hot), or (more or less equivalently) two (Hot)-equivalent topologies T, T' on M; is it enough, for instance, that they give rise to the same notion of "weak equivalences" (namely morphisms in M which are transformed into an isomorphism of (Hot))? And moreover, granting that this equivalence relation between certain full subcategories (say) B of M is understood, how to define, in terms of M, a "natural" equivalence class of such full subcategories, giving rise to a canonical functor $M \to$ (Hot)?[§]

[§]Same question arises for Hot-equivalence on topologies on M...

Recalling that the algebraic structure considered can be described in terms of an arbitrary small category $\mathbb{B}$ where arbitrary finite direct limits exist (namely $\mathbb{B}$ is the full subcategory of M of objects of finite presentation), it seems reasonably to assume that indeed

$$B \to \mathbb{B} \quad \text{(a full embedding)},$$

and the question transforms into describing a natural equivalence class of such full subcategories, in (more or less) any small category $\mathbb{B}$ where finite direct limits exist, and where moreover there exist such full subcategories B. Also, we may have to throw in some extra conditions on $\mathbb{B}$, such as the condition that direct sums be "disjoint" and "universal" already contemplated before.

[p. 28] Maybe I was a little overenthusiastic, when observing for any full embedding of a category B in M (let's call the categories B giving rise to a localization functor $B^\wedge \to$ (Hot) *homotopy-test* categories, or simply *test* categories) we get a localization functor

$$M^\sim = (M, T_B)^\sim \to \text{(Hot)},$$

where T_B is the topology on M corresponding to the full subcategory B. After all, there is a long way in between M itself and the category of sheaves $M^\sim$ – and what we want is to get (Hot) as a localization of M itself, not of $M^\sim$. It is not even clear, without some extra assumptions, that the natural functor from M to $M^\sim$ is fully faithful, namely that M can be identified with a full subcategory of the category $M^\sim$ we've got to localize to get (Hot). We definitely would like this to be true, or what amounts to the same, that the functor $M \to B^\wedge$ defined by $B \to M$ should be fully faithful – which means also that the full subcategory B of M is "generating by strict epimorphisms" namely that for every K in M, there exists a strictly epimorphic family of morphisms $b_i \to K$, with sources b_i in B. This interpretation of full faithfulness of $M \to B^\wedge$ is OK when $B \to M$ is fully faithful, a condition which I gradually put into the fore without really compelling reason, except that in those examples I have in mind and which are *not* connected with the theory of stacks of various kinds, this condition is satisfied indeed. Apparently, with this endless digression on algebraic models for homotopy types, stacks (which I am supposed to be after, after all) are kind of fading into the background! Maybe we should after all forget about the fully faithfulness condition on either $B \to M$ or $M \to B^\wedge$, and just insist that the compositum

$$M \to M^\sim \to B^\wedge \to \text{(Hot)}$$

(which can be described directly in terms of the topology T_B on M associated to the functor $B \to M$) should be a localization functor. I guess that for a given M or $\mathbb{B}$, the mere fact that there should exist a test category B and a functor $B \to M$, or $B \to \mathbb{B}$, having this property is already a very strong condition on the structure species considered, namely on the category $\mathbb{B}$ which embodies this structure. It possibly means that the corresponding functor

$$\mathbb{B} \to (\mathrm{Hot})$$

factors through a functor

$$\mathbb{B} \to (\mathrm{Hot})_{\mathrm{f.t.}} \quad \text{(f.t. means "finite type")}$$

which is itself a localization functor. It is not wholly impossible, after all, that this condition on a functor $B \to \mathbb{B}$ (B a test category) is so stringent, that all such functors (for variable B) must be already "equivalent", namely define Hot-equivalent topologies on $\mathbb{B}$ (or M, equivalently), and hence define "the same" functor $M \to (\mathrm{Hot})$ or $\mathbb{B} \to (\mathrm{Hot})_{\mathrm{f.t.}}$. [p. 29]

Digression on "geometric realization" functors.

27 All this is pretty much "thin air conjecturing" for the time being – quite possibly the notion of a "test category" itself has to be considerably adjusted, namely strengthened, as well as the notion of a "test functor" $B \to \mathbb{B}$ or $B \to M$ – some important features may have entirely escaped my attention. The one idea though which may prove perhaps a valid one, it that a suitable localization functor

$$(*) \qquad M \to (\mathrm{Hot})$$

may be defined, using either various topologies on M (related by a suitable "Hot-equivalence" relation), or various functors $B \to M$ or $B \to \mathbb{B}$ of suitable "test categories" B, and how the two are related. I do not wish to pursue much longer along these lines though, but rather put now into the picture a third way still for getting a functor (*), namely through some more or less natural functor

$$(**) \qquad M \to (\mathrm{Spaces}), \quad K \mapsto |K|,$$

called a "geometric (or topological) realization functor". There is a pretty compelling choice for such a functor, in the case of (semisimplicial or cubical) complexes or multicomplexes of various kinds, and accordingly for the subcategories (Cat), (Preord), (Ord) or (Ss sets), using geometric realization of semisimplicial complexes. In the case of the considerably more sophisticated structure of Gr-stacks though (or the relator[?] structure of stacks, which will be dealt with in much the same way below), although there is a pretty natural choice for geometric realization on the subcategory $\mathbb{B}_\infty$ of M embodying the "primitive structure" (namely the structure of an ∞-graph, see below also); it has been seen that the extension of this to a functor on the whole of M (via its extension to the left coherator defining the structure species, which we denoted by $\mathbb{B}$ at the beginning of these notes, but which is not quite the

$\mathbb{B}$ envisioned here) is by no means unique, that it depends on a pretty big bunch of rather arbitrary choices. This indeterminacy now appears as quite in keeping with the general aspect of a (still somewhat hypothetical) theory of algebraic homotopy models, gradually emerging from darkness. It parallels the corresponding indeterminacy in the choice of an "eligible" topology on M (call these topologies the test topologies), or of a test functor $B \to M$. What I would like now to do, before coming back to stacks, is to reflect a little still about the relations between such choice of a "geometric realization functor", and test topologies or test functors relative to M.

[p. 30] 8.3.

28 *The "inspiring assumption". Modelizers.*

While writing down the notes yesterday, and this morning still while pondering a little more, there has been the ever increasing feeling that I "was burning", namely turning around something very close, very simple-minded too surely, without quite getting hold of it yet. In such a situation, it is next to impossible just to leave it at that and come to the "ordre du jour" (namely stacks) – and even the "little reflection" I was about to write down last night (but it was really too late then to go on) will have to wait I guess, about the "geometric realization functors", as I feel it is getting me off rather, maybe just a little, from where it is "burning"!

There was one question flaring up yesterday (p. 27) which I nearly dismissed as kind of silly, namely whether two localization functors

$$M \to (\mathrm{Hot}) \tag{*}$$

obtained in such and such a way were isomorphic (maybe even canonically so??) provided they defined the same notion of "weak equivalence", namely arrows transformed into isomorphisms by the localization functors. Now this maybe isn't so silly after all, in view of the following
Assumption: The category of equivalences of (Hot) with itself, and of natural isomorphisms (possibly even any morphisms) between such, is equivalent to the one point category.

This means 1) any equivalence $(\mathrm{Hot}) \xrightarrow{\simeq} (\mathrm{Hot})$ is isomorphic to the identity functor, and 2) any automorphism of the identity functor (possibly even any endomorphism?) is the identity.

Maybe these are facts well-known to the experts, maybe not – it is not my business here anyhow to set out to prove such kinds of things. It looks pretty plausible, because if there was any non-trivial autoequivalence of (Hot), or automorphism of its identity functor, I guess I would have heard about it, or something of the sort would flip to my mind. It would not be so if we abelianized (Hot) some way or other, as there would be the loop and suspension functors, and homotheties by -1 of $\mathrm{id}_{(\mathrm{Hot})}$.

This assumption now can be rephrased, by stating that a localization functors (*) from any category M into (Hot) is well determined, up to a unique isomorphism, when the corresponding class $W \subset \mathrm{Fl}(M)$ of weak equivalences is known, in positive response to yesterday's silly question!

Such situation (*) seems to me to merit a name. As the work "model category" has already been used in a somewhat different and more sophisticated sense by Quillen, in the context of homotopy, I rather use another one in the situation here. Let's call a "modelizing category", [p. 31] or simply a "*modelizer*" ("modélisatrice" in French), any category M, endowed with a set $W \subset \mathrm{Fl}(M)$ (the weak equivalences), satisfying the obvious condition:

(Mod) a) W is the set of arrows made invertible by the localization functor $M \to W^{-1}M$, and

b) $W^{-1}M$ is equivalent to (Hot),

or equivalently, there exists a localization functor (*) (necessarily unique up to unique isomorphism) such that W be the set of arrows made invertible by this functor.

Let (M, W), (M', W') be two modelizers, a functor $F : M \to M'$ is called *model-preserving*, or a *morphism* between the modelizers, if it satisfies either of the following equivalent conditions:

(i) $F(W) \subset W'$, hence a functor $F_{W,W'} : W^{-1}M \to W'^{-1}M'$, and the latter is an equivalence.

(ii) The diagram

$$\begin{tikzcd} M \arrow[rr, "F"] \arrow[dr] & & M' \arrow[dl] \\ & (\mathrm{Hot}) & \end{tikzcd}$$

is commutative up to isomorphism (where the vertical arrows are the "type functors" associated to M, M' respectively.

When dealing with a modelizer (M, W), W will be generally understood so that we write simply M. When M is defined in terms of an algebraic structure species, the task arises to find out whether (if any) there exists a *unique* $W \subset \mathrm{Fl}(M)$ turning M into a modelizer, and if not so, if we can however pinpoint one which is a more natural one, and which we would call "canonical".

Here is a diagram including most of the modelizers and model-preserving functors between these which we met so far (not included however those connected with the theory of "higher" stacks and Gr-stacks, which we will have to elaborate upon later on):

$$\begin{tikzcd} & & & \Delta\!\!\!\!\Delta^{\wedge} = (\text{ss sets}) \arrow[dr, "\xi"] & \\ (\mathrm{Ord}) \arrow[r, hook] & (\mathrm{Preord}) \arrow[r, hook] & (\mathrm{Cat}) \arrow[ur, hook, "\alpha"] \arrow[dr, hook, "\beta"'] & & (\mathrm{Cat}) \\ & & & \square^{\wedge} = (\text{cub. sets}) \arrow[ur, "\eta"'] & \end{tikzcd}$$

where the two last functors, with values in (Cat), are the two obvious functors, obtained from

(**) $$i_A : A^{\wedge} \to (\mathrm{Cat}), \quad F \mapsto A_{/F}$$

by particularizing to $A = \Delta\!\!\!\!\Delta$ or $\square$. As we noticed before, the four first among these six functors admit left adjoints, but except for the first, [p. 32] these adjoints are *not* model preserving. The two last functors, and more generally the functor (**), admit right adjoints, namely the functor

$$j_A : (\mathrm{Cat}) \to A\hat{\ },$$

where $j_A(B) = (a \mapsto \mathrm{Hom}(A_{/a}, B))$ $(B \in \mathrm{Ob}(\mathrm{Cat}))$. It should be noted that the functors $j_\Delta, j_\square$ are *not* the two functors α, β which appear in the diagram above, the latter are associated to the familiar functors

$$(***) \qquad \begin{tikzcd} \Delta \arrow[dr] & \\ & (\mathrm{Cat}) \\ \square \arrow[ur] & \end{tikzcd}$$

(factoring in fact through (Ord)), associating to every ordered simplex, or to each multiordered cube, the corresponding 1-skeleton with suitable orientations on the edges, turning the vertices of this graph into an ordered set; while the two former are associated to the functors deduced from (**) by restricting to $A \subset A\hat{\ }$, namely

$$n \mapsto \Delta_{/\Delta_n}$$

in the case of Δ, and accordingly for $\square$. The values of these functors, contrarily to the two preceding ones, are *infinite* categories, and they cannot be described by (i.e., "are" not) (pre)ordered sets. If however we had defined the categories $\Delta, \square$ in terms of iterated boundary operations only, excluding the degeneracy operations (which, I feel, are not really needed for turning $\Delta\hat{\ }$ and $\square\hat{\ }$ into modelators), we would get indeed *finite* ordered sets, namely the full combinatorial simplices or cubes, each one embodied by the ordered set of all its facets of all possible dimensions.

Contrarily to what happens with the functors α, β, I feel that for the two functors ξ, η in opposite direction, not only are they model preserving, but the right adjoint functors $j_\Delta, j_\square$ must be model preserving too, and we will have to come back upon this in a more general context.

We could amplify and unify somewhat the previous diagram of modelizers, by introducing multicomplexes, which after all can be as well "mixed" namely partly semisimplicial, partly cubical. Namely, we may introduce the would-be "test categories"

$$\Delta^p \times \square^q = T_{p,q} \quad (p, q \in \mathbb{N}, p + q \geq 1)$$

giving rise to the category $T_{p,q}\hat{\ }$ of (p,q)-multicomplexes (p times simplicial, q times cubical). We have a natural functor (generalizing the functors (***))

$$T_{p,q} \to (\mathrm{Ord})(\hookrightarrow (\mathrm{Cat})),$$

associating to a system of p standard simplices and q standard cubes (of variable dimensions), the *product* of the $p+q$ associated ordered sets. We get this way a functor

$$\alpha_{p,q} : (\mathrm{Cat}) \to T_{p,q}\hat{\ }$$

which presumably (as I readily felt yesterday, cf. first lines p. 24) is not any less model-preserving than the functors α, β it generalizes. Of course, taking $A = T_{p,q}$ above, we equally get a natural functor [p. 33]

$$i_{p,q} : T_{p,q}{}^\wedge \to (\mathrm{Cat})$$

admitting a right adjoint $j_{p,q}$, and both functors I feel must be model preserving.

The basic modelizer (Cat). *Provisional definition of test categories and elementary modelizers.*

29 It is time now to elaborate a little upon the notion of a test category, within the context of modelizers. Let A be a small category, and consider the functor (**)

$$(**) \qquad i_A : A^\wedge \to (\mathrm{Cat}), \quad F \mapsto A_{/F}.$$

Whenever we have a functor $i : M \to M'$, when M' is equipped with a W' turning it into a modelizer, there is (if any) just one $W \subset \mathrm{Fl}(M)$ turning M into a modelizer and i into a morphism of such, namely $W = i_{\mathrm{Fl}}^{-1}(W')$. In any case, we may define W ("weak equivalences") by this formula, and get a functor

$$W^{-1}M \to W'^{-1}M',$$

which is an equivalence iff (M, W) is indeed a modelizer and i model preserving. We may say shortly that $i : M \to M'$ is model preserving, even without any W given beforehand. Now coming back to the situation (**), the understanding yesterday was to call A a *test category*, to express that the canonical functor (**) is model preserving. (In any case, unless otherwise specified by the context, we will refer to arrows in $A^\wedge$ which are transformed into weak equivalences of (Cat) as "*weak equivalences*".) It may well turn out, by the way, that we will have to restrict somewhat still the notion of a test category.

In any case, the basic modelizer, in this whole approach to homotopy models, is by no means the category (Ss sets) (however handy) or the category (Spaces) (however appealing to topological intuition), but the category (Cat) of "all" (small) categories. In this setup, the category (Hot) is most suitably defined as the category of fractions of (Cat) with respect to "weak equivalences". These in turn are most suitably defined in cohomological terms, via the corresponding notion for topoi – namely a morphism of topoi

$$f : X \to X'$$

is a "weak equivalence" or homotopy equivalence, iff for every locally constant sheaf F' on X', the maps

$$\mathrm{H}^i(X', F') \to \mathrm{H}^i(X, f^{-1}(F'))$$

are isomorphisms whenever defined – namely for $i = 0$, for $i = 1$ if moreover F' is endowed with a group structure, and for any i if F' is moreover commutative (criterion of Artin-Mazur). Accordingly, a functor [p. 34]

$$f : A \to A'$$

between small categories (or categories which are essentially small, namely equivalent to small categories) is called a weak equivalence, iff the corresponding morphism of topoi

$$f\hat{\ } : A\hat{\ } \to A'\hat{\ }$$

is a weak equivalence.

Coming back to test categories A, which allow us to construct the corresponding modelizers $A\hat{\ }$, our point of view here is rather that the test categories are each just as good as the others, and Δ just as good as $\square$ or any of the $T_{p,q}$ and not any better! Maybe it's the one though which turns out the most economical for computational use, the nerve functor (Cat) $\to \Delta\hat{\ }$ being still the neatest known of all model preserving embeddings of (Cat) into categories $A\hat{\ }$ defined by modelizers. Another point, still more important it seems to me, is that the natural functor

$$(\text{Topoi}) \to \text{Pro}\,(\text{Hot})$$

defined by the Čech-Cartier-Verdier process, and which allows for another description of weak equivalences of topoi, namely as those made invertible by this functor, are directly defined via semi-simplicial structures of simplicial structures (of the type "nerve of a covering").

Modelizers of the type $A\hat{\ }$, with A a test category, surely deserve a name – let's call them *elementary* modelizers, as they correspond to the case of an "elementary" or "trivial" algebraic structure species, whose set-theoretical realizations can be expressed as just *any* functors

$$A^{\text{op}} \to (\text{Sets}),$$

without any exactness condition of any kind; in other words they can be viewed as just diagrams of sets of a specified type, with specified commutativity relations. A somewhat more ambitious question maybe is whether on such a category $M = A\hat{\ }$, namely an elementary modelizer, there cannot exist any other modelizing structure. In any case, the one we got is intrinsically determined in terms of M, which is a topos, by the prescription that an arrow $f : a \to b$ within M is a weak equivalence if and only if the corresponding morphism for the induced topoi $M_{/a}$ and $M_{/b}$ is a weak equivalence (in terms of the Artin-Mazur criterion above, see p. 33).

A more crucial question I feel is whether the right adjoint functor j_A to i_A in (**) (cf. p. 33) is equally model preserving, whenever A is a test category. This, as we have seen, is *not* automatic, whenever we have a model preserving functor between modelizers, whenever this functor admits an adjoint functor. In more general terms still, let

$$\begin{tikzcd} M \arrow[r, bend left, "i"] & M' \arrow[l, bend left, "j"] \end{tikzcd}$$

[p. 35] be a pair of adjoint functors, with M, M' endowed with a "saturated" set of arrows W, W'. Then the following are equivalent:

(a) $i(W) \subset W'$, $j(W') \subset W$, and the two corresponding functors

$$W^{-1}M \rightleftarrows W'^{-1}M'$$

are quasi-inverse of each other, the adjunction morphisms between them being deduced from the corresponding adjunction morphisms for the pair (i, j).

(b) $W = i^{-1}(W')$, and for every $a' \in M'$, the adjunction morphism

$$ij(a') \to a'$$

is in W'.

(b') dual to (b), with roles of M and M' reversed.

In the situation we are interested in here, $M = A\hat{}$ and $M' = (\mathrm{Cat})$, we know already $W = i^{-1}(W')$ by definition, and hence all have to see is whether for any category B, the functor

$$\text{(T)} \qquad i_A j_A(B) \to B$$

is a weak equivalence. This alone will imply that not only $i_{W,W'}$, but equally $j_{W',W}$ is an equivalence, and that the two are quasi-inverse of each other. (NB even without assuming beforehand that W, W' are saturated, (b) (say) implies (b) and (a), provided we assume on W' the very mild saturation condition that for composable arrows u', v', if two among $u', v', v'u'$ are in W', so is the third; if we suppose moreover that M' is actually saturated namely made up with all arrows made invertible by $M' \to W'^{-1}M'$, then condition (b) implies that M is saturated too – which ensures that if (M', W') is modelizing, so is (M, W).)

It is not clear to me whether for every test category A, the stronger condition (T) above is necessarily satisfied. This condition essentially means that for any homotopy type, defined in terms of an arbitrary element B in (Cat) namely a category B, we get a description of this homotopy type by an object of the elementary modelizer $A\hat{}$, by merely taking $j_A(B)$. This condition seems sufficiently appealing to me, for reinforcing accordingly the notion of a test-category A, and of an elementary modelizer $A\hat{}$, in case it should turn out to be actually stronger. Of course, any category equivalent to an elementary modelizer $A\hat{}$ will be equally called by the same name. It should appear in due course whether this is indeed the better suited notion. One point in its favor already is that it appears a lot more concrete.

Another natural question, suggested by the use of simplicial or cubical multicomplexes, is whether the product of two test categories is again a test category – which might furnish us with a way to compare directly the description of (Hot) by the associated elementary modelizers, without [p. 36]
having to make a detour by the "basic" modelizer (Cat) we started with. But here it becomes about time to try and leave the thin air conjecturing, and find some simple and concrete characterization of test categories, or possibly some reinforcement still of that notion, which will imply stability under the product operation.

Here it is tempting to use semi-simplicial techniques though, by lack of independent foundations of homotopy theory in terms of the modelizer (Cat). Thus we may want to take the map of semi-simplicial sets corresponding to (T) when passing to nerves, and express that a) this is a fibration and b) the fibers of this fibration are “contractible” (or “aspheric”), which together will imply that we have a weak equivalence in (Ss sets). Or we may follow the suggestion of Quillen’s set-up, working heuristically in (Cat) as though we actually know it is a model category, and expressing that the adjunction morphism in (T), which is a functor between categories, is actually a “fibration”, and that its fibers are “contractible” namely weakly equivalent to a one-point category. In any case, a minimum amount of technique seems needed here, to give the necessary clues for pursuing.

14.3.

Starting the “asphericity game”.

30 Since last week when I stopped with my notes, I got involved a bit with recalling to mind the “Lego-Teichmüller construction game” for describing in a concrete, “visual” way the Teichmüller groups of all possible types and the main relationships between them, which I had first met with last year. This and other non-mathematical occupations left little time only for my reflections on homotopy theory, which I took up mainly last night and today. The focus of attention was the “technical point” of getting a handy characterization of test categories. The situation I feel is beginning to clarify somewhat. Last thing I did before reading last weeks’ notes and getting back to the typewriter, was to get rid of a delusion which I was dragging along more or less from the beginning of these notes, namely that our basic modelizer (Cat), which we were using as a giving the most natural definition of (Hot) in our setting, was a “model category” in the sense of Quillen, more specifically a “closed model category”, where the “weak equivalences” are the homotopy equivalences of course, and where cofibrations are just monomorphisms (namely functors injective on objects and on arrows) – fibrations being defined in terms of these by the Serre-Quillen lifting property. Without even being so demanding, it turns out still that there is no reasonable structure of a model category on (Cat), having the correct weak equivalences, and such that the standard “Kan inclusions” of the following two ordered sets [p. 37]

$$\begin{tikzcd} & b \\ a \arrow[ur] \arrow[rr] & & c \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & b \arrow[dr] \\ a \arrow[rr] & & c \end{tikzcd}$$

into

$$\Delta_2 = a \to b \to c$$

be cofibering. Namely, for a category bC, to say that it is “fibering” (over the final category $\bullet$) with respect to one or the other monomorphism, means respectively that every arrow in $\mathbb{C}$ has a left respectively a right inverse – the two together mean that $\mathbb{C}$ is a groupoid. But groupoids are definitely *not* sufficient for describing arbitrary homotopy types, they give rise only to sums of $K(\pi,1)$ spaces – thus contradicting Quillen's statement that homotopy types can be described by "models" which are both fibering and cofibering!

The feeling however remains that any elementary modelizer, namely one defined (up to equivalence) by a test category, should be a closed model category in Quillen's sense – I find it hard to believe that this should be a special feature just of semi-simplicial complexes!

While trying to understand test categories, the notion of *asphericity* for a morphism of topoi

$$f : X \to Y$$

came in naturally – this is a natural variant of the notion of n-acyclicity (concerned with commutative cohomology) which has been developed in the context of étale topology of schemes in SGA 4. It can be expressed by "localizing upon Y" the Artin-Mazur condition that f be a weak equivalence, by demanding that the same remain true for the induced morphism of topoi

$$X \times_Y Y' \to Y'$$

for any "localization morphism" $Y' \to Y$. In terms of the categories of sheaves E, F on X, Y, Y' can be defined by an object (equally called Y') of F, the category of sheaves on Y' being $F_{/Y'}$, and the fiber-product X' can be defined likewise by an object of E, namely by $X' = f^*(Y')$, hence the corresponding category of sheaves is $E_{/f^*(Y')}$. In case the functor f^* associated to f admits a left adjoint $f_!$ (namely if it commutes to arbitrary inverse limits, not only to finite ones), the category $E_{/f^*(Y')}$ can be interpreted conveniently as $E_{f_!/Y'}$ (or simply $E_{/Y'}$ if $f_!$ is implicit), whose objects are pairs

$$(U, \varphi), \quad U \in \operatorname{Ob} E, \quad \varphi : f_!(U) \to Y',$$

with obvious "maps" between such objects. For the time being I am mainly interested in the case of a morphism of topoi defined by a functor between categories, which I will denote by the same symbol f:

$$f : C' \to C \quad \text{defines } f \text{ or } f^\wedge : C'^\wedge \to C^\wedge.$$

Using the fact that in the general definition of asphericity it is enough to take Y' in a family of generators of the topos Y, and using here the generating subcategory C of $C^\wedge$, we get the following criterion for asphericity of $f^\wedge$: it is necessary and sufficient that for every $a \in \operatorname{Ob} C$, the induced morphism of topoi [p. 38]

$$C'^\wedge_{/a} \approxeq (C'_{/a})^\wedge \to C^\wedge_{/a} \simeq (C_{/}a)^\wedge$$

be a weak equivalence, i.e., that the natural functor $C'_{/a} \to C_{/a}$ be a weak equivalence. But it is immediate that $C_{/a}$ is "contractible", i.e., "aspheric", namely the "map" from $C_{/a}$ to the final category is a weak equivalence (this is true for any category having a final object). Therefore we get the following

Criterion of asphericity for a functor $f : C' \to C$ between categories: namely it is n. and s. that for any $a \in \operatorname{Ob} C$, $C'_{/a}$ be aspheric.

Let's come back to a category A, for which we want to express that it is a test-category, namely for any category B, the natural functor

$$(*) \qquad i_A j_A(B) \to B$$

is a weak equivalence. One immediately checks that for any $b \in \operatorname{Ob} B$, the category $i_A j_A(B)_{/b}$ over $B_{/b}$ is isomorphic to $i_A j_A(B_{/b})$. Hence we get the following

Proposition. For a category A, the following are equivalent:

(i) A is a test category, namely (*) is a weak equivalence for any category B.

(ii) (*) is aspheric for any category B.

(iii) $i_A j_A(B)$ is aspheric for any category B with final element.

This latter condition, which is the most "concrete" one so far, means also that the element

$$F = j_A(B) = (a \mapsto \operatorname{Hom}(A_{/a}, B)) \in A^\wedge$$

is an aspheric element of the topos $A^\wedge$, namely the induced topos $A^\wedge{}_{/F}$ is aspheric (i.e., the category $A_{/F}$ is aspheric), *whenever* B has a final element.

31 *The end of the thin air conjecturing: a criterion for test categories.*

For the notion of a test category to make at all sense, we should make sure in the long last that Δ itself, the category of standard simplices, is indeed a test category. So I finally set out to prove at least that much, using the few reflexes I have in semi-simplicial homotopy theory. A proof finally peeled out it seems, giving clues for handy conditions in the general case, which should be *sufficient* at least to ensure that B is a test category, but maybe not quite necessary. I'll try now to get it down explicitly.

[p. 39] Here are the conditions I got:

(T 1) A is aspheric.

(T 2) For $a, b \in \operatorname{Ob} A$, $A_{/a\times b}$ is aspheric (NB $a \times b$ need not exist in A but it is in any case well defined as an element of $A^\wedge$).

(T 3) There exists an *aspheric* element I of $A^\wedge$, and two subobjects e_0 and e_1 of I which are final elements of $A^\wedge$, such that $e_0 \cap e_1 (\overset{\text{def}}{=} e_0 \times_I e_1) = \varnothing_{A^\wedge}$, the initial or "empty" element of $A^\wedge$.

In case when $A = \Delta$ (as well as in the cubical analogon $\square$), I took $I = \Delta_1$ which is an element of A itself, and moreover A has a final element (which is a final element of $A^\wedge$ therefore) e, thus e_0 and e_1 defined by well defined arrows in A itself, namely δ_0 and δ_1. But it does not seem that these special features are really relevant. In any case *intuitively I stands for the unit interval, with endpoints* e_0, e_1. If F is any element in $A^\wedge$, the standard way for trying to prove it is aspheric would be to prove that we can find a “constant map” $F \to F$, namely one which factors into

$$F \to e \xrightarrow{c} F \quad (e \text{ the final element of } A)$$

for suitable $c : e \to F$ or “section” of F, which be “homotopic” to the identity map $F \to F$. When trying to make explicit the notion of a “homotopy” h between two such maps, more generally between two maps $f_0, f_1 : F \rightrightarrows G$, we hit of course upon the arrow h in the following diagram, which should make it commutative

(D)
$$\begin{tikzcd}
& F \times I \arrow[dd, dashed, "h"'] & \\
F \simeq F \times e_0 \arrow[ur, hook] \arrow[dr, "f_0"'] & & F \simeq F \times e_1 \arrow[ul, hook'] \arrow[dl, "f_1"] \\
& G &
\end{tikzcd} .$$

This notion of a homotopy is defined in any category $\mathcal{A}$ where we’ve got an element I and two subobjects e_0, e_1 which are final objects. Suppose we got such h, and we know moreover (for given F, G, f_0, f_1, h) that the two inclusions of $F \times e_0$ and $F \times e_1$ into $F \times I$ are weak equivalences, and that f_0 is a weak equivalence (for a given set of arrows called “weak equivalences”, for instance defined in terms of a “topology” on $\mathcal{A}$, in the present case the canonical topology of the topos $A^\wedge$), then it follows (with the usual “mild saturation condition” on the notion of weak equivalence) that h, and hence f_1 are weak equivalences. Coming back to the case $F = G, f_0 = \mathrm{id}_F, f_1 =$ “constant map” defined by a $c : e \to F$, we get that this constant map f_1 is a weak equivalence. Does this imply that $F \to e$ is equally a weak equivalence? This is not quite formal for general $(\mathcal{A}, W)$, but it is true though in the case $\mathcal{A} = A^\wedge$ and with the usual meaning of “weak equivalence”, in this case it is true [p. 40] indeed that if we have a situation of inclusion with retraction $E \to F$ and $F \to E$ (E need not be a final element of $\mathcal{A}$), such that the compositum $p : F \to E \to F$ (a projector in F) is a weak equivalence, then so are $E \to F$ and $F \to E$. To check this, we are reduced to checking the corresponding statement in (Cat), in fact we can check it in the more general situation with two topoi E and F, using the Artin-Mazur criterion. (We get first that $E \to F$ is a weak equivalence, and hence by saturation that $F \to E$ is too.)

Thus the assumptions made on $F \in \mathrm{Ob}\, A^\wedge$ imply that $F \to e$ is a weak equivalence, i.e., $A_{/F} \to A$ is a weak equivalence, and if we assume now that A satisfies (T 1) namely A is aspheric, so is F.

We apply this to the case $F = j_A(B) = (a \mapsto \mathrm{Hom}(A_{/a}, B))$, where B is a category with final element. We have to check (using (T 1) to (T 3)):

(a) The inclusions of $F \times e_0, F \times e_1$ into $F \times I$ are weak equivalences (this will be true in fact for any $F \in \mathrm{Ob}\, A^\wedge$),

(b) there exists a “homotopy” h making commutative the previous diagram (D), where $G = F$, $f_0 = \mathrm{id}_F$, and where $f_1 : F \to F$ is the "constant map" defined by the section $c : e \to F$ of F, associating to every $a \in A$ the *constant* functor $e_{a,B} : A_{/a} \to B$ with value e_B (a fixed final element of B), thus $e_{a,B} \in F(a) = \mathrm{Hom}(A_{/a}, B)$ (and it is clear that his is "functorial in a").

Then (a) and (b) will imply that F is aspheric – hence A is a test-category by the criterion (iii) of the proposition above.

To check (b) we do not make use of (T 1) nor (T 2), nor of the asphericity of I. We have to define a "map"

$$h : F \times I \to F,$$

i.e., for every $a \in \mathrm{Ob}A$, a map (functorial for variable a)

$$h(a) : \mathrm{Hom}(A_{/a}, B) \times \mathrm{Hom}(a, I) \to \mathrm{Hom}(A_{/a}, B)$$

(two of the Hom's are in (Cat), the other is in $A^\wedge$). Thus, let

$$f : A_{/a} \to B, \quad u : a \to I,$$

we must define

$$h(a)(f,u) : A_{/a} \to B$$

a functor from $A_{/a}$ to B, depending on the choice of f and u. Now let, for any $u \in \mathrm{Hom}(a, I)$, $u : a \to I$, a_u be defined in $A^\wedge$ as

$$a_u = u^{-1}(e_0) = (a, u) \times_I e_0,$$

viewed as a subobject of a, and hence $C_u = A_{/a_u}$ can be viewed as a subcategory of $C = A_{/a}$, namely the full subcategory of those objects x over a, i.e., arrows $x \to a \in A$, which factor through a_u namely such that the compositum $x \to a \to I$ factors through e_0. This subcategory is
[p. 41] clearly a "crible" in $A_{/a}$, namely for an arrow $y \to x$ in $A_{/a}$, if the target x is in the subcategory, so is the source y. This being so, we define the functor

$$f' = h(f,u) : A_{/a} = C \to B$$

by the conditions that

$$f'|(A_{/a_u} = C_u) = f|C_u$$
$$f'|(C \setminus C_u) = \text{constant functor with value } e_B.$$

(where $C \setminus C_u$ denotes the obvious *full* subcategory of C, complementary to C_u). This defines f' uniquely, on the objects first, and on the arrows too because the only arrows left in C where we got still to define f' are arrows $x \to y$ with x in C_u and $y \in C \setminus C_u$ (because C_u is a crible), but then $f'(y) = e_B$ and we have no choice for $f'(x) \to f'(y)$! It's trivial checking that this way we get indeed a *functor* $f' : C \to B$, thus the map $h(a)$ is defined – and that this map is functorial with respect to a, i.e., comes from a map $h : F \times I \to F$ as we wanted. The commutativity of (D) is easily checked: for the left triangle, i.e., that the compositum $F \simeq F \times e_0 \to F \times I \xrightarrow{h} F$ is the identity, it comes from the fact that if u

factors through e_0, then $C_u = C$ hence $f' = f$; for the right triangle, it comes from the fact that if u factors through e_1, then $C_u = \varnothing$ (here we use the assumption $e_0 \cap e_1 = \varnothing_{A^\wedge}$), and hence f' is the constant functor $C \to B$ with value e_B. This settles (b).

We have still to check (a), namely that for any $F \in \operatorname{Ob} A^\wedge$, the inclusion of the objects $F \times e_i$ into $F \times I$ are weak equivalences, or what amounts to the same, that the projection

$$F \times I \to F$$

is a weak equivalence – this will be true in fact for any object I of $A^\wedge$ which is aspheric. Indeed, we will prove the stronger result that $F \times I \to F$ is *aspheric*, i.e., that the functor

$$A_{/F\times I} \to A_{/F}$$

is aspheric, We use for this the criterion of p. 38, which here translates into the condition that for any a in A (such that we got an $a \to F$, i.e., such that $F(a) \neq \emptyset$, but never mind), the category $A_{/a\times I}$ is aspheric, i.e., the lement $a \times I$ of $A^\wedge$ is aspheric. Again, *as I is aspheric,* we are reduced to checking that $a \times I \to I$ is aspheric, which by the same argument (with F, I replaces by I, a) boils down to the condition that $A_{x\times b}$ is aspheric for any b in A. Now this is just condition (T 2), we are through.

15.3. [p. 42]

32 *Provisional program of work.*

I definitely have the feeling to be out of the thin air – the conditions (T 1) to (T 3) look to me so elegant and convincing, that I have no doubts left they are "the right ones"! A lot of things come to mind what to do next, I'll have to look at them one by one though. Let me make a quick provisional planning.

1) Have a closer look at the conditions (T 1) to (T 3), to see how far they are necessary for A to be a test category in the (admittedly provisional) sense I gave to this notion last week and yesterday, and to pin down the feeling of these being just the right ones.
2) Use these conditions for constructing lots of test categories, including all the simplicial and cubical types which have been used so far.
3) Check that these conditions are stable under taking the product of two or more test categories, and possibly use this fact for comparing the homotopy theories defined by any two such categories.
4) Look up (using (T 1) to (T 3)) if an elementary modelizer $A^\wedge$ is indeed a "closed model category" in Quillen's sense, and maybe too get a better feeling of how far apart (Cat) is from being a closed model category. Visibly there *are* some natural constructions in homotopy theory which do make sense in (Cat).
5) Using the understanding of test categories obtained, come back to the question of which categories $A^\wedge$ associated to algebraic structure species can be viewed as modelizers, and to the question of unicity or canonicity of the modelizing structure $W \subset \operatorname{Fl}(M)$.

That makes a lot of questions to look at, and the theory of stacks I set out to sketch seems to be fading ever more into the background! It is likely though that a better general understanding of the manifold constructions of the category (Hot) of homotopy types will not be quite useless, when getting back to the initial program, namely stacks. Quite possibly too, on my way I will have to remind myself of and look up, in the present setting, the main structural properties of (Hot), including exactness properties and generators and cogenerators. This also reminds me of an intriguing foundational question since the introduction of derived categories and their non-commutative analogs, which I believe has never been settled yet, namely the following:

6) In an attempt to grasp the main natural structures associated to derived categories, and Quillen's non-commutative analogons including (Hot), try to develop a comprehensive notion of a "triangulated category", without the known drawbacks of Verdier's provisional notion. [p. 43]

33 *Necessity of conditions T1 to T3, and transcription in terms of elementary modelizers.*

For the time being, I'll use the word "test category" with the meaning of last week, and refer to categories satisfying the conditions (T 1) to (T 3) as *strict* test categories. (NB A is supposed to be essentially small in any case.)

First of all, the conditions (T 1) and (T 3) are *necessary* for A to be a test category. For (T 1) this is just the "concrete" criterion (iii) of yesterday (page 38), when B is the final element of (Cat). For (T 3), we get even a *canonical choice* for I, e_0, e_1, namely starting with the "universal" choice in (Cat):

$$\mathrm{I} = \mathbb{\Delta}_1, \quad \mathrm{e}_0 \text{ and } \mathrm{e}_1 \text{ the two subobjects of } \mathbb{\Delta}_1 \text{ in (Cat) (or in (Ord)) corresponding to the two unique sections of I over the final element e of (Cat),}$$

i.e., in terms of I as an ordered set $\{0\} \to \{1\}$, e_0 and e_1 are just the two subobjects defined by the two vertices $\{0\}$ and $\{1\}$ (viewed as defining two one-point ordered subsets of $\mathrm{I} = \mathbb{\Delta}_1$). We now apply j_A to get

$$I = j_A(\mathrm{I}) = j_A(\mathbb{\Delta}_1), \quad e_i = j_A(\mathrm{e}_i) \quad \text{for } i = 0, 1.$$

As $\mathrm{I} \to \mathrm{e}$ is a weak equivalence so is $I \to e_{A^\wedge} = j_A(\mathrm{e})$, and hence (as $e_{A^\wedge}$ is aspheric by (T 1)) I is aspheric. As $\mathrm{e}_0 \cap \mathrm{e}_1 = \varnothing_{(\mathrm{Cat})}$ and j_A commutes with inverse limits, and with sums (and in particular transforms the initial element $\varnothing$ of (Cat) in the initial element $\varnothing$ of $A^\wedge$), it follows that $e_0 \cap e_1 = \varnothing$.

It's worthwhile having a look at what this object just constructed is like. For this end, let's note first that for any category C, we have a canonical bijection, functorial in C

$$\mathrm{Hom}(C, \mathbb{\Delta}_1) \xrightarrow{\sim} \mathrm{Crib}(C) \simeq \text{set of all subobjects of } e_{C^\wedge},$$

by associating to any functor $f : C \to \mathbb{\Delta}_1$ the full subcategory $f^{-1}(\mathrm{e}_0)$ of C, which clearly is a "crible" in C. Thus we get, for $a \in \mathrm{Ob}\, A$

$$I(a) \simeq \mathrm{Crib}(A_{/a}) \quad (\text{subobjects of } a \text{ in } A^\wedge).$$

More generally, we deduce from this, for any $F \in \mathrm{Ob} A^\wedge$:

$$\mathrm{Hom}(F, I) = \Gamma(I_F = I \times F/F) = \text{set of all subobjects of } F,$$

in other words, the object I is defined intrinsically in the topos $A^\wedge$ (up to unique isomorphism) as the "*Lawvere element*" representing the functor $F \mapsto$ set of subobjects of F, generalizing the functor $F \mapsto \mathfrak{P}(F)$ in the category of sets (namely sheaves over the one-point topos), represented by the two-point set $\{0, 1\}$. The condition $e_0 \cap e_1 = \varnothing_{\mathcal{A}}$ is automatic in any topos $\mathcal{A}$, provided $\mathcal{A}$ is not the "empty topos", namely corresponding to a category of sheaves equivalent to a one-point category.

There is a strong temptation now to diverge from test categories, to expand on intrinsic versions of conditions (T 1) to (T 3) for any topos, and extract a notion of a (strictly) modelizing topos, generalizing the "elementary modelizers" $A^\wedge$ defined by strict test categories A. But it seems more to the point for the time being to look more closely to the one condition, namely (T 2), which does not appear so far as necessary for A being a test category – and I suspect it is *not* necessary indeed, as no idea occurred to me how to deduce it. (I have no idea of how to make a counterexample though, as I don't see any other way to check a category A is a test category, except precisely using yesterday's criterion via (T 1) to (T 3).) Still, I want to emphasize about the fact that (T 2) is indeed a very natural condition. In this connection, it is timely to remember that in the category (Hot), finite direct and inverse limits exist (and even infinite ones, I guess, but I feel I'll have to be a little careful about these. . .). The existence of such limits, in terms of the description of (Hot)as a category of fractions of (Cat), doesn't seem at all a trivial fact, for the time being I'll admit it as "well known" (from the semisimplicial set-up, say), and probably come back upon this with some care later. Now if (M, W) is any modelizer, hence endowed with a localization functor [p. 44]

$$M \to (\mathrm{Hot}),$$

it is surely not irrelevant to ask about which limits this functor commutes with, and study the case with care. Thus in no practical example I know of does this functor commute with binary amalgamated sums or with fibered products without an extra condition on at least one of the two arrows involved in M – a condition of the type that one of these is a monomorphism or a cofibration (for co-products), or a fibration (for products). However, in all cases known, it seems that the functor commutes to (finite, say) sums and products. For sums, it really seems hard to make a sense out of a localizing functor $M \to (\mathrm{Hot})$, namely playing a "model" game, without the functor commuting at the very least to these! In this respect, it is reassuring to notice that for any category A, the associated functors i_A, j_A between $A^\wedge$ and (Cat) do indeed both commute with (arbitrary) sums – which of course is trivial anyhow for i_A (commuting to arbitrary direct limits), and easily checked for j_A (which apparently does not commute to any other type of direct limits, but of course commutes with arbitrary inverse limits). Now for products too, it is current use to look at products of "models" for homotopy types, as models for the product type – so much so that this fact is surely tacitly used everywhere, without any feeling of a need to comment. It seems too that any category M which one has looked at so far for possible use [p. 45] as a category of "models" in one sense or other for homotopy types, for instance set-theoretic realizations of some specified algebraic structure species, or topological spaces, and the like, do admit arbitrary direct and inverse limits, and surely sums and products therefore, so that the question of commutation of the localization functor to these arises indeed and is felt to be important. Possibly so important even, that the notion I introduced of a "modelizer" should take this into account, and be strengthened to the effect that the canonical functor $M \to (\mathrm{Hot})$ should commute at least to finite sums and products, and possibly even to infinite ones (whether the latter will have to be looked up with care). I'll admit provisionally that in (Hot), finite sums and products can be described in terms of the corresponding operations in (Cat), namely that the canonical functor (going with our very definition of (Hot) as a localization of (Cat))

$$(\mathrm{Cat}) \to (\mathrm{Hot})$$

commutes with finite (presumably even infinite) sums and products. This is indeed reasonably, in view of the fact that the Nerve functor

$$(\mathrm{Cat}) \to \mathbb{\Delta}^\wedge = (\mathrm{Ss\ sets})$$

does commute to sums and products.

In the case of a test category A and the corresponding elementary modelizer $A^\wedge$, the corresponding localizing functor is the compositum

$$A^\wedge \xrightarrow{i_A} (\mathrm{Cat}) \to (\mathrm{Hot}),$$

which therefore commutes with finite (presumably even infinite) sums automatically, because i_A does. Commutation with finite *products* though does not look automatic. The property of commutation with final elements is OK and is nothing but condition (T 1), which we saw is necessary for A to be a test category. Thus remains the question of commutation with binary products, which boils down to the following condition, for any two elements F and G in $A^\wedge$:

$$i_A(F \times G) \to i_A(F) \times i_A(G) \quad \text{should be a weak equivalence,}$$

i.e.,

$$(*) \qquad A_{/F\times G} \to A_{/F} \times A_{/G} \quad \text{a weak equivalence.}$$

This now implies condition (T 2), as we see taking $F = a$, $G = b$, in which case the condition (*) just means asphericity of $a \times b$ in $A^\wedge$, namely (T 2). To be happy, we have still to show that conversely, (T 2) implies (*) for any F, G. As usual, it implies even the stronger condition that the functor in question is aspheric, which by the standard criterion (page 38) just means that the categories $A_{/a\times b}$ (for a, b in A, and such moreover that $F(a)$ and $G(b)$ non-empty, but never mind) are aspheric.

Everything turns out just perfect – it seems worthwhile to summarize [p. 46] it in one theorem:

Theorem. *Let A be an essentially small category, and consider the composed functor*

$$m_A : A\hat{} \xrightarrow{i_A} (\mathrm{Cat}) \to (\mathrm{Hot}),$$

where $i_A(F) = A_{/F}$ for any $F \in \mathrm{Ob}\,A\hat{}$, and where (Cat) → (Hot) *is the canonical functor from* (Cat) *into the localized category with respect to weak equivalences. The following conditions are equivalent:*

(i) *The functor m_A commutes with finite products, and is a localization functor, i.e., induces an equivalence $W_A^{-1}A\hat{} \xrightarrow{\approx} (\mathrm{Hot})$, where W_A is the set of weak equivalences in $A\hat{}$, namely arrows transformed into weak equivalences by i_A.*

(ii) *The functor i_A and its right adjoint j_A define functors between $W_A^{-1}A\hat{}$ and $(\mathrm{Hot}) = W^{-1}(\mathrm{Cat})$ (thus j_A should transform weak equivalences of* (Cat) *into weak equivalences of $A\hat{}$) which are* quasi-inverse *to each other, the adjunction morphisms for this pair being deduced from the adjunction morphisms for the pair i_A, j_A. Moreover, the functor m_A commutes with finite products.*

(iii) *The category A satisfies the conditions* (T 1) *to* (T 3) (*page 39*).

9.4. Presumably, we'll have to add that $j_A = i_A^*$ is model preserving, cf. remark 3 on p. 174.

I could go on with two or three more equivalent conditions, which could be expressed intrinsically in terms of the topos $A\hat{}$ and make sense (and are equivalent) for any topos, along the lines of the reflections of p. 43 and of yesterday. But I'll refrain for the time being!

In the proof of the theorem above, I did not make use of semi-simplicial techniques nor of any known results about (Hot), with the only exception of the assumption (a fact, I daresay, but not proved for the time being in the present framework, without reference to semi-simplicial theory say) that the canonical functor (Cat) → (Hot) commutes with binary products. We could have avoided this assumption, by slightly changing the statement of the theorem, the condition that m_A commute with finite products in (i) and (ii) being replaced by the assumption that i_A commute with finite products "up to weak equivalence", as made explicit in (*) above for the case of binary products (which are enough of course).

I suspect that the notion of a test category in the initial, wider sense will be of no use any longer, and therefore I will reserve this name to the *strict* case henceforth, namely to the case of categories satisfying the equivalent conditions of the theorem above. Accordingly, I'll call [p. 47] "*elementary modelizer*" any category $\mathcal{A}$ equivalent to a category $A\hat{}$, with A a test category. Such a category will be always considered as a modelizer, of course, with the usual notion of weak equivalence $W \subset \mathrm{Fl}(\mathcal{A})$, namely of a "map" $F \to G$ in $\mathcal{A}$ such that the corresponding morphism for the induced topoi is a weak equivalence. The category $\mathcal{A}$ is an elementary modelizer, therefore, iff it satisfies the following conditions:

a) $\mathcal{A}$ is equivalent to a category $A\hat{}$ with A small, which amount to saying that $\mathcal{A}$ is a topos, and has sufficiently many "essential points", namely "points" such that the corresponding fiber-functor $\mathcal{A} \to (\mathrm{Sets})$ commutes with arbitrary products – i.e., there exists a conservative family of functors $\mathcal{A} \to (\mathrm{Sets})$ which commute to arbitrary direct and inverse limits. (Cf. SGA 4 IV 7.5.)

b) The pair $(\mathcal{A}, W)$ is modelizing (where $W \subset \mathrm{Fl}(\mathcal{A})$ is the set of weak equivalences), i.e., the category $W^{-1}\mathcal{A}$ is equivalent to (Hot).

c) The canonical functor $\mathcal{A} \to (\mathrm{Hot})$ (or, equivalently, $\mathcal{A} \to W^{-1}\mathcal{A}$) commutes with finite products (or, equivalently, with binary products – that it commutes with final elements follows already from a), b)).

As I felt insistently since yesterday, there is a very pretty notion of a "modelizing topos" generalizing the notion of an elementary modelizer, where $\mathcal{A}$ is a topos but not necessarily equivalent to one of the type $A^\wedge$ – but where however the aspheric (= "contractible") objects form a generating family (which generalizes the condition $\mathcal{A} \simeq A^\wedge$, which means that the 0-connected projective elements of $\mathcal{A}$ form a generating family). I'll come back upon this notion later I guess – it is not the most urgent thing for the time being...

Examples of test categories.

34 I will now exploit the handy criterion (T 1) to (T 3) for test categories, for constructing lots of such. In all cases I have in mind at present, the verification of (T 1) and (T 3) is obvious, they are consequences indeed of the following stronger conditions:

(T 1') A has a final element e_A.

(T 3') There exists an element $I = I_A$ in A, and two sections e_0, e_1 of I over $e = e_A$

$$\delta_0, \delta_1 : e \to I,$$

such that the corresponding subobjects e_0, e_1 of I satisfy

$$e_0 \cap e_1 = \varnothing_{A^\wedge},$$

namely that for any $a \in \mathrm{Ob}A$, if $p_a : a \to e$ is the projection, we have

$$\delta_0 p_a \neq \delta_1 p_a.$$

[p. 48] Of course, the element I (playing the role of unit interval with endpoints e_0, e_1) is by no means unique, for instance it can be replaced by any cartesian power I^n ($n \geq 1$), provided it is in A – or in the case of $A = \mathbb{\Delta}$ we can take for I any $\mathbb{\Delta}_n$ ($n \geq 1$) and for δ_0, δ_1 any two *distinct* maps from $e = \mathbb{\Delta}_0$ to $\mathbb{\Delta}_n$, instead of the usual choice $\mathbb{\Delta}_1$ for I, and corresponding δ_0 and δ_1. This high degree of arbitrariness in the choice of I should be no surprise, this was already one striking feature in Quillen's theory of model categories.

There remains the asphericity condition (T 2) for the categories $A_{/a\times b}$, with $a, b \in \mathrm{Ob}A$, which is somewhat subtler. There is one very evident way though to ensure this, namely assuming

(T 2') If a, b are in A, so is $a \times b$, i.e., A is stable under binary products.

In other words, putting together (T 1') and (T 2'), we may take categories (essentially small) *stable under finite products*. When such a category satisfies the mild extra condition (T 3') above, it is a test category! This is already an impressive bunch of test categories. For instance, take any category C with finite products and for which there exist I, e_0, e_1 as in (T 3') – never mind whether C is essentially small, for instance any "non-empty" topos will do (taking for I the Lawvere element for instance, if no simpler choice comes to mind). Take any subcategory A (full or not) stable under finite products and containing I and Δ_0, Δ_1 and where the C-products are also A-products – for instance this is OK if A is full. Then A is a test category.

The simplest choice here, the smallest in any case, it to take the subcategory made up with all cartesian powers I^n $(n \geq 0)$. We may take the full subcategory made up with these elements, but instead we may take, still more economically, only the arrows

$$I^n \to I^m$$

where m components $I^n \to I$ are each, either a projection pr_i $(1 \leq i \leq n)$ or of the form $\delta_i(p_{I^n})$ $(i \in \{0, 1\})$. The category thus obtained, up to unique isomorphism, does not depend on the choice of (C, I, e_0, e_1), visibly – it is, in a sense, *the* smallest test category satisfying the stronger conditions (T 1') to (T 3'). Its elements can be visualized as "standard cubes": One convenient way to do so is to take $C = (\mathrm{Ord})$ (category of all ordered sets), $I = \Delta_1 = (\{0\} \to \{1\})$, e_0 and e_1 as usual, thus A can be interpreted as a subcategory of (Ord), but this embedding is not full (there are maps of I^2 to I in (Ord) which are not in A, i.e., do not "respect the cubical structure"). We may also take $C = (\mathrm{Spaces})$, $I =$ unit interval $\subset \mathbb{R}$, e_0 and e_1 defined by the endpoints 0 and 1, thus the elements of A are interpreted as the standard cubes I^n in $\mathbb{R}^n$; the allowable maps between them $I^n \to I^m$ are those whose components $I^n \to I$ are either constant with value $\in \{0, 1\}$, or one of the projections pr_i $(1 \leq i \leq n)$. We may denote this category of standard cubes by $\square$, but recall that the cubes here have symmetry operations, they give rise to the notion of "(fully) cubical complexes" $(K_n)_{n \geq 0}$, in contrast with what might be called "semicubical complexes" without symmetry operations on the K_n's, in analogy with the case of simplicial complexes, where there are likewise variants "full" and "semi". To make the distinction, we better call the corresponding test category (with symmetry operations, hence with more arrows) $\widetilde{\square}$ rather than $\square$, and accordingly for $\widetilde{\Delta}$, Δ. On closer inspection, it seems to me that even apart the symmetry operations, the maps between standard cubes allowed here are rather plethoric – thus we admit diagonal maps such as $I \to I \times I$, which is I guess highly unusual in the cubical game. It is forced upon us though if we insist on a test category stable under finite products, for easier checking of condition (T 2). [p. 49]

A less plethoric looking choice really is

$$A = \textit{non-empty} \text{ finite sets, with arbitrary maps}$$

or, equivalently, the full subcategory $\widetilde{\Delta}$ formed by the standard finite sets $\mathbb{N} \cap [0, n]$, which we may call $\widetilde{\Delta}_n$ in contrast to the Δ_n's (viewed as being endowed with their natural total order, and with correspondingly more restricted maps). This gives rise to the category $\widetilde{\triangle}$ of "(fully) simplicial complexes (or sets)". The elements of A, or of $\widetilde{\triangle}$, can be interpreted in the well-known way as *affine simplices*, and simplicial maps between these.

It is about time now to come to the categories $\triangle$ and $\square$ and check they are test categories, although they definitely do not satisfy (T 2'), thus we are left with checking the somewhat delicate condition (T 2). It is tempting to dismiss the question by saying that it is "well-known" that the categories

$$\triangle_{/\triangle_n \times \triangle_m} \quad (n, m \in \mathbb{N})$$

are aspheric – but this I feel would be kind of cheating. Maybe the very intuitive homotopy argument already used yesterday (pages 39–40) will do. In general terms, under the assumptions (T 1), (T 3), we found a sufficient criterion of asphericity for an element F of $A^\wedge$, which we may

[p. 50] want to apply to the case $F = a \times b$, with a and b in A.

Now let's say that $e_{A^\wedge}$ is a *deformation retract of* F (F any element in $A^\wedge$) if the identity map of F is homotopic (with respect to I) to a "constant" map of F into itself. It is purely formal, using the diagonal map $I \to I \times I$ in $A^\wedge$, that if (more generally) F_0 is a deformation retract of F, and G_0 a deformation retract of G, then $F_0 \times G_0$ is a deformation retract of $F \times G$. Thus, a *sufficient* condition for (T 2) to hold is the following "homotopy-test axiom":

Condition (T H): For any $a \in A$, $e = e_{A^\wedge}$ is a deformation retract of a with respect to (I, e_0, e_1), namely there exists

(1) a section $c_a : e \to a$ (hence a constant map $u_a = p_a c_a : a \to a$) and a homotopy $h_a : a \times I \to a$ from id_a to u_a, i.e., a map h_a such that

$$h_a(\mathrm{id}_a \times \delta_0) = \mathrm{id}_a, \quad h_a(\mathrm{id}_a \times \delta_1) = u_a.$$

We have to be cautious though, it occurs to me now, not to make a "vicious circle", as the homotopy argument used yesterday for proving asphericity of f makes use of the fact that $F \times I \to F$ is a weak equivalence. To check that $F \times I \to F$ is a weak equivalence, and even aspheric, we have seen though that it amounts to the same to prove $a \times I \to a$ is a weak equivalence for any a in A, i.e., $a \times I$ is aspheric for any a in A. This, then, was seen to be a consequence of the assumption that all elements $a \times b$ are aspheric (for a, b in A) – but this latter fact is now what we want to prove! Thus it can't be helped, we have to complement the condition (T H) above (which I will call (T H 1), by the extra condition

(2) For any a in A, $a \times I$ is aspheric,

which, in case I itself is in A, appears as just a particular case of (T 2), which now, it seems, we have to check directly some way or other.

19.3.

The notion of a modelizing topos. Need for revising the Čech-Verdier-Artin-Mazur construction.

35 I really feel I should not wait any longer with the digression on modelizing topoi that keeps creeping into my mind, and which I keep trying to dismiss as not to the point or not urgent or what not! At least this will fix some terminology, and put the notion of an elementary modelizer into perspective, in terms of a wider class of topoi.

A topos X is called *aspheric* if the canonical "map" from X to the "final topos" (corresponding to a one-point space, and whose category of sheaves is the category (Sets)) is aspheric. It is equivalent to say that X is 0-connected (namely non-empty, and not decomposable non-trivially into a direct sum of two topoi), and that for any constant sheaf of groups G on X, the cohomology group $\mathrm{H}^i(X,G)$ $(i \geq 1)$ is trivial whenever defined, namely for $i \geq 1$ if G commutative, and $i = 1$ if G is not supposed commutative. The 0-connectivity is readily translated into a corresponding property of the final element in the category of sheaves $\mathrm{Sh}(X) = \mathcal{A}$ on X. An element U of $\mathcal{A}$ is called *aspheric* if the induced topos X/U or $\mathcal{A}_{/U}$ is aspheric. An arrow $f : U' \to U$ in $\mathcal{A}$ is called a *weak equivalence* resp. *aspheric*, if the corresponding "map" or morphism of the induced topoi [p. 51]

$$X/U' \to X/U \quad \text{or} \quad \mathcal{A}_{/U'} \to \mathcal{A}_{/U}$$

has the corresponding property. Of course, in the notations it is often convenient simply to identify objects U of $\mathcal{A}$ with the induced topos, and accordingly for arrows. Clearly, if f is aspheric, it is a weak equivalence, more specifically, f is aspheric iff it is "universally a weak equivalence", namely iff for any base change $V \to U$ in $\mathcal{A}$, the corresponding map $g : V' = V \times_U U' \to V$ is a weak equivalence. Moreover, it is sufficient to check this property when V is a member of a given generating family of $\mathcal{A}$.

An interesting special case is the one when $\mathcal{A}$ admits a family of aspheric generators, or equivalently a generating full subcategory A whose objects are aspheric in $\mathcal{A}$; we will say in this case that the topos X (or $\mathcal{A}$) is "*locally aspheric*". This condition is satisfied for the most common topological spaces (it suffices that each point admit a fundamental system of contractible neighborhoods), as well as for the topoi $A^\wedge$ associated to (essentially small) categories A (for such a topos, A itself is such a generating full category made up with aspheric elements of $\mathcal{A}$). In the case of a locally aspheric topos endowed with a generating full subcategory A as above, a map $f : U' \to U$ is aspheric iff for any $a \in \mathrm{Ob} A$, and map $a \to U$, the object $U' \times_U a$ is aspheric (because for a map $g : V' \to V$ with V aspheric, g is a weak equivalence iff V' is aspheric, which is a particular case of the corresponding statement valid for any map of topoi).

Using this criterion, we get readily the following

Proposition 1. *Let $\mathcal{A}$ be a locally aspheric topos, A a generating full subcategory made up with aspheric elements. The following conditions on $\mathcal{A}$ are equivalent:*

a) *Any aspheric element of $\mathcal{A}$ is aspheric over e (the final object).*

b) *The product of any two aspheric elements of $\mathcal{A}$ is again aspheric.*

a'), b') *as a), b) but with elements restricted to be in* $\mathrm{Ob}\,A$*.*

These conditions, in case $\mathcal{A} = A\hat{}$ are nothing but (T 2) for A. *It*
[p. 52] *turns out they imply already* (T 1). More generally, let's call a topos *totally aspheric* if is locally aspheric, and satisfies more the equivalent conditions above. It then turns out:

Corollary. *Any totally aspheric topos is aspheric.*

This follows readily from the definition of asphericity, and the Čech computation of cohomology of X in terms of a family $(a_i)_{i\in I}$ covering e, with the a_i in A. As the mutual products and multiple products of the a_i's are all aspheric (a fortiori acyclic for any constant coefficients), the Čech calculation is valid (including of course in the case of non-commutative H^1) and yields the desired result.

The condition for a topos to be totally aspheric, in contrast to *local* asphericity, is an exceedingly strong one. If for instance the topos is defined by a topological space, which I'll denote by X, then it is seen immediately that any two non-empty open subsets of X have a non-empty intersection, in other words X is an *irreducible space* (i.e., not empty and not the union of two closed subsets distinct from X). It is well known on the other hand that any irreducible space is aspheric, and clearly any non-empty open subset of an irreducible space is again irreducible. Therefore:

Corollary 2. *A topological space is totally aspheric* (*i.e., the corresponding topos is t.a.*) *iff it is irreducible. In case X is Hausdorff, means also that X is a one-point space.*

Let's now translate (T 3) in the context of general topoi. We get:

Proposition 2. *Let X be a topos. Then the following two conditions are equivalent:*

a) *The "Lawvere element" L_X of $\mathcal{A} = \mathrm{Sh}(X)$ is aspheric over the final object $e_X = e$.*

b) *There exists an object I in $\mathcal{A}$ which is aspheric over e, and two sections δ_0, δ_1 of I (over e), such that $\mathrm{Ker}(\delta_0, \delta_1) = \varnothing$, i.e., such that $e_0 \cap e_1 = \varnothing$, where e_0, e_1 are the subobjects of I defined by δ_0, δ_1.*

I recall that the Lawvere element is the one which represents the functor

$$F \mapsto \text{set of all subobjects of } F$$

on $\mathcal{A}$, it is endowed with two sections over e, corresponding to the two "trivial" subobjects $\varnothing, e$ of e, and the kernel of this pair of sections is clearly $\varnothing$, thus a) $\Rightarrow$ b). Conversely, by the homotopy argument already
[p. 53] used (p. 39–40), we readily get that b) implies that the projection $L \to e$ is a weak equivalence. As the assumption b) is stable by localization,

this shows that for any U in $\mathcal{A}$, $L_U \to U$ is equally a weak equivalence, and hence L is aspheric over e.

In proposition 2 we did not make any assumption of the topos X being locally aspheric, let alone totally aspheric. The property of total asphericity, and the one of prop. 2 about the existence of a handy substitute for the unit interval, seem to be independent of each other. The case of topological spaces is instructive in this respect. Namely:

Corollary 1. *Assume X is a topological space, and X is* 0*-connected. Then the Lawvere element L_X in $\mathcal{A} = \mathrm{Sh}(X)$ is* 0*-connected, except exactly when X is irreducible, in which case L_X decomposes into two connected components.*

Indeed, L is disconnected iff there is a direct summand L' of L containing e_0 and $\neq L$, or equivalently, if for any open $U \subset X$, and $U \to L$, namely an open subset U_0 of U, we can associate a *direct summand* $U' = L'(U, U_0)$ of U containing U_0, functorially for variable U, and such that

$$L'(U, \emptyset) = \emptyset.$$

The functoriality in U means that for V open in U and $V_0 = U_0 \cap V$, we get

$$(*) \qquad L'(V, V_0) = V \cap L'(U, U_0).$$

This implies that $L'(U, U_0)$ is known when we know $L'(X, U_0)$, namely

$$L'(U, U_0) = U \cap L'(X, U_0).$$

As $L'(X, U_0)$ is a direct summand of X containing U_0, and X is connected, we see that

$$L'(X, U_0) = X \text{ if } U_0 \neq \emptyset, \quad L'(X, \emptyset) = \emptyset;$$

and hence

$$(**) \qquad L'(U, U_0) = U \text{ if } U_0 \neq \emptyset, \quad L'(U, U_0) = \emptyset \text{ if } U_0 = \emptyset,$$

thus the association $(U, U_0) \mapsto L'(U, U_0)$ is uniquely determined, and it remains to see whether the association (**) is indeed functorial, i.e., satisfies (*) for any open $V \subset U$, with $V_0 = V \cap U_0$. It is OK if $V_0 \neq \emptyset$, or if $V_0 = \emptyset$ and $U_0 = \emptyset$. If $V_0 = \emptyset$ and $U_0 \neq \emptyset$, it is OK iff V is empty, in other words, if both open subsets U_0 and V of U are non-empty, so must be their intersection V_0. But this means that X is irreducible.

Corollary 2. *A topological space X cannot be totally aspheric (i.e., irreducible) and satisfy the condition of prop.* 2.

Because for a connected topos, this latter condition implies that L_X is equally connected, which contradicts corollary 1.

Thus, surely local asphericity for a topos *does not* imply the condition of prop. 2 – the simplest counterexample is the final topos, corresponding to one-point spaces. But I don't expect either that condition of prop. 2, [p. 54] even for a locally aspheric topos X, and even granting X is aspheric moreover, implies that X is totally aspheric. The positive result in cor. 1 gives a hint that the condition of prop. 2 may be satisfied for locally aspheric topological spaces which are sufficiently far away from those awful non-separated spaces (including the irreducible ones) of the algebraic geometry freaks – possibly even provided only X is locally Hausdorff.

Now the conjunction of the conditions of prop. 1 and 2, namely total asphericity plus existence of a monomorphism of $e \amalg e$ into an aspheric element, seems an extraordinarily strong assumption – so strong indeed that no topological space whatever can satisfy it! I feel like calling a topos satisfying these conditions a "*modelizing topos*". For a topos of the type $A\hat{}$, this just means it is an "elementary modelizer", i.e., A is a test category, which are notions which, I feel, are about to be pretty well understood. I have not such feeling yet for the more general situation though. For instance, there is another kind of property of a topos, which from the very start of our model story, one would have thought of pinpointing by the name of a "modelizing topos". Namely, when taking $W \subset \mathrm{Fl}\,\mathcal{A}$ to be weak equivalences in the sense defined earlier, we demand that $(\mathcal{A}, W)$ should be a modelizer in the general sense (cf. p. 31), with some exactness reinforcement (cf. p. 45), namely that $w^{-1}\mathcal{A}$ should be equivalent to (Hot), and W saturated, namely equal to the set of maps made invertible by the localization functor – and moreover I guess that this functor should commute with (at least finite) sums and products. Of course, we would even expect a little more, namely that the "canonical functor"

$$\mathcal{A} \to (\mathrm{Hot})$$

can be described, by associating to every $F \in \mathrm{Ob}\,\mathcal{A}$ the (pro-)homotopy type associated to it by the Verdier-Artin-Mazur process. One feels there is a pretty juicy bunch of intimately related properties for a topos, all connected with the "homotopy model yoga", and which one would like to know about.

This calls to my mind too the question of understanding the Verdier-Artin-Mazur construction in the present setting, where homotopy types are thought of in terms of categories as preferential models, rather than
[p. 55] semi-simplicial sets. This will be connected with the question, which has been intriguing me lately, of understanding the "structure" of an arbitrary morphism of topoi

$$X \to \mathrm{Top}(B),$$

where $\mathrm{Top}(B)$ is the topos associated to an arbitrary (essentially small) category B – a situation, it seems, which generalizes the situation of a "fibered topos" over the indexing category B, in the sense of SGA 4 IV. The construction of Verdier in SGA 4 V (appendix) corresponds apparently to the case of categories B of the type $\Delta_{/F}$ with $F \in \mathrm{Ob}\,\Delta\hat{}$, as brought near by the standard Čech procedure. When however the topos X is already pretty near a topos of the type $\mathrm{Top}(B)$, for instance if it is such a topos, to describe its homotopy type in terms of a huge inverse system of semi-simplicial objects of $B\hat{}$, rather than just take B and keep it as it is, seems technically somewhat prohibitive, at least in the present set-up (and with a distance of twenty years!).

36 Now back to test categories, and more specifically to her majesty Δ – we still have to check it is a test category indeed! The method suggested last Tuesday (p. 50) does work indeed, without any reference to "well-known" facts from the semi-simplicial theory. To check (T H 1), we take of course $I = \Delta_1$ and e_0, e_1 as usual, we then have for every $n \in \mathbb{N}$ to construct a homotopy

Characterization of a particular type of test functors with values in (Cat).

$$h_n : \Delta_n \times \Delta_1 \to \Delta_n,$$

where the product and the arrow can be interpreted as one in (Cat) or even in (Ord) of course (of which Δ is a full subcategory), and there is a unique such h if we take for u_a the "constant" endomorphism of Δ_n whose value is the *last* element of Δ_n. There remains only (T H 2) – namely to prove that the objects $\Delta_n \times \Delta_1$ in Δ are aspheric ($n \geq 0$). For this we cover $\Delta'_n = \Delta_n \times \Delta_1$ by maximal representable subobjects, namely maximal flags of the ordered product set (these are I guess what are called the "shuffles" in semi-simplicial algebra). It then turns out that Δ'_n can be obtained by successive gluing of flags, the intersection of each flag we add with the sup or union of the preceding ones being just a subflag, hence representable and aspheric. Thus we only have to make use of the Mayer-Vietoris type easy lemma:

Lemma. *Let U, V be two subobjects of the final object of a topos, assume U, V and $U \cap V$ aspheric, then so is e, i.e., so is the topos.*

This finishes the proof of Δ being a test category. Of course, one cannot help thinking of the asphericity of *all* the products $\Delta_m \times \Delta_n$ as just meaning asphericity of the product ordered set (namely of the corresponding category), which follows for instance from the fact that it has a final object. But we should beware the vicious circle, as we implicitly make the assumption that for an object C of (Cat) (at least an object C such as $\Delta_m \times \Delta_n$, if C is aspheric, then so is the corresponding element $\alpha(C)$ in $\Delta\hat{}$: [p. 56]

$$\alpha(C) : \Delta_n \mapsto \operatorname{Hom}(\Delta_n, C),$$

namely the *nerve* of C. We expect of course something stronger to hold, namely that for any C, $\alpha(C)$ has the same homotopy type as C – in fact that we have a canonical isomorphism between the images of both in (Hot). This relationship has still to be established, as well as the similar statement for $\beta : (\mathrm{Cat}) \to \square\hat{}$, and accordingly with $\tilde{\Delta}$ and $\tilde{\square}$. For all four categories (where the simplicial cases are known to be test categories already), we have, together with the category A (the would-be test category), a functor

$$A \to (\mathrm{Cat}), \quad \text{say } a \mapsto \Delta_a$$

(notation inspired by the case $A = \triangle$), hence a functor

$$(\mathrm{Cat}) \xrightarrow{\alpha} A^\wedge, \quad \alpha(C) = (a \mapsto \mathrm{Hom}_{(\mathrm{Cat})}(\triangle_a, C)),$$

and the question is to compare the homotopy types of C and $\alpha(C)$, the latter being defined of course as the homotopy type of $A_{/\alpha(C)} \simeq A_{/C}$, the category of all pairs (a, p) with $a \in \mathrm{Ob}A$ and $p : \triangle_a \to C$, arrows between pairs corresponding to *strictly* commutative diagrams in (Cat)

$$\begin{tikzcd} \triangle_{a'} \arrow[rr, "\triangle_f"] \arrow[dr, "p'"'] & & \triangle_a \arrow[dl, "p"] \\ & C & \end{tikzcd} .$$

Proposition. *Let the data be as just said, assume moreover that the categories $\triangle_a$ ($a \in \mathrm{Ob}A$) have final elements, that A is aspheric and that $\alpha(\triangle_1)$ is aspheric over the final element of $A^\wedge$ (which is automatic if A is a test category, $A \to$ (Cat) is fully faithful and $\triangle_1$ belongs to its essential image). Then we can find for every C in (Cat) a "map" in (Cat) (i.e., a functor)*

(*) $$\varphi : A_{/C} \to C,$$

functorial in C for variable C, and this map being aspheric, and a fortiori a weak equivalence (hence induces an isomorphism between the homotopy types).

This implies that the compositum $C \mapsto A_{/C}$:

$$(\mathrm{Cat}) \to A^\wedge \xrightarrow{i_A} (\mathrm{Cat})$$

transforms weak equivalences into weak equivalences, and that the functor deduced from it by passage to the localized categories $W^{-1}(\mathrm{Cat}) =$
[p. 57] (Hot) is isomorphic to the identity functor of (Hot).

To define a functor (*), functorial in C for variable C, we only have to choose a final element e_a in each $\triangle_a$. For an element (a, p) of $A_{/C}$, we define

$$\varphi(a, p) = p(e_a),$$

with evident extension to arrows of $A_{/C}$ (NB Here it is important that the e_a be *final* elements of the $\triangle_a$, initial elements for a change wouldn't do at all!), thus we get a functor φ, and functoriality with respect to C is clear. Using the standard criterion of asphericity of a functor φ we have to check that the categories

$$(A_{/C})_{/x}$$

for $x \in \mathrm{Ob}\,C$ are aspheric, but one checks at once that the category above is canonically isomorphic to $A_{/C'}$, where $C' = C_{/x}$. Thus asphericity of φ for arbitrary C is equivalent to asphericity of $A_{/C}$ when C has a final element, or equivalently, to asphericity of the element $\alpha(C)$ in $A^\wedge$ for such C. Now let e_C be the final element of C, and consider the unique homotopy

$$h : \Delta_1 \times C \to C$$

between id_C and the constant functor from C to C with value e_C. Applying the functor α, we get a corresponding homotopy in $A^\wedge$

$$I \times \alpha(C) \to \alpha(C), \quad \text{where } I = \alpha(\Delta_1),$$

between the identity map of $F = \alpha(C)$ and a "constant" map $F \to F$. As we assume that I is aspherical over the final element of $A^\wedge$, hence $I \times F \to F$ is a weak equivalence, it follows that $F \to e$ is a weak equivalence, hence F is aspheric as e is supposed to be aspheric, which was to be proved.

If we apply this proposition to $A = \Delta$ and $n \mapsto \Delta_n$, we are still reduced to check just (T H 2) (which we did above looking at shuffles), whereas (T H 1) appears superfluous now – we get asphericity of the elements $\Delta_m \times \Delta_n$ in $A^\wedge$ as a consequence (but the homotopy argument used in the prop. is essentially the same as the one used for proving (T H 1)). This proposition applies equally to the case when $A = \widetilde{\Delta}$ (category of non-ordered simplices $\widetilde{\Delta}_n$), here the natural functor

$$i : \widetilde{\Delta} \to (\mathrm{Cat}), \quad a \mapsto \mathfrak{P}^*(a)$$

is obtained by associated to any finite set a the combinatorial simplex it defines, embodied by the ordered set $\mathfrak{P}^*(a)$ of its facets of all dimensions, which can be identified with the ordered set of all non-empty subsets of the finite set a. As we know already $\widetilde{\Delta}$ is a test category, all that remains to be done is to check in this case that $I = \alpha(\Delta_1)$ is aspheric (hence aspheric over the final object of $A^\wedge$). Now this follows again from a homotopy argument involving the unit segment substitute in $A^\wedge$. [p. 58]
We could formalize it as follows:

Corollary. *In the proposition above, the condition that $\alpha(\Delta_1)$ be aspheric over $e_{A^\wedge}$ is a consequence of the following assumptions:*

a) *The condition* (T 3) *is valid in A, with the stronger assumption that I is aspheric over the final object e of $A^\wedge$, and moreover:*

b) *for every $a \in \mathrm{Ob}\,A$ and $u : a \to I$, let $a_u = u^{-1}(e_0)$ (subobject of a in $A^\wedge$), $\Delta_{a_u} = \varinjlim_{b \text{ in } A_{/a_u}} \Delta_b$, and $\Delta_{a,u} = \mathrm{Im}(\Delta_{a_u} \to \Delta_a)$. We assume that* $\mathrm{Ob}\,\Delta_{a,u}$ *is a* crible *in Δ_a, say C_u.*

Indeed, interpreting $\alpha(\Delta_1)$ as the functor $a \mapsto \mathrm{Crib}\,\Delta_a$, we define a homotopy

$$h : I \times F \to F \quad (\text{where } F = \alpha(\Delta_1))$$

from the identity id_F to the constant map $c : F \to F$ associating to every crible in some Δ_a the empty crible in the same, by taking for every a in A the map

$$(u, C_0) \mapsto C_u \circ C_0.$$

It follows that $F \to e$ is a weak equivalence, and the same argument in any induced category $A_{/a}$ shows that this is universally so, i.e., $F \to e$ is aspheric, O.K.

In the case above, $A = \widetilde{\triangle}$, we get indeed that for $u : a \to I$, a_u is empty in A and $\triangle_{a_u} \to \triangle_a$ is a full embedding, turning $\triangle_{a_u}$ into a crible in $\triangle_a$, thus b) is satisfied (with the usual choice $I = \widetilde{\triangle}_1$ of course).

21.3.

37 *The "asphericity story" told anew – the "key result".*

By the end of the notes two days ago, there was the pretty strong impression of repeating the same argument all over again, in very similar situations. When I tried to pin down this feeling, the first thing that occurred to me was that the homotopy equivalence I was after $A_{/C} \to C$ (in the proposition of p. 56), when concerned with a general functor $A \to (\mathrm{Cat})$, and the criterion obtained, was applicable to the situation I was concerned with at the very start when defined test categories, namely when looking at the *canonical* functor $A \to (\mathrm{Cat})$ given by $a \mapsto A_{/a}$; and that this gave a criterion in this case for A to be a test category in the wider sense, which I had overlooked when peeling out the characterization of test categories (cf. theorem of p. 46).Finally it becomes clear that it is about time to recast from scratch the asphericity story, and (by one more repetition) tell it anew in a way stripped from [p. 59] its repetitive features!

On a more technical level, I got aware too that the last corollary stated was slightly incorrect, because it is by no means clear that the crible C_u constructed there is functorial in u, i.e., corresponds to a map $I \to \alpha(\triangle_1)$, this in fact has still to be assumed (and turns out to be satisfied in all cases which turned up naturally so far and which I looked up). But let's now "retell the story"!

One key notion visibly in the homotopy technique used, and which needs a name in the long last, is the notion of a *homotopy interval* ("segment homotopique" in French). To be really outspoken about the very formal nature of this notion and the way it will be used, let's develop it in any category $\mathcal{A}$ endowed with a subset $W \subset \mathrm{Fl}(\mathcal{A})$ of the set of arrows of $\mathcal{A}$, and satisfying the usual conditions:

a) W contains all isomorphisms of $\mathcal{A}$,

b) for two composable arrows u, v in $\mathcal{A}$, if two among u, v, vu are in W, so is the third, and

c) if $i : F_0 \to F$, and $r : F \to F_0$ is a left inverse (i.e., a retraction), and if $p = ir : F \to F$ is in W, so is r (and hence i too).

The condition c) here is the ingredient slightly stronger than what we used to call "mild saturation property" of W, meaning a) and b).

We'll call *homotopy interval in* $\mathcal{A}$ (with respect to the notion of "weak equivalence" W, which will generally be implicit and specified by context) a triple (I, e_0, e_1), where I is an object of $\mathcal{A}$, e_0 and e_1 two subobjects, such that the following conditions a) to c) hold:

a) e_0 and e_1 are final objects of $\mathcal{A}$, which implies that $\mathcal{A}$ has a final object, unique up to unique isomorphism, which we'll denote by $e_{\mathcal{A}}$ or simply e, so that the data e_0, e_1 in I are equivalent to giving two sections

$$\delta_0, \delta_1 : e \to I$$

of I over e. Note that $e_0 \cap e_1 = \mathrm{Ker}(\delta_0, \delta_1)$.

b) $e_0 \cap e_1 = \varnothing_{\mathcal{A}}$ (a strict initial element of $\mathcal{A}$),

c) $I \to e$ is "universally in W" or, as we'll say, is W-*aspheric* or simply aspheric,

which just means here that $I \to e$ is "squarable" and that for any F in $\mathrm{Ob}\,\mathcal{A}$, $F \times I \to F$ is in W, i.e., is a "weak equivalence".

It is clear that if (I, e_0, e_1) is a homotopy interval in $\mathcal{A}$, then for any $F \in \mathrm{Ob}\,\mathcal{A}$, the "induced interval" in $\mathcal{A}_{/F}$, namely $(I \times F, e_0 \times F, e_1 \times F)$ is a homotopy interval in $\mathcal{A}_{/F}$. [p. 60]

We now get the (essentially trivial)

Homotopy lemma. *Let $h : F \times I \to F$ be a "homotopy" with respect to (I, e_0, e_1) of id_F with a constant map $c = ir : F \to F$, where $r : F \to e$ and $i : e \to F$ is a section of F over e. Then $F \to e$ is W-aspheric.*

In the proof of this lemma, we make use of a)b)c) for W, but only a)c) for (I, e_0, e_1), namely we don't even need $e_0 \cap e_1 = \varnothing_{\mathcal{A}}$. This is still the case for the proof of the

Comparison lemma for homotopy intervals. *Let L be an object of $\mathcal{A}$, squarable over e and endowed with a composition law $x \wedge y$, let $\delta_i^L : e \to L$ be two sections of L $(i \in \{0, 1\})$, which are respectively a left unit and zero element for the multiplication, namely the corresponding elements e_0^a, e_1^a in any* $\mathrm{Hom}(a, L)$ *satisfy*

$$e_0^a \wedge x = x, \quad e_1^a \wedge x = e_1^a$$

for any x in $\mathrm{Hom}(a, L)$. *Assume moreover we got a homotopy interval (I, e_0, e_1) and a "map of intervals"*

$$\varphi : I \to L$$

(*in the sense: compatible with endpoints*). *Then* id_L *is homotopic* (*with respect to* (I, e_0, e_1)) *to the constant map $c_L : L \to L$ defined by δ_1^L, and hence by the previous homotopy lemma, $L \to e$ is W-aspheric and therefore L endowed with e_0^L, e_1^L defined by δ_0^L, δ_1^L is itself a homotopy interval in $\mathcal{A}$ (provided, at least, we know that $e_0^L \cap e_1^L = \varnothing$).*

The homotopy h is simply given (for $u : a \to I$, $x : a \to L$) by

$$h(x, u) = \varphi(u) \wedge x;$$

when u factors through e_0, then $\varphi(u) = e_0^a$ and hence $h(x, u) = x$; if it factors through e_1, then $\varphi(u) = e_1^a$ and we get $h(x, u) = e_1^a$, qed.

Corollary. *Assume in $\mathcal{A}$ (endowed with W) finite inverse limits exist (i.e., final object and fibered products exist), and moreover that the presheaf on $\mathcal{A}$*

$$F \mapsto \textit{set of all subobjects of } F$$

is representable by an element L of $\mathcal{A}$ (the "Lawvere element"). Assume moreover $\mathcal{A}$ has a strict initial element $\varnothing_{\mathcal{A}}$, i.e., an initial element and that any map $a \to \varnothing_{\mathcal{A}}$ is an isomorphism. Consider the two sections of L over e, δ_0^L and δ_1^L, corresponding to the full and to the empty subobject of e, so that we get visibly $e_0^L \cap e_0^L = \varnothing$. Then, for a homotopy interval to exist in $\mathcal{A}$, it is necessary and sufficient that L be W-aspheric over e, i.e., that (L, e_0^L, e_1^L) be a homotopy interval.

By the comparison lemma, it is enough to show that for any homotopy
[p. 61] interval (I, e_0, e_1) in $\mathcal{A}$, there exists a morphism of "intervals" from I into L, using the fact of course that the intersection law on L is a composition law admitting δ_0^L, δ_1^L respectively as unit and as zero element (still using the fact that the initial object is strict). But the subobject e_0 of I, by definition of L, can be viewed as the inverse image of e_0^L by a uniquely defined map $I \to L$. The induced map $e_1 \to L$ corresponds to the induced subobject $e_0 \cap e_1$ of e_1 which by assumption is $\varnothing_{\mathcal{A}}$, and hence $e_1 \to L$ factors through e_1^L, qed.

Of course, the case for the time being which mainly interests us is the one when $\mathcal{A}$ is a topos (more specifically even, a topos equivalent to a category $A^\wedge$, with A a small category), in which case it is tacitly understood that W is the set of weak equivalences in the usual sense. We now come to the key result:

Theorem. *Let A be a small category, and*

$$i : A \to (\mathrm{Cat})$$

a functor, hence a functor

$$i^* : (\mathrm{Cat}) \to A^\wedge, \quad C \mapsto (a \mapsto \mathrm{Hom}(i(a), C)).$$

Consider the canonical functor $i_A : A^\wedge \to (\mathrm{Cat})$, $F \mapsto A_{/F}$, and the compositum

$$(\mathrm{Cat}) \xrightarrow{i^*} A^\wedge \xrightarrow{i_A} (\mathrm{Cat}), \quad C \mapsto A_{/i^*(C)} \overset{\mathrm{def}}{=} A_{/C}.$$

Assume that for any $a \in \mathrm{Ob} A$, $i(a) \in \mathrm{Ob}(\mathrm{Cat})$ has a final element e_a (but we don't demand that for $u : a \to b$, $i(u) : i(a) \to i(b)$ transforms e_a into e_b nor even into a final element of $i(b)$). Consider the canonical functor

$$(*) \qquad A_{/C} \to C, \quad (a, p : i(a) \to C) \mapsto p(e_a),$$

which is functorial in C, and hence defines a map between functors from (Cat) to (Cat):

$$(**) \qquad i_A i^* \to \mathrm{id}_{(\mathrm{Cat})}.$$

a) *The following conditions are equivalent:*

(i) *For any C in* (Cat)*,* (*) *is aspheric.*

(ii) *For any C in* (Cat)*,* (*) *is a weak equivalence, i.e., $i_A i^*$ transforms weak equivalences into weak equivalences, and* (**) *induces an isomorphism of the corresponding functor* (Hot) $\to$ (Hot) *with the identity functor.*

(iii) *The functor $i_A i^*$ transforms weak equivalences into weak equivalences, and the induced functor* (Hot) $\to$ (Hot) *transforms every object into an isomorphic one, i.e., for any C in* (Cat)*, $A_{/C}$ is isomorphic to C in* (Hot)*.*

(iv) *For any C with a final element, $A_{/C}$ is aspheric.*

b) *The following conditions are equivalent,* and they imply the conditions in a) provided A is aspheric*:* [p. 62]

(i) *For any C in* (Cat) *the functor*

$$(***) \qquad A_{/C} \to A \times C \quad \textit{deduced from } (*) \textit{ and } A_{/C} \xrightarrow{\text{can}} A$$

is aspheric.

(ii) *For any C in* (Cat) *with final element and any a in A, $a \times i^*(C)$ is an aspheric element in $A^\wedge$, i.e., for any such C, $i^*(C)$ is aspheric over the final element e of $A^\wedge$.*

(iii) *The element $i^*(\Delta_1)$ of $A^\wedge$ is aspheric over the final object e.*

Remark. Of course the conditions in a) imply that A is aspheric (take C to be the final category), and hence by an easy lemma (of below, §40) we get that $A \times C \to C$ is a weak equivalence (and even aspheric), and hence $A_{/C} \to A \times C$ is a weak equivalence (because its compositum with the weak equivalence $A \times C \to C$ is a weak equivalence by assumption). It is unlikely however that a) implies the conditions of b), namely asphericity (not merely weak equivalence) of (***) in b(i). But the opposite implication, namely b(i) + asphericity of A implies a(i), is trivial, because $A \times C \to C$ is aspheric.

Proof of theorem. We stated a) and b) in a way to get a visibly decreasing cascade of conditions; and moreover that the weakest in a) implies the strongest, or that b(ii) implies b(i), is an immediate consequence of the standard asphericity criterion for a functor between categories (p. 38). The only point which is a little less formal is that b(iii) implies b(ii). But using the final object in C, we get a (unique) homotopy in (Cat) (relative to Δ_1)

$$\Delta_1 \times C \to C,$$

between the identity map of C and the constant map with value e_C – hence by applying i^*, a homotopy relative to $i^*(\Delta_1)$ (viewed as an "interval" by taking as "endpoints" the arrows deduced from $\delta_0, \delta_1 : e_{(\mathrm{Cat})} = \Delta_0 \rightrightarrows \Delta_1$ by applying i^*), between the identity map of $i^*(C)$ and a constant map of $i^*(C)$, which by the homotopy lemma implies that $i^*(C)$ itself is aspheric over $e_{A^\wedge}$, qed.

Remark 2. The Condition b(iii) can be stated by saying that $i^*(\mathbb{\Delta}_1)$ is a homotopy interval in $A^\wedge$. All which needs to be checked for this, is that condition b) for homotopy intervals (p. 59) namely $e_0 \cap e_1 = \varnothing$ is satisfied in $A^\wedge$, but this follows from the corresponding property for $\mathbb{\Delta}_1$ in (Cat) (as the functor i^* is left exact) and from the fact that i^* transforms initial element $\varnothing_{(\mathrm{Cat})}$ into initial element $\varnothing_{A^\wedge}$. This last fact is equivalent to $i(a) \neq \varnothing$ for any a in A, which is true as $i(a)$ has a final element.

[p. 63] 22.3.

38 *Asphericity story retold (cont'd): generalized nerve functors.*

Let's get back to the "asphericity story retold" – I had to stop yesterday just in the middle, as it was getting prohibitively late.

I want to comment a little about the "key result" just stated and proved. The main point of this result, forgetting the game of givings heaps of equivalent formulations of two kinds of properties, is that the extremely simple condition b(iii), namely that $i^*(\mathbb{\Delta}_1)$ is aspheric over the unit element $e_{A^\wedge}$ of $A^\wedge$, plus asphericity of the latter, ensure already the conditions in a), which can be viewed (among others) as just stating that the compositum

$$i_A i^* : (\mathrm{Cat}) \to A^\wedge \to (\mathrm{Cat})$$

from (Cat) to (Cat) induces an autoequivalence of the localized category (Hot), or (what amounts still to the same) a functor (Hot) $\to$ (Hot) isomorphic to the identity functor. (NB that two do indeed amount to the same follows at once from the implication (iv) $\Rightarrow$ (ii) in a).) It is interesting to note that both properties, the stronger one that $i^*(\mathbb{\Delta}_1)$ is aspheric over $e_{A^\wedge}$, and the weaker one in terms of properties of the compositum $i_A i^*$, make a sense without any reference to the extra assumption that the categories $i(a)$ have a final element each, nor to the corresponding map (*) $A_{/C} \to C$ (functorial in C).

This suggests that there should exist a more general statement than in the theorem, without making the assumption about final objects in the categories $i(a)$, and without the possibility of a *direct* comparison of $A_{/C}$ and C through a functor between them. Indeed I have an idea of a statement in this respect, however for the time being it seems that the theorem as stated is sufficiently general for handling the situations I have in mind.

Applying the theorem to the *canonical* functor i

$$i_0 : a \mapsto A_{/a} : A \to (\mathrm{Cat}),$$

whose canonical extension to $A^\wedge$ (as a functor $A^\wedge \to (\mathrm{Cat})$ commuting to direct limits) is the functor

$$i_A : F \mapsto A_{/F} : A^\wedge \to (\mathrm{Cat}),$$

giving rise to the right adjoint

$$i_0^* = j_A : (\mathrm{Cat}) \to A^\wedge,$$

the condition (i) in a) is nothing but the familiar condition

$$i_A j_A(C) \to C \quad \text{a weak equivalence for any } C \text{ in (Cat)},$$

which we had used for our first (or rather, second already!) definition of so-called "test categories". Later on we considerably strengthened this condition – we now call them "test categories in the wide sense". On the other hand, as we already noticed before, here $i_0^*(\Delta_1) = j_A(\Delta_1)$ *is nothing but the Lawvere element L_A of $A\hat{}$*. Thus the main content of the theorem in the present special case can be formulated thus: [p. 64]

Corollary 1. *Assume the Lawvere element L_A in $A\hat{}$ is aspheric over the final object $e_{A\hat{}}$ of $A\hat{}$, and that moreover the latter be aspheric, i.e., A aspheric. Then A is a test category in the wide sense, namely for any C in* (Cat)*, the canonical functor $i_A j_A(C) \to C$ is a weak equivalence* (*and even aspheric-- see prop. on p. 38, and also p. 35, for equivalent formulations)*.

Thus we did get after all a "handy criterion" sufficient to ensure this basic test-property, which looks a lot less strong a priori than the condition (T 2) of total asphericity of $A\hat{}$.

25.3. or rather, than the set of conditions (T 1) to (T 3)

But let's now come back to the more general situation of the theorem, with a functor

$$i : A \to (\text{Cat})$$

subjected only to the mild condition that the categories $i(a)$ (for a in A) have final objects. Assume condition b(iii) to be satisfied, namely that $i^*(\Delta_1)$ is aspheric over $e_{A\hat{}}$, and therefore it is a homotopy interval in $A\hat{}$. Let again L_A be the Lawvere element in $A\hat{}$, we define (independently of any assumption on i) a morphism of "intervals" in $A\hat{}$, compatible even with the natural composition laws (by intersection) on both members

$$\varphi : J = i^*(\Delta_1) \to L_A.$$

For this we remember that for a in J, we get

$$J(a) \simeq \operatorname{Crib} i(a) \hookrightarrow \text{set of all subobjects of } i(a) \text{ in (Cat)}$$

(this bijection and the inclusion being functorial in a), thus if C is in $J(a)$, i.e., a crible in $i(a)$, we associate to this

$$\varphi(C) = \begin{array}{l}\text{subobject of } a \text{ in } A\hat{}, \text{ corresponding to the}\\ \text{crible in } A_{/a} \text{ of all } b/a \text{ such that } i(b) \to i(a)\\ \text{factors through the crible } C \subset i(a);\end{array}$$

it is immediate that the map $\varphi_a : J(a) \to L(a)$ thus obtained is functorial in a for variable a, hence a map $\varphi : J \to L$, and it is immediately checked too that this is "compatible with endpoints" – namely when C is full, respectively empty, then so is $\varphi(C)$ (for the "empty" case, this comes from the fact that the categories $i(a)$ are non-empty). Applying now the comparison lemma for homotopy intervals (p. 60), we get the following

Corollary 2. *Under the general conditions of the theorem, and assuming moreover that $i^*(\mathbb{\Delta}_1)$ is aspheric over $e_{A^\wedge}$ (condition* b(iii)*), it follows that the Lawvere element L_A of $A^\wedge$ is aspheric over $e_{A^\wedge}$. Assume moreover*
[p. 65] *that $e_{A^\wedge}$ is aspheric, i.e., A aspheric. Then we get:*

a) *The category A is a test category in the wide sense* (*cf. cor.* 1).

b) *Both functors $i^* : (\mathrm{Cat}) \to A^\wedge$ and $i_A : A^\wedge \to (\mathrm{Cat})$ are "modelizing", namely the set of weak equivalences in the source category is the inverse image of the corresponding set of arrows in the target category, and the functor induced on the localizations with respect to weak equivalences is an equivalence of categories.*

c) *Let W* (*resp. W_A*) *be the set of weak equivalences in* (Cat) (*resp. in $A^\wedge$*). *Then the functor*

$$(\mathrm{Hot}) \overset{\mathrm{def}}{=} W^{-1}(\mathrm{Cat}) \to W_A^{-1}A^\wedge$$

induced by i^* is canonically isomorphic to the quasi-inverse of the functor in opposite direction induced by i_A, or equivalently, can. isomorphic to the functor in the same direction induced by $i_0^* = j_A$: $(\mathrm{Cat}) \to A^\wedge$ (*cf. again cor.* 1 *above for the notations*).

Of course a) follows from cor. 1, and implies that i_A has the properties stated in b). That the analogous properties hold for i^* too, and the rest of the statement, i.e., c), follows formally, using the fact that the compositum $i_A i^*$ is canonically isomorphic to the identity functor once we pass to the localized category (Hot) (using the theorem, b(iii) $\Rightarrow$ a)).

This corollary shows that, up to canonical isomorphism, the functor

$$(\mathrm{Hot}) \to W_A^{-1}A^\wedge \quad \text{induced by } i^* : (\mathrm{Cat}) \to A^\wedge$$

does not depend on the choice of the functor $i : A \to$ (Cat), provided only this functor satisfies the two conditions that it takes its values in the full subcategory of (Cat) of all categories with final objects, and that moreover $i^*(\mathbb{\Delta}_1)$ be aspheric over the final object of $A^\wedge$ (plus of course the condition of asphericity on A itself). There *is* of course always the *canonical* choice of a functor $i : A \to (\mathrm{Cat})$, namely $i_A : a \mapsto A_{/a}$ which (because of its canonicity) looks as the best choice theoretically – and it was the first one indeed we investigated into. But in practical terms, the categories $A_{/a}$ are (in the concrete cases one might think of) comparatively big (for instance, infinite) and the corresponding functor $j_A = i_A^*$ gives comparatively clumsy "models" in $A^\wedge$ for describing the homotopy types of given "models" in (Cat), whereas we can get away with considerably neater models in $A^\wedge$, using a functor i giving rise to categories $i(a)$ which are a lot easier to compute with (for instance, finite categories of very specific type). The most commonly used is of course the *nerve functor* i^*, corresponding to the standard embedding of $A = \mathbb{\Delta}$ into (Cat) – and in the general case of the theorem above, complemented by cor. 2, *the functor i^* should be viewed as a generalized*
[p. 66] *nerve functor*.

To be completely happy, we still need a down-to-earth sufficient criterion to ensure that $i^*(\mathbb{\Delta}_1)$ is aspheric over $e_{A^\wedge}$, in the spirit of the somewhat awkward corollary on p. 58. The following seems quite adequate for all cases I have in mind at the present moment:

Corollary 3. *Under the general conditions of the theorem on A and $i : A \to (\mathrm{Cat})$, assume moreover we got a homotopy interval (I, δ_0, δ_1) in $A^\wedge$, that A has a final object e (which is therefore also a final object of $A^\wedge$ so we may view δ_0, δ_1 as maps $e \rightrightarrows I$, i.e., elements in $I(e)$), and let $i_! : A^\wedge \to (\mathrm{Cat})$ be the canonical extension of i to $A^\wedge$ (commuting with direct limits). Assume moreover $i(e) = \mathbb{\Delta}_0$ (the final object in (Cat)), and that we can find a map in (Cat)*

$$i_!(I) \to \mathbb{\Delta}_1$$

compatible with δ_0, δ_1, i.e., whose compositae with $i_!(\delta_n) : \mathbb{\Delta}_0 \to \mathbb{\Delta}_1$ for $n \in \{0,1\}$ are the two standard maps $\boldsymbol{\delta}_0, \boldsymbol{\delta}_1$ from $\mathbb{\Delta}_0$ to $\mathbb{\Delta}_1$. Then the condition b(iii) *of the theorem holds, namely $i^*(\mathbb{\Delta}_1)$ is aspheric over e.*

Indeed, to give a map $i_!(I) \to \mathbb{\Delta}_1$ in (Cat) amounts to the same as giving a map $I \to i^*(\mathbb{\Delta}_1)$ in $A^\wedge$ (namely $i_!$ and i^* are adjoint), moreover the extra condition involving δ_0, δ_1 just means that his map respects endpoints. Using the composition law of intersection on $i^*(\mathbb{\Delta}_1) = (a \mapsto \mathrm{Crib}(i(a)))$, the comparison lemma for homotopy intervals (p. 60 implies that $i^*(\mathbb{\Delta}_1)$ is aspheric over e, qed.

In the cases I have in mind, I is even an element of A, hence $i_!(I) = i(I)$, moreover $i(I) = \mathbb{\Delta}_1$ and the map $i_!(I) \to \mathbb{\Delta}_1$ above is the *identity*! The choice of the functor i is in every case "the most natural one" (discarding however the clumsy $i_0 = i_A$, and trying to get away with categories $i(a)$ which give the simplest imaginable description of objects and arrows of the would-be test category A), and the choice of I itself is still more evident – it is *the* object of A (or *one* among the objects, in cases such as $A = \mathbb{\Delta}^n$, giving rise to simplicial *multi*complexes...) which suggests most strongly the picture of an "interval". Thus the one key verification we are left with (all the rest being "formal" in terms of what precedes) is the asphericity of I over e, i.e., that all the products $I \times a$ are aspheric.

Returning upon terminology: strict test categories, and strict modelizers.

39 Maybe it is time now to come back to the property of total asphericity of $A^\wedge$, expressed by the condition (T 2) on A, namely that the product in $A^\wedge$ of any two elements in A is aspheric. As we saw, this implies already asphericity of A, i.e., of the topos $A^\wedge$. In our present setting, total asphericity is of special interest only when coupled with the property (T 3), which amounts to saying that the Lawvere element L_A in $A^\wedge$ is aspheric over $e_{A^\wedge}$, or (what now amounts to the same) that L_A is aspheric. [p. 67] However, when faced with the question to decide whether a given A does indeed satisfy (T 2), it will be convenient to use a system (I, e_0, e_1) in $A^\wedge$ of which we know beforehand it is a homotopy interval (the delicate part of this notion being the asphericity of all products $I \times a$ for a in A). As part of the "story retold", I recall now the most natural geometric assumption which will ensure (T 2), i.e., total asphericity of $A^\wedge$:

Proposition. *Let A be a small category, (I, e_0, e_1) a homotopy interval in $A^\wedge$. Assume that for any a in A, there exists a homotopy*

$$h_a : I \times a \to a$$

between id_a *and a constant map $c_a : a \to a$ (defined by a section of a over the final element of $A^\wedge$, i.e., by a map $e \to a$). Then $A^\wedge$ is totally aspheric, i.e., every $a \in \mathrm{Ob}\,A$ is aspheric over e (or, equivalently, the product in $A^\wedge$ of any two elements a, b in A is aspheric).*

This is a particular case of the "homotopy lemma" (p. 60). In fact we don't even use the condition $e_0 \cap e_1 = \varnothing_{A^\wedge}$ on the "interval" (I, e_0, e_1), but we easily see that this condition follows from the existence of the homotopy and the sections of any a in A over e.

Before returning to the investigation of specific test categories, I want to come back on some terminology. The condition on $A^\wedge$ that the Lawvere element L_A should be aspheric over e has taken lately considerable geometric significance, and merits a name. I will say from now on that A is a *test category*, and that $A^\wedge$ is an *elementary modelizer*, if this condition is satisfied, and if moreover A is aspheric. This condition (which amounts to our former (T 1) + (T 3)) is weaker than what we had lately called a test category, as we had overlooked so far the fact that (T 1) and (T 3) alone already imply the basic requirement about $i_A j_A(C) \to C$ being a weak equivalence for any C, and hence $W_A^{-1} A^\wedge$ being canonically equivalent to (Hot). Thus it seemed for a while that the only handy conditions we could get for ensuring this requirement were (T 1)(T 2)(T 3), all together (in fact, it turned out later that (T 2) already implies (T 1)). When all three conditions (T 1) to (T 3) are satisfied, [p. 68] I'll now say that A is a *strict test category*, and that $A^\wedge$ is an *elementary strict modelizer*. Here the notion of a "strict modelizer" (not necessary an elementary one) makes sense independently, it means a category M endowed with a set $W \subset \mathrm{Fl}(M)$ satisfying conditions a)b)c) of p. 59, such that $W^{-1}M$ is equivalent to (Hot), and such moreover that the localization functor $M \to W^{-1}M$ *commutes with finite products* (perhaps we should also insist on commutation with finite sums, and possibly include also infinite sums and products in the condition – whether this is adequate is not quite clear yet). The mere condition that $i_A j_A(C) \to C$ should be a weak equivalence for any C in (Cat), or equivalently that $i_A j_A(C)$ should be aspheric when C has a final element, will be referred to by saying that A is a *test category in the wide sense*. It means little more than the fact that $(A^\wedge, W_A)$ is a modelizer. Finally, if we merely assume that $A^\wedge$ admits a homotopy interval, or equivalently, that L_A is aspheric over $e_{A^\wedge}$, we will say that A is a *local test category* (because it just means that the induced categories $A_{/a}$ for a in A are test categories), and accordingly $A^\wedge$ will be called a *local elementary modelizer*. More generally, we call a topos $\mathcal{A}$ such that the Lawvere element $L_{\mathcal{A}}$ be aspheric over the final element a *locally modelizing topos*, and we call it a *modelizing topos* if it is moreover aspheric. When the topos is locally aspheric, i.e., admits a generating family made up with aspheric objects of $\mathcal{A}$, then $\mathcal{A}$ is indeed a locally modelizing topos iff the final object can be covered by elements U_i such that the induced topoi $\mathcal{A}_{/U_i}$ be modelizing topoi. This terminology for topoi more general than of the type $A^\wedge$ is possibly somewhat hasty, as the relation with actual homotopy models, namely with the question whether $(\mathcal{A}, W_{\mathcal{A}})$ is a modelizer, has not been investigated yet. Still I have the feeling the relation should be a satisfactory one, much along the same lines as we got in the case of topoi of the type $A^\wedge$. We will not dwell upon this now any longer.

For completing conceptual clarification, we should still make sure that a test category A need not be a strict test category, i.e., need not be totally aspheric. As a candidate for a counterexample, one would think about a category A endowed with a final element and an element I, together with $\delta_0, \delta_1 : e \to I$, satisfying $\mathrm{Ker}(\delta_0, \delta_1) = \varnothing_{A^\wedge}$ in $A^\wedge$, I being squarable in A, namely the products $I \times a$ ($a \in \mathrm{Ob} A$) are in A – this alone will imply that I is aspheric over e in $A^\wedge$, hence A is a test category, but it is unlikely that this alone will imply equally total asphericity of $A^\wedge$. We would think, as the most "economical" example, one where A is made up with elements of the type I^n ($n \in \mathbb{N}$) namely cartesian powers of I, [p. 69] plus products $a_0 \times I^n$ ($n \in \mathbb{N}$), where a_0 is an extra element, and those maps between these all those (and not more) which can be deduced from δ_0, δ_1 and the assumption that the elements I^n and $a \times I^n$ are cartesian products indeed. This is much in the spirit of the construction of a variant of the category $\square$ of "standard cubes", which naturally came to mind a while ago (cf. p. 48–49). I very much doubt A satisfies (T 2), and would rather bet that $a_0 \times a_0$ in $A^\wedge$ is *not* aspheric.

Another type of example comes to my mind, starting with a perfectly good (namely *strict*) test category A, and taking an induced category $A_{/a_0}$, with a_0 in A. This is of course a test category (it would have been enough that A be just a local test category), however it is unlikely that it will satisfy (T 2), namely the induced topos be totally aspheric. This would imply for instance that for any two non-empty subobjects of a_0 in $A^\wedge$ (namely subobjects of the final element in $A^\wedge{}_{/a_0} \simeq (A_{/a_0})^\wedge$) have a non-empty intersection. Now this is an exceedingly strong property of a_0, which is practically never satisfied, except for the final object of A. Now here is a kind of "universal" counterexample. Take A any test category and I a homotopy interval in $A^\wedge$, thus I is aspheric and hence $A_{/I}$ is again a test category (namely locally test and aspheric), but it is *never* a strict test category, because the two standard subobjects e_0, e_1 given with the structure of I are non-empty, and however their intersection is empty!

These reflections bring very near how much stronger the strictness requirement (T 2) is for test categories, than merely the conditions (T 1), (T 3) without (T 2).

Digression on cartesian products of weak equivalences in (Cat)*; 4 weak equivalences* relative *to a given base object.*

40 Yesterday I incidentally made use of the fact that if A is an aspheric element in (Cat), then for any other object C, $C \times A \to C$ is aspheric (and a fortiori a weak equivalence). The usual asphericity criterion for a functor shows that it is enough to prove that for any C with final element, $C \times A$ is aspheric. For this again, it is enough to prove that the projection $C \times A \to A$ is aspheric, which by localization upon A means that the product categories $C \times (A_{/a})$ (with a in A) are aspheric. Finally we reduced to checking that the product of two categories with final element is aspheric, which is trivial because such a category has itself a final element.

The result just proved can be viewed as a particular case of the following

Proposition. *In* (Cat)*, the cartesian product of two weak equivalences is a weak equivalence.*

[p. 70] (We get the previous result by taking the weak equivalences $A \to e_{(\mathrm{Cat})}$ and $\mathrm{id}_C : C \to C$.) To prove the proposition, we are immediately reduced to the case when one of the two functors is an identity functor, i.e., proving the

Corollary 1. *If* $f : A' \to A$ *is a weak equivalence in* (Cat)*, then for any* C *in* (Cat)*,* $f \times \mathrm{id}_C : A' \times C \to A \times C$ *is a weak equivalence.*

For proving this, we view the functor $f \times \mathrm{id}_C$ as a morphism of categories "over C", which corresponds to a situation of a morphism of topoi over a third one

$$\begin{tikzcd} X' \arrow[rr, "f"] \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd},$$

we will say that f is a *weak equivalence relative to* Y, if not only this is a weak equivalence, but remains so by any base change by a localization functor

$$Y_{/U} \to U$$

giving rise to

$$f_{/U} : X'_{/U} \to X_{/U}.$$

As usual, standard arguments prove that it is enough to take U in a set of generators of the topos Y. In case $Y = C\hat{}$, we may take U in C. In case moreover X, X' are defined by small categories P, P' and a functor of categories over C

$$\begin{tikzcd} P' \arrow[rr, "F"] \arrow[dr] & & P \arrow[dl] \\ & C & \end{tikzcd},$$

this condition amounts to demanding that for any c in C, the induced functor

$$F_{/c} : P'_{/c} \to P_{/c}$$

be a weak equivalence. In our case $P = A \times C$, $P' = A' \times C$, $F = f \times \mathrm{id}_C$, the induced functor can be identified with $f \times \mathrm{id}_{C_{/c}}$. This reduces us, for proving the corollary, to the case when C has a final element. But consider now the commutative diagram

$$\begin{tikzcd} A' \times C \arrow[r, "f \times \mathrm{id}_C"] \arrow[d] & A \times C \arrow[d] \\ A' \arrow[r, "f"] & A \end{tikzcd},$$

where the vertical arrows are the projections and hence, by what was proved before, weak equivalences. As f is a weak equivalence, it follows that $f \times \mathrm{id}_C$ is a weak equivalence too, qed.

The proposition above goes somewhat in the direction of looking at "homotopy properties of (Cat)" and "how far (Cat) is from being a closed model category in Quillen's sense". It is very suggestive for having a [p. 71] closer look at functors $f : B \to A$ in (Cat) which are "universally weak equivalences", i.e., $W_{(\mathrm{Cat})}$-aspheric, namely such that for *any* map in (Cat) $A' \to A$ (not only a localization $A_{/a} \to A$), the induced functor $B' = B \times_A A' \to A'$ is a weak equivalence. This property is a lot stronger than just asphericity, and reminds of the "trivial fibrations" in Quillen's theory. The usual criterion of asphericity for $B' \to A'$ shows that f is W-aspheric iff for any A' with final element, and any functor $A' \to A$, the fiber product $B' = B \times_A A'$ is aspheric. The feeling here is (suggested partly by Quillen's terminology) that this property is tied up some way with the property for f to be a fibering (or cofibering?) functor, in the sense of fibered and cofibered categories, with aspheric fibers moreover. Presumably, fibered or cofibered categories, with "base change functors" which are weak equivalences, will play the part of Serre-Quillen's fibrations – and it is still to be guessed what kind of properties of a functor will play the part of *co*fibrations. Apparently they will have to be a lot more stringent than just monomorphisms in (Cat), cf. p. 37.

However, I feel it is not time yet to dive into the homotopy theory properties of the all-encompassing basic modelizer (Cat), but rather come back to the study of general (and less general) test categories.

41

Role of the "inspiring assumption", and of saturation conditions on "weak equivalences".

One comment still, upon the role played in the theory I am developing of the assumption (p. 30) that the category of autoequivalences of (Hot) is equivalent to the final category. This assumption has been a crucial guide for putting the emphasis where it really belongs, namely upon the set $W \subset \mathrm{Fl}(M)$ of weak equivalences within a category M which one would like to take in some sense as a category of models for homotopy types – the functors $M \to (\mathrm{Hot})$ following along automatically. However, in no statement whatever I proved so far, was this assumption ever used. On the other hand, the notion of a *modelizer* introduced in the wake of the "assumption" (cf. p. 31) was tacitly changed during the reflection, by dropping altogether the condition a) of (strong) saturation of W, namely that W is just the set of arrows made invertible by the localization functor $M \to W^{-1}M$. Instead of this, it turned out that the saturation condition we really had at hands and which was adequate for working, was the conditions a) to c) I finally wrote down explicitly yesterday (p. 59). As for the strong saturation condition, for the time being (using nothing but what has actually been proved so far, without reference to "well-known facts" from homotopy theory), it is not even clear that the basic modelizer (Cat) is one in the initial sense, namely that the set of weak equivalences in (Cat) is strongly saturated. However, [p. 72] from the known relation between (Cat) and an elementary modelizer $A^\wedge$, it follows that $W_{(\mathrm{Cat})}$ is strongly saturated iff $W_A \subset \mathrm{Fl}(A^\wedge)$ is. This implies that it is enough to prove strong saturation in *one* elementary modelizer $A^\wedge$, to deduce it in all others, as well as in (Cat). However, in the case at least when $A = \triangle$, it is indeed "well-known" that the weak equivalences in $A^\wedge$ satisfy the strong saturation condition. In terms of Quillen's set-up, it follows from the fact that $\triangle^\wedge$ is a "closed model category", and prop. 1 in I 5.5 of Quillen's exposé. The only thing which is not quite understood, not by me at any rate, is why $\triangle^\wedge$ *is* indeed a closed model category – Quillen's proof it seems relies strongly on typical simplicial techniques. I'll have to look if the present set-up will suggest a more conceptual proof, valid possibly for any test category (or at least, any strict test category). I'll have to come back upon this later. For the time being, I feel a greater urge still to understand about the relationship between different test categories – also, I did not really finish with my review of what may be viewed as the "standard" test categories such as $\triangle$, $\square$ and their variants.

25.3.

Terminology revised (model preserving functors). Submodelizers of the basic modelizer (Cat).

42 In the notes last time I made clear what finally has turned out for me to correspond to the appelation of a "modelizer", as prompted by the internal logics of the situations I was looking at, in terms of the information available to me. I should by then have added what a *model-preserving* (or *modelizing*) *functor* between modelizers (M, W) and (M', W') has turned out to mean, which has undergone a corresponding change with respect to what I first contemplated calling by that name (p. 31). Namely, here it turned out that I should be more stringent for this notion, replacing the condition $f(W) \subset W'$ by the stronger one

$$W = f^{-1}(W'),$$

and moreover, of course, still demanding that the corresponding functor

$$W^{-1}M \to W'^{-1}M'$$

should be an equivalence. It is by now established, with the exception of just the two first that, all the functors occurring in the diagram on p. 31 are indeed model preserving, namely it is so for α, β, ξ, η – and also the right adjoints to ξ, η, as expected by then. It should be more or less trivial that the first of the functors in this diagram, namely the canonical inclusion (Ord) → (Preord), as well as the left adjoint from (Preord) to (Ord), are model preserving – except for the fact that it has not been yet established that these two categories are indeed modelizers (for the natural notion of weak equivalences, induced from (Cat))), namely that
[p. 73] the localized categories with respect to weak equivalences are indeed equivalent to (Hot). The only natural way one might think of this to be proved, is by proving that the inclusion functor from either into (Cat) induces an equivalence between the localizations, which would imply at the same time that this inclusion functor is indeed model-preserving, and hence that all the functors in the diagram of p. 31 are model preserving functors between modelizers. I still do believe this should be so, and want to give below a reflection which might lead to a proof of this.

Before, there is still one noteworthy circumstance I want to emphasize. Namely, it occurred in a number of instances that we got in a rather natural way several modelizing functors between two modelizers (M, W) and (M', W'), in one direction or the other – for instance from (Cat) to $A^\wedge$ using different functors $i : A \to (\mathrm{Cat})$, or from $A^\wedge$ to (Cat) using $a \mapsto A_{/a}$. It turned out that the corresponding functors between $W^{-1}M$ and $W'^{-1}M'$ were always canonically isomorphic when in the same direction, and quasi-inverse of each other when in opposite directions. This is indeed very much in the spirit of the "inspiring assumption" of p. 30, that the category of autoequivalences of (Hot) is equivalent to the unit category, which implies indeed that for two categories H, H' equivalent to (Hot), for two equivalences from H to H', there is a unique isomorphism between them. Quite similarly, there have been a number of situations (more or less summed up in the end in the "key theorem" of p. 61) when by localization we got a natural functor f from some $W^{-1}M$ to another $W'^{-1}M'$, and another F which is known already to be an equivalence, and it turns out that f is isomorphic to F (and in fact, then, canonically so) iff for any object x in the source category, $f(x)$ and $F(x)$ are isomorphic. This suggests that *presumably, every functor from* (Hot) *into itself, transforming every object into an isomorphic one, is in fact isomorphic to the identity functor*.

I want now to make a comment, implying that there are many full subcategories M of (Cat), such that for the induced notion of weak equivalence, M becomes a modelizer, and the inclusion functor a modelizing functor – or, what amounts to the same, that the canonical functor

$$W_M^{-1}M \to W_{(\mathrm{Cat})}^{-1}(\mathrm{Cat}) = (\mathrm{Hot})$$

is an equivalence. To see this, let more generally (M, W) be any category endowed with a subset $W \subset \mathrm{Fl}(M)$, and let $h : M \to M$ be a functor such that $h(W) \subset W$, and such that the induced functor $W^{-1} \to W^{-1}M$ is an equivalence. Let now M' be any full subcategory of M such that h factors through M', let $h' : M \to M'$ be the corresponding induced functor, and $W' = W \cap \mathrm{Fl}(M')$. Then it is formal that the inclusion $g : M' \to M$ and [p. 74] $h' : M \to M'$ induce functors between $W'^{-1}M'$ and $W^{-1}M$, which are quasi-inverse to each other – hence these two categories are equivalent. In case (M, W) is a modelizer, this implies that (M', W') is a modelizer too and the inclusion functor g, as well as h', are model preserving.

We can apply this remark to the modelizer (Cat), and to the functor

$$i_A i^* : (\mathrm{Cat}) \to (\mathrm{Cat})$$

defined by any functor $i : A \to (\mathrm{Cat})$ satisfying the conditions of the "key theorem" (p. 61), for instance the functor $i_A \restriction A : a \mapsto A_{/a}$ (where A is any test category). We get the following

Proposition. *Let M be any full subcategory of* (Cat)*, assume there exists a test-category A such that for any F in $A^\wedge$, the category $A_{/F}$ belongs to M (i.e., the functor $i_A : A^\wedge \to$* (Cat)*, $F \mapsto A_{/F}$, factors through M). Let W_M be the set of weak equivalences in M. Then (M, W_M) is a modelizer and the inclusion functor $M \to$* (Cat) *is model preserving.*

The same of course will be true for any full subcategory of (Cat) containing M – which makes an impressive bunch of modelizers indeed! When the test category A is given, one natural choice for M is to take all categories C which are "locally isomorphic to A", namely such that for any x in C, the induced category $C_{/x}$ be isomorphic to a category of the type $A_{/a}$, with a in A.

It would be tempting to apply this result to the full subcategory (Ord) of (Cat) – but for this to be feasible, would mean exactly that there exists a test-category A defined by an ordered set (or at least "locally ordered"). To see whether there exists indeed such an ordered set looks like a rather interesting question – maybe it would give rise to algebraic models for homotopy types, simpler than those used so far, namely simplicial and cubical complexes and multicomplexes. It is interesting to note that if such a test category should exist, it will *not* be in any case a strict test category. Indeed, the topos $A^\wedge$ associated to an ordered set A can be viewed also, as we saw before (p. 18), as associated to a suitable topological space (namely A endowed with a suitable topology, the open sets being just the "cribles" in A). But we have seen that the topos associated to a topological space *cannot* be *strictly* modelizing (cor. 2 on page 53).

This remark confirms the feeling that it was worth while emphasizing the notion of a test category (just (T 1) to (T 3)) by a simple and striking name as I finally did, rather than bury it behind the notion I now call a strict test category, which is considerably more stringent and, moreover, more "rigid". For instance, it is not stable under localization $A_{/a}$, whereas the notion of a test category is – indeed, for any aspheric I in $A^\wedge$, $A_{/I}$ is still a test category. [p. 75]

The category Δ^{f} of simplices without degeneracies as a weak test category – or "face complexes" as models for homotopy types.

43 Now let's come back for a little while again to the so-called "standard test categories", and check how nicely the "story retold" applies to them.

Not speaking about multicomplexes, there are essentially two variants for "categories of simplices" as test categories. The smaller, more commonly used one, is the category Δ of "ordered simplices", most conveniently described as the full subcategory of (Cat) defined by the family of simplices Δ_n ($n \in \mathbb{N}$). Here the most natural choice for $i : \Delta \to (\mathrm{Cat})$ is of course the inclusion functor. As i is fully faithful and Δ_1 is in the image, it follows that $i^*(\Delta_1) = \Delta_1$, and we have only to check (for A to be a test-category with "test-functor" i) that Δ_1 is aspheric over $e = \Delta_0$, namely that all products $\Delta_1 \times \Delta_n$ are aspheric – which we did. The extra condition of the "total asphericity criterion" (proposition on p. 67), namely existence of a homotopy in $\Delta^\wedge$ from the identity map to a constant map, for any Δ_n, is indeed satisfied: it is enough to define such a homotopy in (Cat), which is trivial using the final element of Δ_n. Thus Δ is in fact a *strict* test category.

As for $\widetilde{\Delta}$, the most elegant choice theoretically is to take the category of all non-empty finite sets, but his leads to set-theoretic difficulties, as this category is not small - thus we take again the standard non-ordered

simplices
$$\widetilde{\Delta}_n = \mathbb{N} \cap [0, n] \quad (n \in \mathbb{N}),$$
so as to get a "reduced" category with a countable set of objects. This time, as $\widetilde{\Delta}$ is stable under finite products, and contains the "interval" $(\widetilde{\Delta}_1, \delta_0, \delta_1)$ (which is necessarily then a homotopy interval, as all elements of $\widetilde{\Delta}$ are aspheric over $e = \widetilde{\Delta}_0$), the fact that $\widetilde{\Delta}$ is a strict test category is trivial. As for a test functor, the neatest choice is the one we said before, namely associating to every finite set the ordered set of all non-empty subsets. We thus get

$$\widetilde{i} : \widetilde{\Delta}^{\wedge} \to (\mathrm{Cat}) \quad \text{(factoring in fact through (Ord), as does } i : \Delta \to (\mathrm{Cat}) \text{ above)}$$

To prove it is indeed a test functor, the corollary 3 to the "key theorem" (p. 66) applies, taking of course $I = \widetilde{\Delta}_1$, hence

$$\widetilde{i}_!(I) = i(\widetilde{\Delta}_1) = \begin{matrix} \{1\} & \searrow & \\ & & \{0,1\} \\ \{0\} & \nearrow & \end{matrix} \; .$$

We map this into the object Δ_1 of (Cat), by taking $\{0\}$ into 0, $\{1\}$ and $\{0, 1\}$ into 1, we do get indeed a morphism compatible with endpoints, which implies that $\widetilde{i}$ is a test functor. [p. 76]

If we denote still by $i, \widetilde{i}$ the functors from $\Delta, \widetilde{\Delta}$ to the category (Ord) of ordered sets factoring the previous two test functors, we get a commutative diagram of functors

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d, "i"'] & \widetilde{\Delta} \arrow[d, "\widetilde{i}"] \\ (\mathrm{Ord}) \arrow[r, "\mathrm{bar}"] & (\mathrm{Ord}) \end{tikzcd} ,$$

where the first horizontal arrow is the inclusion functor (bijective on objects, and injective but not bijective on arrows), and the second is the "barycentric subdivision" functor, or "flag"-functor, associating to every ordered set the set of all "flags", namely non-empty subsets which are totally ordered for the induced order (here, all subsets, as the simplices are totally ordered).

For a while I thought there was an interesting third variant, namely ordered simplices with *strictly* increasing maps between them – which means ruling out degeneracy operators. This feeling was prompted of course by the fact that the face operators in a complex are enough for computing homology and cohomology groups, which are felt to be among the most important invariants of a complex. Equally, the fundamental groupoid of a semi-simplicial set can be described, using only the face operators. As a consequence, for a map between semisimplicial complexes $K_* \to K'_*$, to check whether this is a weak equivalence, in terms of the Artin-Mazur cohomological criterion, depends only on the underlying map between "simplicial face complexes" (namely, forgetting degeneracies). These *are* indeed striking facts, which will induce us to put greater emphasis on the face operators than on degeneracies. It seems, though that the degeneracies play a stronger role than I suspected, even though it is a somewhat hidden one. In any case, as soon as we try to check ("par acquit de conscience") that the category Δ^{f} of simplices with strictly increasing maps is a test category, it turns out that it is very far from it! Thus, as there is no map from any Δ_n with $n > 1$ into Δ_1, it follows that in $(\Delta^{\mathrm{f}})^\wedge$, we get

$$\Delta^{\mathrm{f}}{}_{/\Delta_0\times\Delta_n} = \text{discrete category with } n+1 \text{ elements},$$

thus these products are by no means aspheric, poor them! Even throwing out Δ_0 (a barbarous thing to do anyhow!) doesn't rule out the trouble, and restricting moreover to products $\Delta_1 \times \Delta_n$ (to have at least a test category, if not a strict one). In any case, Δ_1 wouldn't be of much use, because it has got no "section" anymore (nor does any other element of Δ^{f}) – because this would imply that *any* element of Δ^{f} maps into it –
[p. 77] but for given Δ_n, only the Δ_m's with $m \leq n$ map into it.

Maybe I am only being imprisoned still by the preconception of finding a homotopy interval in Δ^{f} itself, rather than in $(\Delta^{\mathrm{f}})^\wedge$. After all, just applying the definition of a test-category A with test-functor $i : A \to (\mathrm{Cat})$, all we have to care about is whether a) A is aspheric and b) $i^*(\Delta_1)$ is an aspheric element of $A^\wedge$. We just got to apply this to the case of the functor

$$i^{\mathrm{f}} : \Delta^{\mathrm{f}} \to (\mathrm{Cat})$$

induced by $i : \Delta \to (\mathrm{Cat})$ above, taking into account of course the extra trouble that i^{f} is no longer fully faithful.

I just stopped to look, with a big expectation that Δ^{f} *is* a test category after all – but it turns out it definitely isn't! Indeed, $A_{/\Delta_0\times I}$ (where $I = (i^{\mathrm{f}})^*(\Delta_1)$) is again a discrete two-point category, not aspheric. Taking the canonical functor $a \mapsto A_{/a}$ from $A = \Delta^{\mathrm{f}}$ to (Cat), which is the ultimate choice for checking whether or not A is a test category, finally gives the answer: it is not. Because with I now the Lawvere element, we still have that $A_{/\Delta_0\times I}$ is a two-point discrete category. Thus the topos $A^\wedge = (\Delta^{\mathrm{f}})^\wedge$ isn't locally modelizing, i.e., it hasn't got any homotopy interval, which at any rate is a very big drawback I would think. The only hope which still remains, to account for the positive features of face-complexes recalled above, is that Δ^{f} is at least a test category *in the wide sense*, namely that for any category C with final element, the category $i_A j_A(C) = A_{/j_A(C)}$ (= category of all pairs (n,u), with u a map of the (ordered) category $A_{/\tilde{\Delta}_n}$) is aspheric. The first case to check is for $C =$ final category Δ_0, i.e., asphericity of A, next step would be $C = \Delta_1$, i.e., asphericity of the Lawvere element L_A (but of course *not* asphericity over the final element $e_{A^\wedge}$!).

The question certainly deserves to be settled. If the answer is affirmative, i.e., Δ^{f} *is* a test category in the wide sense, then the proposition stated earlier (p. 74), which clearly applies equally when A is a test category in the wide sense, implies that if M is any full subcategory of

(Cat) containing all those C which are "locally isomorphic" to Δ^{f}, (i.e., such that for every $x \in C$, $C_{/x}$ is isomorphic to the ordered category of all subsimplices of some simplex), then for the induced notion W_M of weak equivalences, M is a modelizer and the inclusion functor from M into (Cat) is modelizing. This does not yet apply to (Ord), however, it reopens the question whether the full subcategory of (Ord) made up by all ordered sets J which are locally isomorphic to Δ^{f} in the sense above (namely for any $x \in J$, the ordered subset $J_{\leq x}$ is isomorphic to the ordered set of subsimplices of some simplex) is a modelizer. To ensure this, it would be enough to find an ordered set J satisfying the previous condition (for instance on stemming from a "simplicial maquette"), such that the corresponding category is a test category, or at least a test category in the wide sense. The first candidate that comes to my mind, is to take any *infinite* set S of vertices, and take J to be the ordered set of all *finite non-empty* subsets (called the simplices – thus the elements of S can be interpreted in terms of J as the minimal simplices). By the way, the category associated to J, in case $S = \mathbb{N}$, can be interpreted in terms of $A = \Delta^{\mathrm{f}}$ as the category $A_{/\Delta_\infty}$, where Δ_∞ is defined as the filtering direct limit in $A^\wedge$ of the Δ_n's, arranged into a direct system in the obvious way: [p. 78]

$$\Delta_\infty = \varinjlim_n \Delta_n \quad \text{in } (\Delta^{\mathrm{f}})^\wedge.$$

Asphericity of J looks intuitively evident, and should be easy by a direct limit argument, as a matter of fact any filtering category (the next best to having a final element) should be aspheric, at least if it has a countable cofinal family of objects. The Lawvere element L_A in $A^\wedge$ is not aspheric though over the final object, because when inducing over a zero simplex, we get the same contradiction as before. As a matter of fact, I am getting aware I have been very silly and prejudiced not to see one trivial common reason, applicable to Δ^{f} as to J, showing that they are not test categories nor even local test categories: namely the induced categories $A_{/a}$ should be test categories too, but among these there are one-point categories (take $a = \Delta_0$, or a zero-simplex), and such a category is *not* a test-category!

Still, J may be a test category in the wide sense – the basic test here, as we know already asphericity of J itself, would be (absolute) asphericity of L_J, or equivalently, of the category $J_{/L_J}$, an ordered set in fact (as is the case for any category $A_{/F}$ for A defined by an ordered set and F in $A^\wedge$). This is now the ordered set of all pairs $(K \subset T)$ of finite subsets of S, with T in J, namely T non-empty, with the rule

$$(K', T') \leq (K, T) \quad \text{iff} \quad T' \subset T \text{ and } K' = K \cap T'.$$

[p. 79] 26.3.

Overall review of the basic notions.

44 I finally convinced myself that Δ^{f}, the category of standard ordered simplices with face operations (and no degeneracies) *is* a "test category in the wide sense" after all – although definitely not a test category, as was seen yesterday (turning out to be a practically trivial observation). This now does rehabilitate the notion of a test category in the wide sense, which I expected to be of little or no interest – much the way as previously, the notion I now call by the name "test category" was rehabilitated or rather, discovered, after I expected that the only one proper notion for getting modelizers of the form $A^{\wedge}$ was in terms of conditions (T 1) to (T 3) on A (including the very strong condition (T 2) of total asphericity). This notion I finally called by the name "*strict* test category", and I was fortunately cautious enough to reserve a name too for the notion which appeared then as rather weak and unmanageable, of test categories in the wide sense, or, as I will say now more shortly, *weak test categories* (by which of course I do not mean to exclude the possibility that it be even a test category), thus getting the trilogy of notions with strict implications

weak test categories $\Leftarrow$ test categories $\Leftarrow$ strict test categories.

In order not to get confused, I will recall what exactly each of these notions means.

a) Weak test categories. For a given small category A, we look at the functor

$$i_A : A^{\wedge} \to (\mathrm{Cat}), \quad F \mapsto A_{/F}, \tag{1}$$

which commutes with direct limits, and at the right adjoint functor

$$j_A = i_A^* : (\mathrm{Cat}) \to A^{\wedge}, \quad C \mapsto j_A(C) = (a \mapsto \mathrm{Hom}(A_{/a}, C)). \tag{2}$$

We get an adjunction morphism

$$i_A j_A(C) \to C \quad \text{in } (\mathrm{Cat}), \tag{3}$$

and another

$$F \to j_A i_A F \quad \text{in } A^{\wedge}, \tag{4}$$

functorially in C resp. in F. Presumably, the following are equivalent (I'll see in a minute how much I can prove about these equivalences):

(i) The functors (1) and (2) are compatible with weak equivalences, and the two induced functors between the localized categories

$$W_A^{-1} A^{\wedge} \rightleftarrows W^{-1}(\mathrm{Cat}) \overset{\mathrm{def}}{=} (\mathrm{Hot}) \tag{5}$$

are equivalences.

(ii) As in (ii), and moreover the two equivalences are quasi-inverse of each other, with adjunction morphism in $W^{-1}(\mathrm{Cat}) = (\mathrm{Hot})$ deduced from (3) by localization.

(iii) As in (ii), but moreover the adjunction morphism in $W_A^{-1}A\hat{}$ for the pair of quasi-inverse equivalences in (5) being likewise deduced from (4). [p. 80]

(iv) For any C in (Cat), (3) is a weak equivalences.

(v) Same as (iv), with C restricted to having a final element, i.e., for any such C, $j_A i_A(C) = A_{/j_A(C)}$ is aspheric.

(vi) For any F in $A\hat{}$, (4) is a weak equivalence, moreover

$$W_{(\mathrm{Cat})} = j_A^{-1}(W_A),$$

i.e., a map $f : C' \to C$ in (Cat) is a weak equivalence iff $j_A(f)$ is a weak equivalence in $A\hat{}$ – which means, by definition (more or less) that $i_A j_A(f)$ is a weak equivalence.

(vii) The functor

$$i_A j_A : (\mathrm{Cat}) \to (\mathrm{Cat}), \quad C \mapsto A_{/j_A(C)}, \tag{6}$$

transforms weak equivalences into weak equivalences, i.e., gives rise to a functor (Hot) $\to$ (Hot), and moreover the latter respects final object (i.e., A is aspheric).

(viii) Same as (vii), but restricting to weak equivalences of the type $C \to e$, where C has a final element and e is the final object in (Cat) (the one-point category), plus asphericity of A.

The trivial implications between all these conditions can be summarized in the diagram

$$\begin{tikzcd}
\text{(iii)} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{(vi)} \arrow[r, Rightarrow] & \text{(vii)} \arrow[r, Rightarrow] & \text{(viii)} \arrow[dl, Rightarrow] \\
\text{(ii)} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{(iv)} \arrow[r, Rightarrow] & \text{(v)} & \\
\text{(i)} & , & &
\end{tikzcd}$$

the only slightly less obvious implication here is (viii) $\Rightarrow$ (v), which is seen by looking at the commutative square deduced from $C \to e$ (C in (Cat) is in (v) namely with final element) by applying (3)

$$\begin{tikzcd}
i_A j_A(C) \arrow[r] \arrow[d] & C \arrow[d] \\
A = i_A j_A(e) \arrow[r] & e
\end{tikzcd},$$

by assumption the vertical arrows are weak equivalences, and so is $A \to e$ (because A is supposed to be aspheric), therefore the same holds for the fourth arrow left. On the other hand, an easy asphericity argument showed us that (v) $\Rightarrow$ (iii), hence all conditions (ii) to (viii)

are equivalent, and they are equivalent to the stronger form of (iv), say (iv'), saying that (3) is *aspheric* for any C in (Cat). The only equivalence which is not quite clear yet is that (i) implies the other conditions.
[p. 81] But it is so if we grant the "inspiring assumption", implying that any autoequivalence of (Hot) is isomorphic to the identity functor – in this case it is clear that even the weaker form (i') of (i), demanding only that $i_A j_A$ induce an autoequivalence of (Hot) and nothing on either factor i_A, j_A in (5), implies (v). Also, when we assume moreover A aspheric, it is clear that (i) (and even (i')) implies (vii), i.e., all other conditions. Thus, instead of the assumption on (Hot), it would be enough to know that the particular autoequivalence of (Hot) induced by $i_A j_A$ transforms the final element of (Hot) (represented by the element e of (Cat)) into an isomorphic one, which looks like a very slight strengthening of (i) indeed.

27.3. But this is true for *any* equivalence of categories – I'm really being very dull!

In any case, the conditions (ii) to (viii) are equivalent, and equivalent to (i) (or (i')) *plus*[†] asphericity of A. The formally strongest form is (iii), the formally weakest one (with the exception of (i) or (i')) is (v), which is also the one which looks the most concrete, namely amenable to practical verification. This was indeed what at the very beginning was attractive in the condition, in comparison to the first one that came to my mind when introducing the notion of a modelizer and of model preserving functors – namely merely that i_A should induce an equivalence between the localized categories, or equivalently, that $A^\wedge$ should be a "modelizer" and i_A should be model-preserving. That we should be more demanding and ask for j_A to be equally model preserving crept in first rather timidly – and I still don't know (and didn't really stop to think) if the first one implies the other.[‡]

[†] The "plus" is unnecessary visibly, see note above.

[‡] If fact, it is the condition that $j_A = i_A^*$, not i_A, should be model preserving, which is "the right" condition on A, equivalent to A being a weak test category, if A is assumed aspheric.

In any case, as far as checking a property goes, I would consider (v) to be *the* handy definition of a weak test category, wheres (iii) is the best, when it goes to making use of the fact that A is indeed a weak test category. As for (i), it corresponds to the main intuitive content of the notion, which means that homotopy types can be "modelled" by elements of $A^\wedge$, using i_A for describing which homotopy type is described by an element F of $A^\wedge$, and using j_A for getting a model in $A^\wedge$ for a given homotopy type, described by an object C in (Cat).

b) Test categories and local test categories. Let A still be a small category. Then the following conditions are equivalent:

[p. 82]
(i) All induced categories $A_{/a}$ (with a in A) are weak test categories.

(ii) For any aspheric F in $A^\wedge$, $A_{/F}$ is a weak test category.

(iii) There exists a "homotopy interval" in $A^\wedge$, namely an element I in $A^\wedge$, *aspheric over the final element* $e_{A^\wedge}$ (i.e., such that all products $I \times a$ (for a in A) are aspheric), endowed with two sections δ_0, δ_1 ever $e = e_{A^\wedge}$, i.e., with two subobjects e_0, e_1 isomorphic to e, such that $\mathrm{Ker}(\delta_0, \delta_1) = \varnothing_{A^\wedge}$, i.e., $e_0 \cap e_1 = \varnothing_{A^\wedge}$.

(iv) The Lawvere element L_A or $L_{A^\wedge}$ in $A^\wedge$ (i.e., the presheaf $a \mapsto$ subobjects of a in $A^\wedge \simeq$ cribles of $A_{/a}$) is aspheric over the final element e (in other words, it is a homotopy interval, when endowed with the two sections δ_0, δ_1 corresponding to the full and the empty crible in A).

(v) For any C in (Cat), the canonical map in (Cat)

$$i_A j_A(C) = A_{/j_A(C)} \to A \times C \tag{7}$$

(with second component (3)) is (not only a weak equivalence, which definitely isn't enough, but even) *aspheric*.

By the usual criterion of asphericity for a map in (Cat), condition (v) is equivalent with the condition (v'): For any a in A and any C in (Cat) with final element, $A_{/a\times j_A(C)}$ is aspheric, i.e., $a \times j_A(C)$ is an aspheric element in $A\hat{}$; now this is clearly equivalent to (i) (by the checking-criterion (v) above for weak test categories, applied to the categories $A_{/a}$). Thus we get the purely formal implications

$$\begin{array}{ccccccc} \text{(ii)} & \Longrightarrow & \text{(i)} & \Longrightarrow & \text{(iv)} & \Longrightarrow & \text{(iii)} \\ & & \Updownarrow & & & & \\ & & \text{(v)} & & & & , \end{array}$$

where (i) ⇒ (iv) is obtained by applying the criterion (v') just recalled to the case of $C = \Delta_1$. As the condition (iii) is clearly stable by localizing to a category $A_{/F}$ (using the equivalence $(A_{/F})\hat{} \simeq A\hat{}_{/F}$), we see that (iii) implies (ii), we are reduced (replacing A by $A_{/F}$) to the case when F is the final element in $A\hat{}$, and thus to proving that if A has a final element, then (iii) implies that A is a weak test category, i.e., that the categories $A_{/j_A(C)}$, with C having a final element, are aspheric. This was done by a simple homotopy argument in two steps. One step ("the comparison lemma for homotopy intervals" on p. 60) shows that (iii) ⇒ (iv), i.e., existence of a homotopy interval implies that the Lawvere interval is a homotopy interval, the other step (presented in a more general set-up in the "key result" on page 61) proving that (iv) implies asphericity of the elements $j_A(C)$ (C with final object) over $e_A = e_{A\hat{}}$. [p. 83]

We express the conditions (i) to (v) by saying that A is a *local test category*, or that $A\hat{}$ is a *locally modelizing topos*, or (if we want to recall that this topos is of the type $A\hat{}$) an *elementary local modelizer*. If A is moreover aspheric (i.e., $e_{A\hat{}}$ aspheric), or what amounts to the same, if A is moreover a weak test category, we say that A is a *test category*, or that $A\hat{}$ is a *modelizing topos*, or (to recall it is an $A\hat{}$ and not just any topos) an *elementary modelizer*. These are intrinsic properties on the topos $A\hat{}$, the first one of a local nature, the second not (as asphericity of $A\hat{}$ is a global notion).

Here the question arises whether the condition for A to be a weak test category can be likewise expressed intrinsically as a property of the topos $A\hat{}$ – which we then would call a *weakly modelizing topos*, or a *weak elementary modelizer*. This doesn't look so clear, as all conditions stated in a) make use of at least one among the two functors i_A, j_A, which do not seem to make much sense in the more general case, except possibly when using a specified small generating subcategory A of the given topos – and possibly checking that the condition obtained (if something reasonable comes out, as I do expect) does not depend on the choice of the generating site. This should be part of a systematic reflection on modelizing topoi, to make sure for instance they are modelizing indeed with respect to weak equivalences – but I'll not enter into such reflection for the time being.

c) Strict test categories. The following conditions on the small category A are equivalent:

(i) A is a weak test category (cf. a) (iii) above), and thus induces a localization functor

$$A\hat{\ } \to (\mathrm{Hot}) = W^{-1}(\mathrm{Cat}),$$

and this functor moreover *commutes with binary products*; or equivalently, the canonical functor $A\hat{\ } \to W_A^{-1}A\hat{\ }$ commutes with binary products.

(ii) A is a test category (a local test category would be enough, even), and moreover the topos $A\hat{\ }$ is totally aspheric, namely (apart having a generating family of aspheric generators, which is clear anyhow) the product of any two aspheric elements in $A\hat{\ }$ is aspheric, i.e., any aspheric element of $A\hat{\ }$ is aspheric over the final object.

[p. 84] (iii) A satisfies the two conditions:

T 2) The product in $A\hat{\ }$ of any two elements in A is aspheric.

T 3) $A\hat{\ }$ admits a homotopy interval (which is equivalent to saying that A is a local test category).

Of course, (ii) implies (iii). The condition (iii) can be expressed in terms of the topos $\mathcal{A} = A\hat{\ }$ by saying that this is a totally aspheric and locally modelizing topos, which implies already (as we saw by the Čech computation of cohomology) that the topos is aspheric, and therefore modelizing, i.e., A a test category, which is just (ii). Thus (ii) $\Leftrightarrow$ (iii). Total asphericity of $\mathcal{A}$, and the property that $\mathcal{A}$ be a local modelizer, are expressed respectively by the two neat conditions T 2) and T 3), neither of which implies the other even for an $A\hat{\ }$, with A a category with final object.

For expressing in more explicit terms condition (i), and check equivalence with the two equivalent conditions (ii), (iii), we need to admit that the canonical functor from (Cat) to its localization (Hot) commutes with binary products. This being so, the condition of commutation of $A\hat{\ } \to (\mathrm{Hot})$ with binary products can be expressed by the more concrete condition that for F, G in $A\hat{\ }$, the canonical map in (Cat)

$$i_A(F \times G) \to i_A(F) \times i_A(G), \quad \text{i.e., } A_{/F\times G} \to A_{/F} \times A_{/G} \tag{8}$$

is a weak equivalence. In fact, the apparently stronger condition that the maps (8) are aspheric will follow, because the usual criterion of asphericity for a map in (Cat) shows that it is enough for this to check that (8) is a weak equivalence when F and G are representable by elements a and b in A, which also means that $A_{/a\times b}$ is aspheric, namely $a \times b$ in $A^\wedge$ aspheric – which is nothing but condition T 2) in (iii), i.e., total asphericity of $A^\wedge$. On the other hand, the condition that A be a weak test category means that the elements $j_A(C)$ in $A^\wedge$, for C in (Cat) admitting a final object, are aspheric, or (what amount to the same when $A^\wedge$ is totally aspheric) that they are aspheric over the final object $e_{A^\wedge}$, i.e., that the products $a \times j_A(C)$ are aspheric, which also means A is a local test category. Thus (i) is equivalent to (ii).

The equivalent conditions (i) to (iii) are expressed by saying that A is a *strict test category*, or that $\mathcal{A} = A^\wedge$ is a *strictly modelizing topos*, or also that it is an *elementary strict modelizer* (when emphasizing the topos $\mathcal{A}$ should be of the type $A^\wedge$ indeed). This presumably will turn out to be the more important among the three notions of a "test category" and the weak and strict variant. The two less stringent notions, however, seem interesting in their own right. The notion of a weak test category mainly (at present) because it turns out that the category $\triangle^{\mathrm{f}}$ of standard ordered simplices with only "face-like" maps between them, namely strictly increasing ones (that is, ruling out degeneracies) is a weak test category, and not a test category. On the other hand, for any test category A, we can construct lots of test categories which are not strict, namely all categories $A_{/F}$ where F is any aspheric object in $A^\wedge$ which admits two "non-empty" subobjects whose intersection is "empty" – take for instance for F any homotopy interval. In the case of $A = \triangle$, we may take for F any objects $\triangle_n$ $(n \geq 1)$ in A, except just the final object $\triangle_0$. [p. 85]

d) Test functors. Let A be a weak test category, and let

$$i : A \to (\mathrm{Cat}) \tag{9}$$

be a functor, giving rise to a functor

$$i^* : (\mathrm{Cat}) \to A^\wedge, \quad C \mapsto i^*(C) = (a \mapsto \mathrm{Hom}(i(a), C)). \tag{10}$$

We'll say that i is a *weak test functor* if i^* is a morphism of modelizers, i.e., model-preserving, namely

$$\begin{cases} W_{(\mathrm{Cat})} = (i^*)^{-1}(W_{A^\wedge}) \quad \text{and} \\ W_{(\mathrm{Cat})}^{-1}(\mathrm{Cat}) = (\mathrm{Hot}) \to W_{A^\wedge}^{-1}A^\wedge \ \text{an equivalence.} \end{cases}$$

As by assumption we know already that (1)

$$i_A : A^\wedge \to (\mathrm{Cat}), F \mapsto A_{/F}$$

is model-preserving, this implies that i is a weak test functor iff the compositum

$$i_A i^* : (\mathrm{Cat}) \to (\mathrm{Cat}), \quad C \mapsto A_{i^*(C)} \stackrel{\mathrm{def}}{=} A_{/C} \tag{11}$$

is model-preserving, i.e., (essentially) induces an *autoequivalence* of (Hot).

The basic example is to take $i = i_A$, hence $i^* = j_A$ (2), the fact that A is a weak test category can be just translated (p. 80, (vii)) by saying that this functor (11) = (6) transforms weak equivalences into weak equivalences, and the localized functor (Hot) → (Hot) is an equivalent (or, which is enough, transforms final object into final object of (Hot)). But as we say before, this weak test functor gives rise to categories $i(A) = A_{/a}$ which are prohibitively large, and one generally prefers working [p. 86] with weak test functors more appropriate for computations – including the customary ones for categories of simplices or cubes, such as Δ^{f}, Δ, $\widetilde{\Delta}$ and the cubical analogons. If we admit the "inspiring assumption" that any autoequivalence of (Hot) is uniquely isomorphic to the identity functor, it will follow that the autoequivalence (Hot) → (Hot) induced by (11) (where again i is any weak test functor) is canonically isomorphic to the identity. This we checked directly (as part of the "key result" p. 61 and following) when we make on i the extra assumption that each category $i(a)$ has a final object. In practical terms, *the role of a weak test functor i is to furnish us with a quasi-inverse of*

$$W_A^{-1}A^\wedge \to W_{(\mathrm{Cat})}^{-1}(\mathrm{Cat}) = (\mathrm{Hot})$$

induced by i_A, more handy than the one deduced from j_A by localization, namely taking i^* instead of $j_A = i_A^*$. In other terms, for every homotopy type, described by an object C in (Cat), we get a ready model in $A^\wedge$, just taking $i^*(C) = (a \mapsto \mathrm{Hom}(i(a), C))$.

In the theorem on p. 61 just referred to, one point was that we did not assume beforehand that A was a weak test category, but we were examining two sets (a) and (b) of mutually equivalent conditions, where (a) just boils down to $i_A i^*$ (11) inducing an autoequivalence of (Hot), while (b) would be expressed (in the present terminology) by stating that moreover A is a test category (if we assume beforehand in (b) that A is aspheric, plus a little more still on i). It was not quite clear by then, as it is now, that this is actually strictly stronger. The somewhat bulky statement essentially reduces, with the present background, to the following

Proposition. *Let A be a small category, $i : A \to (\mathrm{Cat})$ a functor, such that for any a in A, $i(a)$ has a final object. Consider the following conditions:*

(i) *A is a test category (NB not only a weak one), and i is a weak test functor.*

(ii) *The element $i^*(\Delta_1)$ in $A^\wedge$ (namely $a \mapsto \mathrm{Crib}(i(a))$) is aspheric over $e_{A^\wedge}$, i.e., all products $i^*(\Delta_1) \times a$ ($a \in \mathrm{Ob}\,A$) are aspheric; moreover $e_{A^\wedge}$ is aspheric, i.e., A itself is aspheric.*

(iii) *There exists a homotopy interval (I, δ_0, δ_1) in $A^\wedge$, and a map in* (Cat) *compatible with δ_0, δ_1*

$$(12) \qquad i_!(I) \to \Delta_1,$$

where $i_! : A^\wedge \to (\mathrm{Cat})$ is the canonical extension of $i : A \to (\mathrm{Cat})$; moreover, A is aspheric.

The conditions (ii) and (iii) are equivalent and imply (i).

That (ii) implies (iii) is trivial by taking $I = i^*(\Delta_1)$; the converse, as we saw, is a corollary of the comparison lemma for homotopy intervals (p. 60), applied to $I \to i^*(\Delta_1)$ corresponding to (12), which is a morphism of intervals, namely compatible with endpoints. [p. 87] (However, in practical terms (12) is the more ready-to-use criterion, because in most cases I will be in A and therefore $i_!(I) = i(I)$, and (12) will be more or less trivial, for instance because I will be chosen so that $i(I)$ is either $\Delta_1 = \{0\} \to \{1\}$ or its barycentric subdivision $\begin{smallmatrix}\{0\}\\ \{1\}\end{smallmatrix} \rightrightarrows \{0,1\}$.) The same argument shows that (iii) implies that A is a test category, namely the Lawvere element L_A in $A^\wedge$ is a homotopy interval in $A^\wedge$, namely is aspheric over $e_{A^\wedge}$. This being so, we only got to prove (under the assumption (iii)) that $i_A i^*$ (11) induces an autoequivalence of (Hot), and more specifically a functor isomorphic to the identity functor. This we achieved by comparing directly $A_{/C}$ to C by the functor $(a,p) \mapsto p(e_a)$

$$A_{/C} \to C, \tag{13}$$

where e_a is the final element in $i(a)$. It is enough to check this is a weak equivalence for any C, a fortiori that it is aspheric, and the usual asphericity criterion reduces us to showing it is a weak equivalence when C has a final object, namely that in this case $A_{/C} = A_{/i^*(C)}$ is aspheric, i.e., $i^*(C)$ is aspheric in $A^\wedge$. This was achieved by an extremely simple homotopy argument, using a homotopy in (Cat)

$$\Delta_1 \times C \to C$$

between the identity functor in C and the constant functor with value e_C (cf. p. 62).

I feel like reserving the appellation of a *test functor* $A \to$ (Cat) (in contract to a *weak* test functor) to functors satisfying the equivalent conditions (ii) and (iii), implying that A is a test category (not merely a weak one!), and which seem stronger a priori than assuming moreover that i is a weak test functor (condition (i)). I confess I did not make up my mind if for a test category A, there may be weak test functors, with all $i(a)$ having final objects, which are not test functors in the present sense, namely whether it may be true that for any C in (Cat) with final object, $i^*(C)$ is aspheric, and therefore $i^*(\Delta_1)$ aspheric, without the latter being aspheric over $e_{A^\wedge}$, i.e., all products $i^*(\Delta_1) \times a$ ($a \in \operatorname{Ob} A$) being aspheric (which would be automatic though if A is even a *strict* test category). The difficulty here is that it looks hard to check the condition for arbitrary C with final object, except through the homotopy argument using relative asphericity of $i^*(\Delta_1)$. Of course, the distinction between weak test functors and test functors is meaningful only as long as it is expected that the two are actually distinct, when applied to test categories. Besides this, the restriction to categories $i(a)$ with final objects looks theoretically a little awkward, and it shouldn't be too hard I believe to get rid of it, if need be. But for the time being the only [p. 88] would-be test functors or weak test functors which have turned up, do satisfy this condition, and therefore it doesn't seem urgent to clean up the notion in this respect.

Remarks.

1. I regret I was slightly floppy when translating the condition that $\varphi = i_A i^*$ (11) be model-preserving, by the (a priori less precise) condition that it induce an autoequivalence of the localized category (Hot) (which is of course meant to imply that it transforms weak equivalences into weak equivalences, and thus does induce a functor from (Hot) into itself). It isn't clear that if this functor is an equivalence, $W = \varphi^{-1}(W)$, except if we admit that W is strongly saturated, i.e., an arrow in (Cat) which becomes an isomorphism in (Hot), is indeed a quasi-equivalence. This, as we saw, follows from the fact that there are weak test categories (such as $\blacktriangle$, which is even a strict one), such that $A^\wedge$ be a "closed model category" in Quillen's sense – but I didn't check yet in the present set-up that $\blacktriangle^\wedge$ does indeed satisfy Quillen's condition. *Which* elementary modelizers are "closed model categories" remains one of the intriguing questions in this homotopy model story, which I'll have to look into pretty soon now.

2. It should be noted that if $i : A \to (\mathrm{Cat})$ is a test functor, with a strict test category A (such as $\blacktriangle$, with the standard embedding i of $\blacktriangle$ into (Cat) say), whereas $i^* : (\mathrm{Cat}) \to A^\wedge$ is model preserving by definition, it is by no means always true that the left adjoint functor

$$i_! : A^\wedge \to (\mathrm{Cat})$$

(which can be equally defined as the canonical extension of i to $A^\wedge$, commuting with direct limits) is equally model preserving. This seems to be in fact an extremely special property of just the "canonical" test functor $i_A \restriction A$.

3. On the other hand, I do not know if for any small category A, such that $(A, W_A = W_{A^\wedge})$ is a modelizer and $i_A : A^\wedge \to (\mathrm{Cat})$ is model preserving, is a weak test category. Assume that $j_A = i_A^* : (\mathrm{Cat}) \to A^\wedge$ is model preserving, and moreover A aspheric, then A is a weak test category, because j_A transforms weak equivalences into weak equivalences and we apply criterion (vii) (p. 80). More generally, if we have a functor $i : A \to (\mathrm{Cat})$ such that $i^* : (\mathrm{Cat}) \to A^\wedge$ is model preserving (assuming already $(A^\wedge, W_A)$ to be a "modelizer"), I wonder whether this implies that A is already a weak test category, as it does when i is the canonical functor i_A and hence $i^* = j_A$. The answer isn't clear to me even when i_A, or equivalently $\varphi = i_A i^*$ model-preserving too.

Part III

Grinding my way towards canonical modelizers

27.3. [p. 89]

It's burning again! Review of some "recurring striking features" of modelizers and standard modelizing functors.

45 The review yesterday of the various "test notions", turning around test categories and test functors, turned out a lot longer than expected, so much so to have me get a little weary by the end – it was clear though that this "travail d'intendance" was necessary, not only not to get lost in a morass of closely related and yet definitely distinct notions, but also to gain perspective and a better feeling of the formal structure of the whole set-up. As has been the case so often, during the very work of "grinding through", there has appeared this characteristic feeling of getting close to something "burning" again, something very simple-minded surely which has kept showing up gradually and more and more on all odds and ends, and which still is escaping, still elusive. These is an impressive bunch of things which are demanding pressingly more or less immediate investigation – still I can't help, I'll have to try and pin down some way or other this "burning spot".

There seem to be recurring striking features of the modelizers met with so far – namely essentially (Cat) and the elementary modelizers $A^\wedge$ and possibly their "weak" variants. In all of them, these is a very strong interplay between the following notions, which seem to be the basic ones and more or less determine each other mutually: *weak equivalences* (which define the modelizing structure of the given modelizer M), *aspheric objects* (namely such that $x \to e_M$ is a weak equivalence), *homotopy intervals* (I, δ_0, δ_1), and last not least, the notion of a test functor $A \to M$, where A is a test category (or more generally weak test functors of weak test categories into M). The latter so far have been defined only when $M = \text{(Cat)}$, and initially they were viewed as being mainly more handy substitutes to j_A, for getting a model-preserving functor $\text{(Cat)} \to A^\wedge$ quasi-inverse to the all-important model-preserving functor $i_A : A^\wedge \to \text{(Cat)}$, $F \mapsto A_{/F}$. I suspect however that their role is a considerably more basic one than just computational convenience – and this reminds me of the analogous feeling I had, when first contemplating using such a thing as (by then still vaguely conceived) "test categories", for investigating ways of putting modelizing structures on categories M such as categories of algebraic structures of some kind or other. (Cf. notes of March 7, and more specifically par. 26 – this was the very day, by the way, I first had this feeling of being "burning"...)

[p. 90] Test categories seem to play a similar role here as (the spectra of) discrete valuation rings in algebraic geometry – they can be mapping into anywhere, to "test" what is going on there – here it means, they can be sent into any modelizer (M, W) (at least among the ones which we feel are the most interesting), by "test functors" $i : A \to M$ giving rise to a model-preserving functor $i^* : M \to A\hat{}$, allowing comparison of M with an elementary modelizer $A\hat{}$. As for the all-encompassing basic modelizer (Cat), it seems to play the opposite role in a sense, at least with respect to elementary modelizers, $A\hat{}$, which all admit modelizing maps $i_A : A\hat{} \to (\mathrm{Cat})$. As a matter of fact, for given test category A, i.e., a given elementary modelizer $A\hat{}$, I see for the time being just *one* way to get a modelizing functor from it to (Cat), namely just the canonical i_A. There is also a striking difference between the exactness properties of the functors

$$i^* : M \to A\hat{} \tag{1}$$

one way, which commute to inverse limits, and the functors

$$i_A : A\hat{} \to (\mathrm{Cat}) \tag{2}$$

in the other direction, commuting to direct limits. Another difference is that we should not expect that the left adjoint $i_!$ to i^* be model preserving too (with the exception of the very special case when $M = (\mathrm{Cat})$ and $i : A \to M = (\mathrm{Cat})$ is the canonical functor $i_A \restriction A$, which appears as highly non-typical in this respect), whereas the right adjoint $j_A = i_A^*$ of i_A is model preserving, this i_A is part of a pair (i_A, j_A) of model preserving adjoint functors.

Of course, we may want to compare directly an arbitrary modelizer M to (Cat) by sending it into (Cat) by a modelizing functor $M \to (\mathrm{Cat})$; we get quite naturally such a functor (for any given choice of test-functor $i : A \to M$)

$$i_A i^* : M \to (\mathrm{Cat}), \tag{3}$$

but this functor is not likely any more to commute neither to direct nor inverse limits, even finite ones – and it isn't too clear that for a modelizer M which isn't elementary, we have much chance to get a modelizing functor to (Cat) which is either left or right exact. However, the functors (3) we've got, whenever modelizing and if M is a *strict* modelizer (namely $M \to W_M^{-1} M \simeq (\mathrm{Hot})$ commutes with finite products), will commute to finite products "up to weak equivalence". Also the functors i^*, although not right exact definitely, have a tendency to commute to sums, and hence the same will hold (not only up to weak equivalence) for (3).

As for getting a modelizing functor (Cat) $\to M$, for a modelizer M which isn't elementary, in view of having a standard way for describing a given homotopy type (defined by an object C in (Cat)) by a "model" in M, depending functorially on C, there doesn't seem to be any general process for finding one, even without any demand on exactness properties, except of course when M is supposed to be elementary; in this case $M = A^\wedge$ we get the functors [p. 91]

$$i^* : (\mathrm{Cat}) \to A^\wedge \tag{4}$$

associated to test functors $A \to$ (Cat), which can be viewed as a particular case of (1), applied to the case $M =$ (Cat). Using such functors (4), we see that the question of finding a modelizing functor

$$\varphi : (\mathrm{Cat}) \to M, \tag{*}$$

for a more or less general M, is tied up with the question of finding such a functor from an elementary modelizer $A^\wedge$ into M

$$\psi : A^\wedge \to M. \tag{**}$$

More specifically, if we got a ψ, we deduce a φ by composing with i^* in (4), and conversely, if we got a φ, we deduce a ψ by composing with the canonical functor i_A in (2). Maybe it's unrealistic to expect modelizing functors (*) or (**) to exist for rather general M. (Which modelizers will turn out to be really "the interesting ones" will appear in due course presumably...) There is one interesting case though when we got such functors, namely when

$$M = (\mathrm{Spaces})$$

is the category of topological spaces, and taking for ψ one of the manifold avatars of "geometric realization functor", associated to a suitable functor

$$r : A \to (\mathrm{Spaces}) \tag{***}$$

by taking the canonical extension $r_!$ to $A^\wedge$, commuting with direct limits. This is precisely the "highly non-typical" case, when we get a pair of adjoint functors $r_!, r^*$

$$A^\wedge \underset{r^*}{\overset{r_!}{\rightleftarrows}} (\mathrm{Spaces}) \tag{5}$$

which are *both* modelizing. The situation here mimics very closely the situation of the pair $(i_A, j_A = i_A^*)$ canonically associated to the elementary modelizer $A^\wedge$, with the "basic modelizer" (Cat) being replaced by (Spaces), which therefore can be considered as another "basic modelizer" of sorts. In this case the corresponding functor

$$r_! r^* : (\mathrm{Cat}) \to (\mathrm{Spaces}) \tag{6}$$

mimics the functor $i_A i^*$ of (3) (where on the left hand side M is taken to be just (Cat), and on the right (Cat) as the basic modelizer is replaced by its next best substitute (Spaces)). Here as in (3), the modelizing functor we got is neither left nor right exact, it has a tendency though to commute to sums, as usual. [p. 92]

I wouldn't overemphasize the capacity of (Spaces) to serve the purpose of a "basic modelizer" as does (Cat), despite the attractive feature of more direct (or at any rate, more conventional) ties with so-called "topological intuition". One drawback of (Spaces) is the relative sophistication of the structure species "topological spaces" it corresponds to, which is by no means an "algebraic structure species", and fits into algebraic formalisms only at the price of detours. More seriously still, or rather as a reflection of this latter feature, only for some rather special elementary modelizers $A\hat{}$, namely rather special test categories A, do we get a geometric realization functor $r_! : A\hat{} \to$ (Spaces) which can be view as part of a pair of mutually adjoint modelizing functors, mimicking the canonical pair (i_A, j_A); still less does there seem to be anything like a really canonical choice (although some choices are pretty natural indeed, dealing with the standard test categories such as $\mathbb{\Delta}$ and its variants). At any rate, it is still to be seen whether there exists such a pair $(r_!, r^*)$ for some rather general class of test categories – this is one among the very many things that I keep pushing off, as more urgent matters are calling for attention...

To sum up the outcome of these informal reflections about various types of modelizing functors between modelizers, the two main types which seem to overtower the whole picture, and are likely to be the essential ones for a general understanding of homotopy models, are the two types (1) and (2) above. The first one i^* is defined in terms of an arbitrary modelizer M. The second i_A, with opposite exactness properties to the previous one, is canonically attached to any test category, and maps the corresponding elementary modelizer $A\hat{}$ into the basic modelizer (Cat), without any reference to more general types of modelizers M. The right adjoint of the latter, which is still model preserving, is in fact of the type (1) again, for the canonical test functor $A \to$ (Cat) induced by i_A, namely $a \mapsto A_{/a}$.

This whole reflection was of course on such an informal level, that there was no sense at that stage to bother with distinctions between weaker or stricter variants of the test-notion. Maybe it's about time now to start getting a little more specific. [p. 93]

46 *Test functors with values in any modelizer: an observation, with an inspiring "silly question".*

First thing to do visibly is to define the notion of a test-functor

$$i : A \to M,$$

where M is any modelizer. Thus M is endowed with a subset $W_M \subset \mathrm{Fl}(M)$, i.e., a notion of weak equivalence, satisfying the "mild saturation conditions" of p. 59, and moreover we assume that $W_M^{-1}M$ is equivalent to (Hot) – but the choice of an equivalence, or equivalently, of the corresponding localization functor

$$M \to (\mathrm{Hot}),$$

is not given with the structure. (If we admit the "inspiring assumption", there is no real choice, as a matter of fact – but we don't want to use this in a technical sense, but only as a guide and motivation.)

Let's start with the weak variant – we assume A to be a weak test category, and want to define what it means that i is a *weak test functor*. In all this game, it is understood that in case $M = (\mathrm{Cat})$, the notions we want to define (of a weak test functor and of a test functor) should reduce to the ones we have pinpointed in yesterday's notes.

The very first idea that comes to mind, is to demand merely that the corresponding i^* (1)

$$i^* : M \to A\hat{}$$

should be modelizing, which means (I recall)

a) $W_M = (i^*)^{-1}(W_A)$.

b) The induced functor $W_M^{-1}M \to W_A^{-1}A\hat{}$ is an equivalence.

This, I just checked, does correspond to the definition we gave yesterday (p. 85), when $M = (\mathrm{Cat})$. There is a very interesting extra feature though in this special case, which appears kind of "in between the lines" in the "key result" on p. 61, and which I want now to state in the more general set-up.

As usual in related situations, the notion of weak equivalence in M gives rise to a corresponding notion of "aspheric" elements in M – namely those for which the unique map

$$x \to e_M$$

is a weak equivalence. We assume now the existence of a final object e_M in M, and will assume too, if necessary, that it's the image in the localization $W_M^{-1}M = H_M$ is equally a final object. Thus, if x in M is aspheric, its image in H_M is a final object, and the converse holds provided as assume W_M strongly saturated, namely any map in M which becomes an isomorphism in H_M is a weak equivalence.

29.3. This assumption will be verified if there exists a weak test functor $i : A \to M$.

I can now state the "interesting extra feature".

[p. 94]

Observation. *For a functor $i : A \to M$ of a weak test category A into the modelizer M (with final object e_M, giving rise to the final object in $H_M = W_M^{-1}M$), and in the special case when $M = (\mathrm{Cat})$, the following conditions are equivalent:*

(i) *i^* transforms weak equivalence into weak equivalences, i.e., induces a functor $H_M \to H_{A\hat{}}$.*

(ii) *i^* transforms aspheric objects into aspheric objects.*

(iii) *i is a weak test functor, namely $W_M = (i^*)^{-1}(W_{A\hat{}})$ (a stronger version of* (i)) and *the induced functor $H_M \to H_{A\hat{}}$ is an* equivalence*.*

Here the obvious implications are of course

$$\text{(iii)} \Rightarrow \text{(i)} \Rightarrow \text{(ii)},$$

the second implication coming from the fact that i^* is compatible with final objects, and that $e_{A\hat{}}$ is aspheric. Of course, (ii) means that for any aspheric x in M, $A_{/i^*(x)}$ is aspheric in (Cat). In case $M =$ (Cat), and when moreover the elements $i(a)$ in (Cat) have final objects (a condition I forgot to include in the statement of the observation above, sorry), this condition was seen to imply (iii) (cf. "key result" on p. 61, (a iv) $\Rightarrow$ (a ii) – indeed, it is even enough to check that for any C *with final element* in (Cat), $i^*(C)$ is aspheric. The proof moreover turns out practically trivial, in terms of the usual asphericity criterion for a functor between categories. So much so that the really amazing strength of the statement, which appears clearly when looked at in a more general setting, as I just did, was kind of blurred by the impression of merely fastidiously grinding through routine equivalences. We got there at any rate quite an interesting class of functors between modelizers (an elementary and the basic one, for the time being), for which the mere fact that the functor be compatible with weak equivalences, or only even take aspheric objects into aspheric ones, implies that the functor in modelizing, namely that the functor $H_M \to H_{A^\wedge}$ it induces (and the very *existence* of this functor was all we demanded beforehand!) is actually an *equivalence of categories*.

The question that immediately comes to mind now, is if this "extra feature" is indeed an extremely special one, strongly dependent on the assumption $M =$ (Cat) and the categories $i(a)$ having final objects – or if it may not have a considerably wider significance. This suggests the still more general questions, involving two modelizers M, M', neither
[p. 95] of which needs by elementary or by (Cat) itself:

Question. Let

$$f : M \to M'$$

be a functor between modelizers (M, W) and (M', W'), assume if needed that f commute with inverse limits, or even has a left adjoint, and that inverse limits (and direct ones too, as for that!) exist in M, M'. Are there some natural conditions we can devise for M and M' (which should be satisfied for elementary modelizers and for the basic modelizer (Cat)), plus possibly some mild extra conditions on f itself, which will ensure that whenever f transforms weak equivalences into weak equivalences, or even only aspheric objects into aspheric objects, f is model-preserving, i.e., $W_M = f^{-1}(W_{M'})$ *and* the induced functor $H_f : H_M \to H_{M'}$ on the localizations is an equivalence of categories?

Maybe it's a silly question, with pretty obvious negative answer – in any case, I'll have to find out! The very first thing to check is to see what happens in case of a functor

$$i^* : M \to A^\wedge,$$

where $A^\wedge$ is a weak elementary modelizer, and where M is either (Cat) or another weak elementary modelizer $B^\wedge$, i^* in any case being associated of course to a functor

$$i : A \to M,$$

with a priori no special requirement whatever on i. In case $M =$ (Cat), this means looking up in the end the question we have postponed for quite a while now, namely of how to rescue the "key result" of p. 61, when dropping the assumption that the categories $i(a)$ (for a in A) have final objects. We finally got a strong motivation for carrying through a generalization, if this is indeed feasible.

30.3.

An approach for handling (Cat)-valued test functors, and promise of a "key result" revised. The significance of contractibility.

47 It had become clear that the most urgent thing to do now was to come to a better understanding of test functors with values in (Cat), when dropping the assumption that the categories $i(a)$ have final objects, and trying to replace this (if it should turn out that something *is* needed indeed) by a kind of assumption which should make sense when (Cat) is replaced by a more or less arbitrary modelizer M. I spent a few hours pondering over the situation, and it seems to me that in the case at least when A is a *strict*, namely when $A^\wedge$ is totally aspheric, there is now a rather complete understanding of the situation, with a generalization of the "key result" of p. 61 which seems to be wholly satisfactory.

The basic idea of how to handle the more general situation, namely how to compare the categories $A_{/i^*(C)} = A_{/C}$ and C, and show (under suitable assumptions) that there is a canonical isomorphism between their images in the localized category $W_{(\mathrm{Cat})}^{-1}(\mathrm{Cat}) = (\mathrm{Hot})$, was around since about the moment I worked out the "key result". It can be expressed by a diagram of "maps" in (Cat) [p. 96]

(1)
$$\begin{tikzcd} A_{/C} \arrow[r] & A_{/\!\!/C} \\ A \times C \arrow[r] & C \arrow[u] \end{tikzcd} \quad ,$$

where $A_{/\!\!/C}$ is the fibered category over A, associated to the functor

$$A^{\mathrm{op}} \to (\mathrm{Cat}), \quad a \mapsto \mathbf{Hom}(i(a), C).$$

Here one should be careful with the distinction between the *set*

$$\mathrm{Hom}(i(a), C) = \mathrm{Ob}\,\mathbf{Hom}(i(a), C),$$

and the *category* $\mathbf{Hom}(i(a), C)$, both depending bi-functorially on a in A and C in (Cat). The former (as a presheaf on A for fixed C) gives rise to $A_{/C}$, a fibered category over A with *discrete* fibers, whereas the latter gives rise to $A_{/\!\!/C}$, which is fibered over A with fibers that need not be discrete. Identifying a set with the discrete category it defines, we get a canonical functor

$$\mathrm{Hom}(i(a), C) \to \mathbf{Hom}(i(a), C),$$

which is very far from being an equivalence nor even a weak equivalence; being functorial for varying a, it gives rise to the first map in (1). The second is deduced from the canonical functor

$$C \to \mathbf{Hom}(i(a), C),$$

identifying C with the full subcategory of *constant* functors from $i(a)$ to C. This map is functorial in a, and gives rise again to a cartesian functor between the corresponding fibered categories over A, the first one (which corresponds to the constant functor $A^{\mathrm{op}} \to (\mathrm{Cat})$ with value C) is just $A \times C$ fibered over A by pr_1, hence the second arrow in (1). The third arrow is just pr_2.

It seems that, with the introduction of $A_{/\!/C}$, this is the first time since the beginning of these reflections that we are making use of the *bicategory structure* of (Cat), namely of the notion of a morphism or map or "homotopy" (the tie with actual homotopies will be made clear below), between two "maps" namely (here) functors $C' \rightrightarrows C$. There is of course a feeling that such a notion of homotopy should make sense in a more or less arbitrary modelizer M, and that the approach displayed
[p. 97] by the diagram (1) may well generalize to mere general situations still, with (Cat) replaced by such an M.

In the situation here, the work will consist in devising handy conditions on $i : A \to (\mathrm{Cat})$ and A that will ensure that all three maps in (1) are weak equivalences, for any choice of C. This will imply that the corresponding maps in (Hot) are isomorphisms, hence a canonical isomorphism between the images in (Hot) of $A_{/C}$ and C, which will imply that a) the functor $i_A i^*$ from (Cat) to (Cat) carries weak equivalences into weak equivalences, and hence induces a functor

$$(\mathrm{Hot}) \to (\mathrm{Hot}),$$

and b) that this functor is isomorphic to the identity functor. If moreover A is a weak test category, and therefore the functor

$$W_A^{-1} A^{\wedge} \to (\mathrm{Hot})$$

induced by i_A is an equivalence, it will follow that the functor

$$(\mathrm{Hot}) \to W_A^{-1} A^{\wedge}$$

induced by i^* is equally an equivalence, namely that i^* is indeed a test-functor.

For the map

$$A \times C \to C \tag{2}$$

to be a weak equivalence for any C in (Cat), it is necessary and sufficient that A be aspheric (cf. par. 40, page 69), a familiar condition on A indeed! For handling the map

$$A \times C \to A_{/\!/C} \quad \text{associated to } C \to \mathbf{Hom}(i(a), C), \tag{3}$$

we'll use the following easy result (which I'll admit for the time being):

Proposition. *Let F and G be two categories over a category A, and $u : F \to G$ a functor compatible with projections. We assume*

a) *For any a in A, the induced map on the fibers*

$$u_a : F_a \to G_a$$

is a weak equivalence.

b) *Either F and G are both cofibering over A and u is cocartesian, or F and G are fibering and u is cartesian.*

Then u is a weak equivalence.

This shows that a sufficient condition for (3) to be a weak equivalence, is that the functors $C \to \mathbf{Hom}(i(a), C)$ be a weak equivalence, for any a in A and C in (Cat). We'll see in the next section that this amounts to demanding that the objects $i(a)$ in (Cat) should be "contractible", in the most concrete sense of this expression, which is actually stronger than just asphericity. (Earlier in these notes I was a little floppy with the terminology, by using a few times the word "contractible" as synonymous to "aspheric" as in the context of topological spaces, or CW spaces at any rate, the two notions do indeed coincide, finally I came to use rather the word "aspheric" systematically, as it fits nicely with the notion of an aspheric morphism of topos. . .) The most evident example of contractible categories are the categories with final object. Thus I have the strong feeling that the condition of contractibility of the objects $i(a)$ in (Cat) is "the right" generalization of the assumption made in the "key result", namely that the $i(a)$ have final elements. Also, it seems now likely that the numerous cases of statements, when to check some property for arbitrary C, it turned out to be enough to check it for C with a final object, may well generalize to more general cases, with (Cat) replaced by some M and the reduction is from arbitrary C in M to contractible ones.

9.4. Actually, this is done only a lot later, on page 121 and (for the converse) on page 143.

[p. 98]

The next thing to do is to develop a little the notion of contractibility of objects and of homotopies between maps, and to get the criterion just announced for (3) to be a weak equivalence for any C. After this, handling the question of the first map in (1)

$$A_{/C} \to A_{/\!\!/C} \tag{4}$$

being a weak equivalence for any C, in terms of the asphericity criterion for a functor, will turn out pretty much formal, and we'll finally be able to state a new version of the "key result" about test functors $i : A \to$ (Cat), with this time twice as many equivalent formulations of the same property. On n'arrête pas le progrès!

31.3.

48 *A journey through abstract homotopy notions (in terms of a set W of "weak equivalences").*

It's time now to develop some generalities about homotopy classes of maps, the relation of homotopy between objects of a modelizer, and the corresponding notion of contractibility. For the time being, it will be enough to start with any category M, endowed with a set $W \subset \mathrm{Fl}(M)$ of arrows (the "weak equivalences"), satisfying the mild saturation assumptions a)b)c) of par. 37 (p. 59). On M we'll assume for the time being that there exists (at least one) homotopy interval $\mathbb{I} = (I, \delta_0, \delta_1)$ in M (loc. sit.) which implies also that M has a final object, which I denote by e_M or simply e, and equally an initial element $\varnothing_M$. I'm not too sure yet whether we'll really need that the latter be strict initial element, as required in the definition of a homotopy interval on page 59 (it was used in the generalities of pages 59 and 60 only for the corollary on the Lawvere element...). Whether or not will appear soon enough! I'll assume it till I am forced to. To be safe, we'll assume on the other hand that M admits binary products.

[p. 99] Let X, Y be objects of M, and

$$f, g : X \rightrightarrows Y$$

two maps in M from X to Y. One key notion constantly used lately (but so far only when f is an identity map, and g a "constant" one – which is equally the case needed for defining contractibility) is the notion of an $\mathbb{I}$*-homotopy from f to g*, namely a map $X \times I \xrightarrow{h} Y$ making commutative the diagram

$$\begin{tikzcd} & X \times I \arrow[d, "h"] & \\ X \arrow[ur, "\mathrm{id}_X \times \delta_0"] \arrow[r, phantom] \arrow[dr, "f"'] & {} & X \arrow[ul, "\mathrm{id}_X \times \delta_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd} .$$

Let's first restate the "homotopy lemma" of page 60 in a slightly more complete form:

Homotopy lemma reformulated. *Assume f and g are $\mathbb{I}$-homotopic. Then:*

a) $\gamma(f) = \gamma(g)$, *where*

$$\gamma : M \to W^{-1}M = H_M$$

is the canonical functor

b) *If f is a weak equivalence, so is g (and conversely of course, by symmetry of the roles of δ_0 and δ_1).*

c) *Assume f is an isomorphism, and g constant – then $X \to e$ and $Y \to e$ are W-aspheric.*

It is important to notice that the relation "f is $\mathbb{I}$-homotopic to g" in $\mathrm{Hom}(X, Y)$ is not necessarily symmetric nor transitive, and that it depends on the choice of the homotopy interval $\mathbb{I} = (I, \delta_0, \delta_1)$. Thus the symmetric relation from $\mathbb{I}$-homotopy is $\check{\mathbb{I}}$-homotopy, where $\check{\mathbb{I}}$ is the homotopy interval "opposite" to $\mathbb{I}$ (namely with δ_0, δ_1 reversed). The example we are immediately interested in is $M = (\mathrm{Cat})$, with $W = W_{(\mathrm{Cat})}$ the usual notion of weak equivalence. A homotopy interval is just an aspheric small category I, endowed with two *distinct* objects e_0, e_1. (The condition $e_0 \neq e_1$ just expresses the condition $e_0 \cap e_1 = \varnothing$ on homotopy intervals – if it were not satisfied, $\mathbb{I}$-homotopy would just mean equality of f and g...) For the usual choice $I = \Delta_1 = (e_0 \to e_1)$, an $\mathbb{I}$-homotopy from f to g is just a morphism between functors $f \to g$ – the $\mathbb{I}$-homotopy relation between f and g is the existence of such a morphism, it is a transitive, and generally non-symmetric relation. If we take I to be a category with just two objects e_0 and e_1, equivalent to the final category, an $\mathbb{I}$-homotopy between f and g is just an isomorphism from f to g – the $\mathbb{I}$-homotopy relation now is both transitive and symmetric, and it is a lot more restrictive than the previous one. If we take I to be the barycentric subdivision of Δ_1, which can also be interpreted as an amalgamated sum of Δ_1 with itself, namely [p. 100]

$$I = \begin{matrix} e_0 \searrow \\ \qquad e_2 \\ e_1 \nearrow \end{matrix},$$

an $\mathbb{I}$-homotopy from f to g is essentially a triple (k,u,v), with $k : X \to Y$ and $u : f \to k$ and $v : g \to k$ maps in $\mathbf{Hom}(X,Y)$; this time, the relation of $\mathbb{I}$-homotopy is symmetric, but by no means transitive. Returning to general M, it is customary to introduce the equivalence relation in $\mathrm{Hom}(X,Y)$ generated by the relation of $\mathbb{I}$-homotopy – we'll say that f and g are $\mathbb{I}$-*homotopic in the wide sense*, and we'll write

$$f \underset{\mathbb{I}}{\sim} g,$$

if they are equivalent with respect to this relation. As seen from the examples above (where $M = (\mathrm{Cat})$), this relation still depends on the choice of the homotopy interval $\mathbb{I}$. Let's first look at what we can do for fixed $\mathbb{I}$, and then how what we do depends on $\mathbb{I}$.

If f and g are $\mathbb{I}$-homotopic, then so are their composition with any $Y \to Z$ or $T \to X$. This implies that the relation $\underset{\mathbb{I}}{\sim}$ of $\mathbb{I}$-homotopy in the wide sense is compatible with compositions. If we denote by

$$\mathrm{Hom}(X,Y)_{\mathbb{I}}$$

the quotient set of $\mathrm{Hom}(X,Y)$ by the equivalence relation of $\mathbb{I}$-homotopy in the wide sense, we get composition between the sets $\mathrm{Hom}(X,Y)_{\mathbb{I}}$, and hence a structure of a category $M_{\mathbb{I}}$ having the same objects as M, and where maps from X to Y are $\mathbb{I}$-homotopy classes (in the wide sense – this will be understood henceforth when speaking of homotopy classes) of maps from X to Y in M. Two objects X, Y of M, i.e., of $M_{\mathbb{I}}$ which are isomorphic as objects of $M_{\mathbb{I}}$ will be called $\mathbb{I}$-*homotopic*. This means also that we can find maps in M (so-called $\mathbb{I}$-*homotopisms* – namely $M_{\mathbb{I}}$-isomorphisms)

$$f : X \to Y, \quad g : Y \to X$$

such that we get $\mathbb{I}$-homotopy relations in the wide sense

$$gf \underset{\mathbb{I}}{\sim} \mathrm{id}_X, \quad fg \underset{\mathbb{I}}{\sim} \mathrm{id}_Y.$$

We'll say X is $\mathbb{I}$-*contractible* if X is $\mathbb{I}$-homotopic to the final object $e_M = e$ of M (which visibly is also a final object of $M_{\mathbb{I}}$), i.e., if X is a final object of $M_{\mathbb{I}}$. In terms of M, this means that there exists a section f of X over e, such that $f p_X$ is $\mathbb{I}$-homotopic in the wide sense to id_X (where p_X is the unique map $X \to e$). In fact, if there is such a section f, any other section will do too.

[p. 101] From the homotopy lemma a) it follows that the canonical functor $M \to W^{-1}M = H_M$ factors into

$$M \to M_{\mathbb{I}} \to H_M = W^{-1}M,$$

and from b) it follows that if $f, g : X \rightrightarrows Y$ are in the same $\mathbb{I}$-class, then f is in W iff g is, hence by passage to quotient a subset

$$W_{\mathbb{I}} \subset \mathrm{Fl}(M_{\mathbb{I}})$$

of the set of arrows in $M_{\mathbb{I}}$, namely a notion of weak equivalence in $M_{\mathbb{I}}$. It is evident from the universal property of H_M that the canonical functor $M_{\mathbb{I}} \to H_M$ induces an isomorphism of categories

$$H_{M_{\mathbb{I}}} = W_{\mathbb{I}}^{-1} M_{\mathbb{I}} \xrightarrow{\sim} H_M = W^{-1}M.$$

It's hard at this point not to expect that $W_{\mathbb{I}}$ should satisfy the same mild saturation conditions as W, so let's look into this in the stride (even though I have not had any use of this so far). Condition b) of saturation, namely that if f, g are composable and two among f, g, gf are in W, so is the third, carries over trivially. Condition a), namely the tautological looking condition that W should contain all isomorphisms, makes already a problem, however. It is OK though if W satisfies the following saturation condition, which is a strengthening of condition c) of page 59:

c') Let $f : X \to Y$ and $g : Y \to X$ such that $gf \in W$ and $fg \in W$, then $f, g \in W$.

This condition c') carries over to $M_{\mathbb{I}}$ trivially. This suggests to introduce a strengthening of the "mild saturation conditions", which I intend henceforth to call by the name of "saturation", reserving the term of "*strong saturation*" to what I have previously referred to occasionally as "saturation" – namely the still more exacting condition that W consists of *all* arrows made invertible by the localization functor $M \to H_M = W^{-1}M$, or equivalently, by some functor $M \to H$. Thus, we'll say W is *saturated* iff it satisfies the following:

a') For any X in M, $\mathrm{id}_X \in W$.

b') Same as b) before: if two among f, g, gf are in W, so is the third.

c') If $f : X \to Y$ and $g : Y \to X$ are such that $gf, fg \in W$, then $f, g \in W$.

Each of these conditions carries over from W to a $W_{\mathbb{I}}$ trivially.

I can't help either having a look at the most evident exactness properties of the canonical functor $M \to M_{\mathbb{I}}$: Thus one immediately sees that for two maps

$$f, g : X \rightrightarrows Y = Y_1 \times Y_2,$$

with components f_i, g_i $(i \in \{1,2\})$, f and g are $\mathbb{I}$-homotopic in the wide sense iff so are f_i and g_i (for $i \in \{1,2\}$). The analogous statement is valid for maps into any product object $Y = \prod Y_i$ on a finite set of indices. [p. 102] The dual statement so to say, when X is decomposed as a sum $X = \coprod X_i$, is valid too, provided taking products with I is distributive with respect to finite direct sums. Thus we get that $M \to M_{\mathbb{I}}$ commutes with finite products, and with finite sums too provided they are distributive with respect to multiplication with any object (or with I only, which would be enough).

The notion of $\mathbb{I}$-homotopy in the wide sense between $f, g : X \rightrightarrows Y$ can be interpreted in terms of strict $\mathbb{I}'$-homotopy with variable $\mathbb{I}'$, as follows, provided we make some mild extra assumptions on (M, W), namely:

a) (Just for memory) M is stable under finite products.

b) M is stable under amalgamated sums $I \amalg_e J$ under the final object e (I and J endowed with sections over e).

c) If moreover I and J are aspheric over e, then so is $I \amalg_e J$.

Conditions b) and c) give a means of constructing new homotopy intervals $\mathbb{K}$, by amalgamating two homotopy intervals $\mathbb{I}$ and $\mathbb{J}$, using as sections of $\mathbb{I}$ and $\mathbb{J}$ for making the amalgamation, either δ_0 or δ_1, which gives four ways of amalgamating – of course we take as endpoints of the amalgamated interval, the sections over e coming from the two endpoints of $\mathbb{I}$ (giving rise to δ_0 for $\mathbb{K}$) and $\mathbb{J}$ (giving rise to δ_1 for $\mathbb{K}$) which have not been "used up" in the amalgamation. Maybe the handiest convention is to define *the* amalgamated interval $\mathbb{I} lor \mathbb{J}$, without any ambiguity of choice, as being

$$\mathbb{I} \vee \mathbb{J} = (I, \delta_1^{\mathbb{I}}) \amalg_e (J, \delta_0^{\mathbb{J}}) \qquad \text{endowed with the two sections coming from } \delta_0^{\mathbb{I}} : e \to I,\ \delta_1^{\mathbb{I}} : e \to J,$$

and defining the three other choices in terms of this operation, by replacing one or two among the summands $\mathbb{I}$, $\mathbb{J}$ by the "opposite" interval $\check{\mathbb{I}}$ or $\check{\mathbb{J}}$. The operation of amalgamation of intervals, and likewise of homotopy intervals, just defined, is clearly associative up to a canonical isomorphism, and we have a canonical isomorphism of intervals

$$(\mathbb{I} \vee \mathbb{J})^{\vee} \simeq \check{\mathbb{I}} \vee \check{\mathbb{J}}.$$

I forgot to check, for amalgamation of homotopy intervals, the condition b) of page 59, namely $e_0 \cap e_1 = \varnothing_M$ (which has not so far played any role, anyhow). To get this condition, we'll have to be slightly more specific in condition b) above on M of existence of the relevant amalgamations, by demanding (as suggested of course by the visual intuition of the

situation) that I and J should become subobjects of the amalgamation K, and their intersection should be reduced to the tautological part e of it. More relevant still for the use we have in mind is to demand that those amalgamations should commute to products by an arbitrary
[p. 103] element X of M. This I'll assume in the interpretation of $\underset{\mathbb{I}}{\sim}$ in terms of strict homotopies. Namely, let

$$\mathrm{Comp}(\mathbb{I})$$

by the set of all homotopy intervals deduced from $\mathbb{I}$ by taking amalgamations of copies of $\mathbb{I}$ and $\check{\mathbb{I}}$, with an arbitrary number $n \geq 1$ of summands. Thus we get just $\mathbb{I}$ and $\check{\mathbb{I}}$ for $n = 1$, four intervals for $n = 2$, $\dots$, 2^n intervals for n arbitrary. It is now immediately checked that for f, g in $\mathrm{Hom}(X, Y)$, the relation $f \underset{\mathbb{I}}{\sim} g$ is equivalent to the existence of $\mathbb{K} \in \mathrm{Comp}(\mathbb{I})$, such that f and g be $\mathbb{K}$-homotopic (in the strict sense).

Remark. The saturation conditions a')b')c') on W are easily checked for the usual notion of weak equivalence for morphisms of topoi, and hence also in the categories (Cat) and in any topos, and therefore in any category $A^\wedge$ (where it boils down too to the corresponding properties on $W_{(\mathrm{Cat})}$, as $W_A = i_A^{-1}(W_{(\mathrm{Cat})})$). Thus it seems definitely reasonable henceforth to take these as the standard notion of saturation (referring to its variants by the qualifications "mild" or "strong"). On the other hand, the stability conditions a)b)c) on (M, W) are satisfied whenever M is a topos, with the usual notion of weak equivalence – the condition c) being a consequence of the more general Mayer-Vietoris type statement about amalgamations of topoi under closed embeddings of such (cf. lemma on page [?]). The same should hold in (Cat), with a similar Mayer-Vietoris argument – there is a slight trouble here for applying the precedent result on amalgamation of topoi, because a section $e \to C$ of an object C of (Cat), namely an embedding of the one-point category $e = \mathbb{\Delta}_0$ into C by choice of an object of C, does not correspond in general to a closed embedding of topoi. (In geometrical language, we get a "point" of the topos $C^\wedge$ defined by C, but a point need not correspond to a subtopos, let alone a closed one...) This shows the asphericity criterion for amalgamation of topoi, and hence also for amalgamation of categories, has not been cut out with sufficient generality yet. As this whole $\mathrm{Comp}(\mathbb{I})$ story is just a digression for the time being, I'll leave it at that now.

More important than amalgamation of intervals, is to compare the notions of homotopy defined in terms of a homotopy interval $\mathbb{I}$, to the corresponding notions for another interval, $\mathbb{J}$. Here the natural idea first is to see what happens if we got a morphism of intervals (compatible with endpoints, by definition)

$$\mathbb{J} \to \mathbb{I}.$$

[p. 104] It is clear then, for $f, g \in \mathrm{Hom}(X, Y)$, that any $\mathbb{I}$-homotopy from f to g gives rise to a $\mathbb{J}$-homotopy; hence if f and g are $\mathbb{I}$-homotopic, they are

$\mathbb{J}$-homotopic, and hence the same for homotopy in the wide sense. We get thus a canonical functor

$$M_{\mathbb{I}} \to M_{\mathbb{J}}$$

which is the identity on objects, entering into a cascade of canonical functors

$$M \to M_{\mathbb{I}} \to M_{\mathbb{J}} \to H,$$

where $H = H_M = W^{-1}M$ can be viewed as the common localization of $M, M_{\mathbb{I}}, M_{\mathbb{J}}$ with respect to the notion of weak equivalences in these categories. We may view $M_{\mathbb{J}}$ as a closer approximation to H than $M_{\mathbb{I}}$. There is of course an evident transitivity relation for the functors corresponding to two composable morphisms of intervals

$$\mathbb{K} \to \mathbb{J} \to \mathbb{I}.$$

Remark. In order to get that $\underset{\mathbb{I}}{\sim}$ implies $\underset{\mathbb{J}}{\sim}$, it is sufficient to make a much weaker assumption than existence of a morphism of intervals $\mathbb{J} \to \mathbb{I}$ – namely it suffices to assume that the two sections $\delta_i^{\mathbb{I}} : e \to I$ are $\mathbb{J}$-homotopic. More generally, let $\sim$ be an equivalence relation in $\mathrm{Fl}(M)$, compatible with compositions and with cartesian products (this is the case indeed for $\underset{\mathbb{J}}{\sim}$), and let $\mathbb{I} = (I, \delta_0, \delta_1)$ any object I of M (not necessarily aspheric over e) endowed with two sections over e, such that $\delta_0 \sim \delta_1$. Then the interval $\mathbb{I}$ gives rise to an equivalence relation $\underset{\mathbb{I}}{\sim}$ in $\mathrm{Fl}(M)$, whose definition is quite independent of W – and a priori, if $f \underset{\mathbb{I}}{\sim} g$ and $f \in W$, this need not imply $g \in W$. However, the condition $\delta_0 \sim \delta_1$ implies immediately that the relation $\underset{\mathbb{I}}{\sim}$ implies the relation $\sim$. When the latter is $\underset{\mathbb{J}}{\sim}$, we get moreover that W is the inverse image of a set of arrows in $M_{\mathbb{I}}$, i.e., $f \underset{\mathbb{I}}{\sim} g$ and $f \in W$ implies $g \in W$.

An interesting particular case is the one when we can find a homotopy interval $\mathbb{I}_0$ in M, which has the property that for any other homotopy interval $\mathbb{I}$ in M, its structural sections satisfy

$$\delta_0^{\mathbb{I}} \underset{\mathbb{I}_0}{\sim} \delta_1^{\mathbb{I}}.$$

This implies that the homotopy relation $\underset{\mathbb{I}_0}{\sim}$ is implies by all other similar relations $\underset{\mathbb{I}}{\sim}$, i.e., it is the coarsest among all relations $\underset{\mathbb{I}}{\sim}$. We may then view $\mathbb{I}_0$ as a "*fundamental*" or "*characteristic*" *homotopy interval* in M, in the sense that the relation $f \underset{W}{\sim} g$ in the sense below, namely existence of a homotopy interval $\mathbb{I}$ such that $f \underset{\mathbb{I}}{\sim} g$, is equivalent to $f \underset{\mathbb{I}_0}{\sim} g$, i.e., can be check using the one and unique $\mathbb{I}_0$. In the case of $M = (\mathrm{Cat})$, we get readily $\triangle_1$ as a characteristic homotopy interval. More specifically, if $\delta_0, \delta_1 : e \to I$ are two sections of an object I of (Cat), i.e., two objects e_0, e_1 of the small category I, then these are $\triangle_1$-homotopic iff e_0, e_1 [p. 105] belong to the same connected component of I, which is automatic if I is 0-connected, and a fortiori if I is aspheric. This accounts to a great extent, it seems, for the important role $\mathbb{\Delta}_1$ is playing in the homotopy theory of (Cat), and consequently in the whole foundational set-up I am developing here, using (Cat) as the basic "modelizer".

It is not clear to me for the time being whether it is reasonable to expect in more or less any modelizer (M, W) the existence of a characteristic homotopy interval (provided of course a homotopy interval exists). This is certainly the case for the elementary modelizers met so far. Presumably, I'll have to come back upon this question sooner or later.

We'll now see that the set of equivalence relations $\underset{\mathbb{I}}{\sim}$ on $\mathrm{Fl}(M)$, indexed by the set of homotopy intervals $\mathbb{I}$, is "filtrant décroissant", namely that for two such relations $\underset{\mathbb{I}}{\sim}$ and $\underset{\mathbb{J}}{\sim}$, there is a third "wider" one, $\underset{\mathbb{K}}{\sim}$, implied by both. It is enough to construct a $\mathbb{K}$, endowed with morphisms

$$\mathbb{K} \to \mathbb{I}, \quad \mathbb{K} \to \mathbb{J}$$

of homotopy intervals. Indeed, there is a universal choice, namely the category of homotopy intervals admits binary products – we'll take thus

$$\mathbb{K} = \mathbb{I} \times \mathbb{J},$$

where the underlying object of $\mathbb{K}$ is just $I \times J$, endowed with the two sections $\delta_i^{\mathbb{I}} \times \delta_i^{\mathbb{J}}$ $(i \in \{0, 1\})$.

As usual, we'll denote by $\underset{W}{\sim}$ or simply $\sim$ the equivalence relation on $\mathrm{Fl}(M)$, which is the limit or union of the the equivalence relations $\underset{\mathbb{I}}{\sim}$ – in other words

$$f \underset{W}{\sim} g \quad \text{iff exists } \mathbb{I}\text{, a homotopy interval in } M\text{, with } f \underset{\mathbb{I}}{\sim} g.$$

For X, Y in M, we'll denote by

$$\mathrm{Hom}(X, Y)_W$$

the quotient of $\mathrm{Hom}(X, Y)$ by the previous equivalence relation. This relation is clearly compatible with compositions, and hence we get a category M_W, having the same objects as M, which can be equally viewed as the filtering limit of the categories $M_{\mathbb{I}}$,

$$M_W = \varinjlim_{\mathbb{I}} M_{\mathbb{I}},$$

where we may take as indexing set for the limit the set of all $\mathbb{I}$'s, pre-ordered by $\mathbb{I} \le \mathbb{J}$ iff there exists a morphism of intervals $\mathbb{J} \to \mathbb{I}$ (it's the
[p. 106] preorder relation opposite to the usual one on the set of objects of a category...).

We now get canonical functors

$$M \to M_{\mathbb{I}} \to M_W \to H,$$

where H can again be considered as the common localization of the three categories M, $M_{\mathbb{I}}$ (any $\mathbb{I}$), M_W with respect to the notion of weak equivalence in each. It is clear that the set of weak equivalences in M_W, say $\overline{W}$, is saturated provided W is. Also, the canonical functor $M \to M_W$ commutes with finite products, and also with finite sums provided formation of such sums in M commutes with taking products with any fixed element of M.

A map $f : X \to Y$ is called a *homotopy equivalence* or a *homotopism* (with respect to W) if it is an isomorphism in M_W, namely if there exists $g : Y \to X$ such that

$$gf \underset{W}{\sim} \mathrm{id}_X, \quad fg \underset{W}{\sim} \mathrm{id}_Y.$$

This implies that f and g are weak equivalences. If such an f exists, namely if X and Y are isomorphic objects of M_W, we'll say they are W*-homotopic*, or simply *homotopic*. This implies that there exist weak equivalences $X \to Y$ and $Y \to X$, but the converse of course isn't always true. If Y is the final object e_M, we'll say X is W*-contractible* (or simply *contractible*) instead of W-homotopic to $Y = e$. In case there exists a characteristic homotopy interval $\mathbb{I}_0$, all these W-notions boil down to the corresponding $\mathbb{I}_0$-notions considered before.

Remark. If occurs to me that in all what precedes, we never made any use really of the existence of a homotopy interval! The only notion we have effectively been working with, it seems to me, is the notion of "weak homotopy interval", by which I mean a triple $\mathbb{I} = (I, \delta_0, \delta_1)$ satisfying merely the condition that I be aspheric over e (which is the condition c) on page 59). Such an $\mathbb{I}$ always exists of course, we need only take $I = e$ itself! In case however an actual homotopy interval (with $\mathrm{Ker}(\delta_0, \delta_1) = \varnothing_M$) does exist, to make sure that the notion of W-homotopy obtained by using all weak homotopy intervals is the same when using only actual homotopy intervals, we should be sure that for any I aspheric over e, any two sections δ_0, δ_1 of I over e are W-homotopic in the initial meaning. This is evidently so when $M = (\mathrm{Cat})$ and $W = W_{(\mathrm{Cat})}$ (indeed, it is enough that I be 0-connected, instead of being aspheric), and in the case when M is one among the usual elementary modelizers. The question of devising notions which still [p. 107]
make sense when there is no homotopy interval in M isn't perhaps so silly after all, if we remember that the category of semi-simplicial face complexes (namely without degeneracies) is indeed a modelizer, but it hasn't hot a homotopy interval. However, for the time being I feel it isn't too urgent to get any more into this.

49 I would like now to elaborate a little on the notion of a contractible element X in M, which (I recall) means an object, admitting a section $e_0 : e \to X$, such that the constant map

Contractible objects. Multiplicative intervals.

$$c = p_X e_0 : X \to e \to X$$

is homotopic to id_X, i.e., there exists a homotopy interval I such that $\mathrm{id}_X \underset{\mathbb{I}}{\sim} c$ ($\mathbb{I}$-homotopy in the wide sense).

If X is contractible, then the constant map $c : X \to X$ is a weak equivalence (as it is homotopic to id_X which is) and hence by the saturation condition c') (in fact the mild saturation condition c) suffices) it follows that $p_X : X \to e$ is a weak equivalence. In fact, one would expect it is even universally so, i.e., that p_X is an *aspheric* map, as a plausible generalization of the homotopy lemma c) above (which we have used already a number of times as our main asphericity criterion in elementary $A\hat{}\ldots$). The natural idea to prove asphericity of p_X, namely that for any base change $S \to e$, the projection $X_S = X \times S \to S$ is a weak equivalence, is to apply the precedent criterion to X_S, viewed as an element of $M_{/S}$. As the base change functor commutes with products, it is clear indeed that for two maps $f, g : X \rightrightarrows Y$ in M, $\mathbb{I}$-homotopy of f and g will imply $\mathbb{I}_S$-homotopy of f_S and g_S, with evident definition of base change for an "interval". On the other hand, asphericity of $\mathbb{I}$ over e implies tautologically asphericity of $\mathbb{I}_S$ over S, the final object of $M_{/S}$, which is all we need to care about to get the result that $X_S \to S$ is a weak equivalence. (The condition b) on homotopy intervals is definitely misleading here, as it would induce us to put the extra condition that the base change functor is compatible with initial elements, which is true indeed if $\varnothing_M$ is strict, but here a wholly extraneous condition...)

Thus X *contractible implies* X *aspheric over* e. It is well-known that the converse isn't true, already for $M = (\text{Ss sets})$, or $M = (\text{Spaces})$ with the usual notions of weak equivalences, taking aspheric complexes which are not Kan complexes, or "aspheric" spaces (in the sense of singular cohomology) which are not CW-spaces. The same examples show, too,
[p. 108] that even a good honest homotopy interval need not be contractible, contrarily to what one would expect from the intuitive meaning of an "interval". In those two examples, the condition of getting two "disjoint" sections of the aspheric I over e looks kind of trivial – kind of unrelated to the question of whether or not I is actually contractible, and not only aspheric. What comes to mind here is to look for *contractible homotopy intervals*, namely homotopy intervals (I, δ_0, δ_1) (including condition b) that $\mathrm{Ker}(\delta_0, \delta_1) = \varnothing_M$) such that I is not only aspheric over e, but even contractible. Existence of contractible homotopy intervals seems a priori a lot stronger than just existence of homotopy intervals, but in the cases I've met so far, the two apparently coincide. As a matter of fact, the first choice of homotopy interval that comes to mind, in all those examples, *is* indeed a contractible one – so much so that till the notes today I was somewhat confused on this matter, and was under the tacit impression that homotopy intervals are all contractible, and trivially so! But probably, as usual with most evidently false impressions, underneath, something correct should exist, worth being explicitly stated.

First thing that comes to mind is here to look at the simplest case of an aspheric interval $\mathbb{I}$ which is contractible – namely when $\mathbb{I}$ is even $\mathbb{I}$-contractible, and more specifically still, when there exists an $\mathbb{I}$-homotopy (an "elementary", not a "composed" one!) from id_I to one of the two constant maps of I into itself, defined by the two sections δ_0, δ_1 – say by δ_1, to be specific. Such a homotopy is a map

$$h : I \times I \to I$$

in M, having the two properties expressed symbolically by

$$h(e_0, x) = x, \quad h(e_1, x) = e_1,$$

where x may be viewed as any "point" of I with "values" in an arbitrary parameter object T of M, i.e., $x : T \to I$, and e_0, e_1 are the constant maps $T \to I$ defined by δ_0, δ_1. If we view h as a composition law on I, these relations just mean that e_0 acts as a left unit, and e_1 as a left zero element. Such composition law in an interval I, a would-be homotopy interval as a matter of fact, has been repetitively used before, and this use systematized in the "comparison lemma for homotopy intervals" (page 60); there we found that if I is an object of M with such a composition law, and if there exists an *aspheric* interval $\mathbb{J}$ and a morphism of intervals from $\mathbb{J}$ into $\mathbb{I}$, then I is equally aspheric. This we would now see as coming from the fact that I being $\mathbb{I}$-contractible is $\mathbb{J}$-contractible, and a fortiori (as $\mathbb{J}$ is aspheric over e) aspheric over e. In any case, we see that when I is endowed with a composition law as above, then I is aspheric over e iff it is contractible, and equivalently iff δ_0 and δ_1 are homotopic.

In case when there exists a Lawvere element L_M in M, for instance if M is a topos, this element is automatically endowed with an idempotent composition law, coming from intersection of subobjects, and in case M admits a strict initial object, L_M is endowed with two canonical sections which are indeed "disjoint", corresponding respectively to the "full" and the "empty" subobjects. Then (as already noticed) previously in the corollary of the comparison lemma) M admits a homotopy interval iff L_M is such an interval, namely is aspheric over e_M. But we can now add that in this case, there exists even a *contractible* homotopy interval, namely L_M itself. It is even a "*strict homotopy interval*", namely one admitting a composition $L \times L \to L$ having the properties above (where δ_0 and δ_1, for the first time, play asymmetric roles!). Thus when a Lawvere element exists in M, there is an equivalence between the three properties we may expect from a pair (M, W), with respect to homotopy intervals, namely: [p. 109]

$$(\exists\text{homotopy int.}) \Leftarrow (\exists\text{contractible hom. int.}) \Leftarrow (\exists\text{strict hom. int.}).$$

It seems that in all cases I've in mind at present, when there exists a homotopy interval, there exists even a strict one. The only case besides topoi (where it is so, because of the existence of a Lawvere element) which I have looked up so far, is the case of (Cat) and various full subcategories, all containing Δ_1 which is indeed a strict homotopy interval, as it represents the presheaf on (Cat)

$$C \mapsto \text{set of all cribles in } C.$$

If we take the choice $I =$ two-point category equivalent to final one, this also is a strict homotopy interval, as it represents the functor

$$C \mapsto \text{set of all full subcategories of } C,$$

hence again an intersection law. The first homotopy interval though is a lot more important than the second, because the first one is "characteristic", namely sufficient for checking the homotopy relation between any two maps in (Cat), whereas the second isn't. A "*perfect*" homotopy interval would be one which is both *strict* (hence contractible) and *characteristic*.

2.4.

50

Reflection on some main impressions. The foresight of an "idyllic picture" (of would-be "canonical modelizers").

While writing the notes last time, and afterwards while pondering a little more about the matter, a few impressions came gradually into the fore. One was about the interplay of four basic "homotopy notions" which more or less mutually determine each other, namely the *homotopy relation* between maps, the notion of *homotopy interval*, the notion of homotopy equivalences or *homotopisms* (which has formal analogy to weak equivalences the was it is handled), and the notion of *contractible objects*. Another impression was about the dependence of these notions upon a preliminary notion of "weak equivalence", namely upon $W \subset \mathrm{Fl}(M)$, being a rather loose one. Thus the construction of homotopy notions in terms of a given interval $\mathbb{I}$ (including the category $M_{\mathbb{I}}$ with the canonical functor $M \to M_{\mathbb{I}}$) is valid for any interval in any category M with final object and binary products (instead of binary products, it is even enough that I be "squarable", namely all products $X \times I$ exist in M). As for the W-homotopy notions, they depend on W via the corresponding notion of W-asphericity over e, which is at first sight *the* natural condition to impose upon an interval $\mathbb{I}$, in order for the corresponding $\mathbb{I}$-homotopy notions to fit nicely with W (as expressed in the homotopy lemma). But then we noticed that a much weaker condition than asphericity on $\mathbb{I}$ suffices – namely that the two sections δ_0, δ_1 of I over e be W-homotopic, which means essentially that the "points" of I they define can be "joined" by a finite chain of W-aspheric intervals mapping into I. This strongly suggests (in view of the main application we have in mind, namely to the study of modelizers) that the natural condition to impose upon intervals, in most contexts of interest to us, will be merely 0-connectedness. But this notion is *intrinsic to the category M*, irrespective again of the choice of any W; and therefore the corresponding homotopy notions in M will turn out (in the cases at least of greatest interest to us) to be equally intrinsic to the category M. On the other hand (and here fits in the third main impression that peeled out two day ago), the work carried through so far in view of the "observation" and the (naive) "question" of last week (pages 94 and 95) strongly suggests that in the nicest modelizers (including (Cat) and the elementary modelizers, presumably), the notion of weak equivalence W can be described in terms of the homotopy notions, more specifically in terms of the notion of contractible objects (when exactly and how should appear in due course). Thus it will follow that for those modelizers, the modelizing structure W itself is uniquely determined in terms of the intrinsic category structure – thus any equivalence between the underlying categories of any two such modelizers should automatically

[p. 110]

be model-preserving! It will be rather natural to call the modelizers which fit into this idyllic picture *canonical modelizers*, as their modelizing structure W is indeed canonically determined by the category structure. Next thing then would be to try to gain an overall view of how to get "all" canonical modelizers, if possible in as concrete terms as the overall view we got upon elementary modelizers $A^\wedge$ in terms of the corresponding test categories A. [p. 111]

51 First thing though I would like to do now, is to elaborate "from scratch" on the four basic homotopy notions and their interplay, much in the style of a "fugue with variations" I guess, and without interference of a pregiven notion W of weak equivalence – relationship with a W will be examined only after the intrinsic homotopy notions and their interrelations are well understood.

The four basic "pure" homotopy notions with variations.

We start with a category M, without for the time being any specific assumptions on M. The strongest we're going to introduce, I guess, is existence of finite products, and incidently maybe finite sums and fiber products. In the cases we have in mind, M is a "large" category, therefore it doesn't seem timely here introduce $M^\wedge$ and the embedding of M into $M^\wedge$.

The most trivial implications between the four basic homotopy notions are symbolized by the plain arrows in the diagram below, the somewhat more technical ones by dotted arrows. It is understood these notions correspond to a given "homotopy structure" on M, symbolized by the letter h, and which (in the most favorable cases) may be described at will in terms of any one of the four notions. I'll first describe separately each of these basic notions, and afterwards the relationships symbolized by the arrows in the diagram. I recall that in *interval* in M is just an object I, endowed with two subobjects e_0, e_1 which are final objects of M, or equivalently, with two sections δ_0, δ_1 of I over a fixed final object $e_M = e$ of M. I definitely want to forget entirely for the time being about any condition of the type $e_0 \cap e_1 = \varnothing_M$ (initial object of M) – we may later refer to these as "*separated*" intervals (namely the endpoints e_0, e_1 are "separated"). We'll denote by $\operatorname{Int}(M)$ the set of all intervals in M, by $\mathbf{Int}(M)$ the corresponding category (the notion of a morphism of intervals being the obvious one). Now here's the organigram:

(D)
$$\begin{tikzcd}[column sep=large]
\begin{array}{c}\text{1) homotopy relation}\\ R_h \subset \operatorname{Fl}(M)\times\operatorname{Fl}(M)\end{array} \arrow[r, Rightarrow] \arrow[d, Rightarrow, shift right] & \begin{array}{c}\text{2) homotopism}\\ W_h \subset \operatorname{Fl}(M)\end{array} \arrow[l, Rightarrow, dashed, bend right, "1"'] \arrow[d, Rightarrow] \\
\begin{array}{c}\text{3) homotopy intervals}\\ \Sigma_h \subset \operatorname{Int}(M)\end{array} \arrow[u, Rightarrow, dashed, shift right, "2"'] & \begin{array}{c}\text{4) contractible objects}\\ C_h \subset \operatorname{Ob}(M)\end{array} \arrow[l, Rightarrow, dashed, "3"']
\end{tikzcd}$$

A) Homotopy relation between maps. As a *type of structure*, a homotopy relation between maps in M is a subset [p. 112]

$$R_h \subset \operatorname{Fl}(M) \times \operatorname{Fl}(M),$$

namely a relation in the set $\mathrm{Fl}(M)$ or arrows of M, the *basic assumption* being that whenever f and g are "homotopic" arrows, then they have the same source, and the same target. Thus, the data R_h is equivalent to giving a "homotopy relation" in any one of the sets $\mathrm{Hom}(X,Y)$, with X and Y objects in M. The relevant *saturation condition* is twofold:

a) the relation R_h is an equivalence relation, or equivalently, the corresponding relations in the sets $\mathrm{Hom}(X,Y)$ are equivalence relations;

b) stability under composition: if f and g are homotopic, then so are vf and vg, and so are fu and gu, for any arrow v or u such that the relation makes sense.

When these conditions are satisfied, we'll say we got a *homotopy relation* between maps of M. This relation between f and g will be denoted by a symbol like

$$f \underset{h}{\sim} g.$$

If the basic assumption is satisfied, but not the saturation condition, there is an evident way of "saturating" the given relation, getting one $\overline{R}_h$ which is saturated, i.e., a homotopy relation in M (in fact, the smallest one containing R_h).

Given a homotopy relation R_h, we denote by

$$\mathrm{Hom}(X,Y)_h$$

the corresponding quotient sets of the set $\mathrm{Hom}(X,Y)$, they compose in an evident way, so as to give rise to a category M_h having the same objects as M, and to a canonical functor

$$M \to M_h$$

which is the identity on objects, and surjective on arrows. We may view thus M_h as a *quotient category* of M, having the same objects as M. Clearly, $R_h \mapsto M_h$ is a bijective correspondence between the set of homotopy relations in M, and the set of quotient categories of M satisfying the aforesaid property. By abuse of language, we may even consider that considering a homotopy relation in M, amounts to the same as giving a functor $M \to M_h$ from M which is bijective on objects and surjective on arrows.

When we got a homotopy relation R_h in M, we deduce a notion of *homotopisms*

$$W_h \subset \mathrm{Fl}(M),$$

[p. 113] namely those arrows in M which become isomorphisms in M_h. Also we deduce a notion of *homotopy interval*, i.e.,

$$\Sigma_h \subset \mathrm{Int}(M),$$

namely those intervals $\mathbb{I}$ in M such that the two marked sections be homotopic. (NB Σ_h is non-empty if and only if M has a final element, in this case Σ_h contains all intervals such that $\delta_0 = \delta_1$ – which we may call *trivial* intervals, for instance the *final interval* with $I = e\dots$) This notion of a homotopy interval is considerably wider than the one we have worked with so far, however it is clearly the right one in the context of pure homotopy notions. To avoid any confusion, we better call this notion by the name of *weak homotopy intervals* – funnily there won't be any unqualified "homotopy intervals" in our present set-up of "pure" homotopy notions!

The two prescriptions above account for two among the plain arrows in our organigram.

We'll often make use of an *accessory assumption* on R_h, which can be expressed by demanding that *the canonical functor* $M \to M_h$ *commute to binary products*, in case we assume already such products exist in M. This can be expressed also by the property that for two maps

$$f, g : X \rightrightarrows Y_1 \times Y_2,$$

f and g are homotopic iff so are f_i and g_i ($i \in \{1,2\}$) (where the "only if" part is satisfied beforehand anyhow). This implies too that in M_h binary products exist, and that $f \sim f'$, $g \sim g'$ implies $f \times g \sim f' \times g'$. On the other hand, it is trivial that if M admits a final object, this is equally a final object of M_h (and hence, under the accessory assumption on R_h, the functor $M \to M_h$ commute to finite products).

B) Homotopisms. As a type of structure on M, we got just a subset

$$W_h \subset \mathrm{Fl}(M),$$

without any basic assumption to make. The natural *saturation condition* is just the strong saturation for a subset of $\mathrm{Fl}(M)$, which can be expressed by stating that W_h can be obtained as the set of arrows made invertible by some functor from M into a category M', or equivalently, by the localization functor

$$M \to W_h^{-1} M.$$

We may refer to a strongly saturated W_h as a "*homotopism structure*" (or "homotopy equivalence structure") in M – but as in the case A), we'll have soon enough to make pretty strong extra assumptions. Maybe we should, at the very least, demand for the notion of homotopy structure that the canonical functor above, which is bijective on objects in any case, should be moreover *surjective on arrows* – thus I'll take this as a *basic assumption* [p. 114]
after all. This assumption makes sense of course independently of any saturation condition. If W_h is not strongly saturated, then denoting by $\overline{W}_h$ the subset of $\mathrm{Fl}(M)$ of all arrows made invertible by the canonical functor, this will now be a strongly saturated set of arrows (in fact the smallest one containing W_h, and giving rise to the same localized category, and hence satisfying the basic assumption too) – thus $\overline{W}_h$ is indeed a homotopism structure on M. When M admits a final object, this will equally be a final object of $W_h^{-1}M$. We may now define in terms of W_h the notion of contractible objects in M, forming a subset

$$C_h \subset \mathrm{Ob}(M),$$

as those objects X in M such that the projection $p_X : X \to e$ is in W_h, or equivalently, such that X is a final object in the localized category $W_h^{-1}M$. This accounts for the third plain arrow of the organigram.

We'll now dwell a little more on the first dotted arrow, namely the description of a homotopy relation

$$R_h \subset \mathrm{Fl}(M) \times \mathrm{Fl}(M)$$

in terms of W_h: the natural choice here is to define $f, g \in \mathrm{Fl}(M)$ to be *homotopic* (or W_h-homotopic, if ambiguity may arise) iff their images in the category $W_h^{-1}M$ are equal. This relation between maps in M clearly satisfies the basic assumption on source and target, as well as the saturation condition – it is therefore a "homotopy relation" in M, namely the one associated to W_h^{-1}, viewed as a quotient category of M. It is clear that we recover W_h from R_h, consequently, by the process described in A).

To make the relationship between the notions 1) and 2) still clearer, let's denote respectively by

$$\mathrm{Hom}_1(M), \mathrm{Hom}_2(M)$$

the set of all homotopy relations, resp. of homotopism notions, in M. We got maps

3.4. There *is no map* r_{21}, only r_{12}, see correction in §52.

$$\mathrm{Hom}_1(M) \underset{r_{12}}{\overset{r_{21}}{\rightleftarrows}} \mathrm{Hom}_2 M,$$

and the relevant fact here is that

$$r_{12} : \mathrm{Hom}_2 \to \mathrm{Hom}_1$$

is injective, and admits r_{21} as a left inverse. Thus, we may view Hom_2 as a subset of Hom_1, i.e., the structure of a "homotopism notion" on M as a particular case of the structure of a "homotopy relation" on M. Namely, a structure of the latter type can be described in terms of a notion of homotopism in M, iff the canonical functor $M \to M_h$ it gives

[p. 115] to is a localization functor.

For a general $R_h \in \mathrm{Hom}_1(M)$, if we consider the corresponding W_h $(= r_{21}(R_h))$ in $\mathrm{Hom}_2(M)$, it is clear that the canonical functor $M \to M_h$ of A) factors into

$$M \to W_h^{-1} \to M_h,$$

and R_h "is in $\mathrm{Hom}_2(M)$", i.e., $R_h = r_{12}(W_h)$, iff the second functor

$$W_h^{-1}M \to M_h$$

(which is anyhow bijective on objects and surjective on arrows) is an isomorphism, or equivalently, faithful. Here is a rather direct *sufficient* condition on R_h for this to be so,[†] namely:

[†]3.4. See §52 for a [?] of this rash statement!

C_{12}) If $f, g : X \to Y$ are homotopic, there exists a homotopism $X' \to X$, two sections s_0, s_1 of X' over X, and a map $h : X' \to Y$, such that

$$f_0 = hs_0, \quad f_1 = hs_1.$$

Remark. Intuitively, we are thinking of course of X' as a product $X \times I$, where $\mathbb{I} = (I, e_0, e_1)$ is a weak homotopy interval, and s_0, s_1 are defined in terms of e_0, e_1. In Quillen's somewhat different set-up, X' is referred to as a "cylinder object for X", suitable for defining the "left homotopy relation" associated to a given W_h. The condition C_{12} is not autodual, we could state a dual sufficient condition in terms of a "path object for Y", namely a homotopism $Y \to Y'$ endowed with two retractions t_0, t_1 upon Y – but we don't have any use for this in the present set-up, which (as for as the main emphasis is concerned) is by no means autodual, as is Quillen's.

The condition C_{12} above can be viewed equally as a condition on a $W_h \in \mathrm{Hom}_2(M)$.

We may interpret the set $\mathrm{Hom}_2(M)$ of homotopism notions in M as the set of all quotient categories M_h of M, having the same objects as M, and such that moreover the canonical functor $M \to M_h$ be a localizing functor. As in A), the relevant "*accessory assumption*" on W_h (a particular case indeed of the corresponding one for R_h) is that this functor commute to products. I don't see any simple computational way though to express this condition directly in terms of W_h, as previously in terms of R_h. I would only like to notice here a consequence of this assumption (I doubt it is equivalent to it), namely that *the cartesian product of two homotopisms is again a homotopism* – which implies, for instance, that the product of a finite family of contractible objects of M [p. 116] is again contractible.

C) Weak homotopy intervals.

We assume M stable under finite products. The type of structure we've in view is a set of intervals in M,

$$\Sigma_h \subset \mathrm{Int}(M),$$

called the "weak homotopy intervals". No basic assumption on this set, it seems; the natural "*saturation condition*" is the following:

(Sat 3) Any interval $\mathbb{I} = (I, \delta_0, \delta_1)$ in M, such that the sections δ_0, δ_1 of I be Σ_h-homotopic (see below), is in Σ_h.

The assumption on $\mathbb{I}$ means, explicitly, that there exists a finite chain of sections of I

$$s_0 = \delta_0, s_1, \ldots, s_N = \delta_1,$$

joining δ_0 to δ_1, and for two consecutive s_i, s_{i+1} an interval $\mathbb{J}$ in Σ_h, and a map of intervals from $\mathbb{J}$ or $\check{\mathbb{J}}$ to (I, s_i, s_{i+1}), i.e., a map $J \to I$, mapping the two given sections of J, one into s_i, the other into s_{i+1} (without specification which is mapped into which).

The significance of this saturation condition becomes clear in terms of the second dotted arrow of the organigram. Namely, in terms of any subset Σ_h of $\mathrm{Int}(M)$, we get a corresponding homotopy relation between maps, say R_h, which is the equivalence relation in $\mathrm{Fl}(M)$ generated by the "elementary" homotopy relation (with respect to Σ_h) between maps

f, g in M, namely the relation R_0

$$f \underset{R_0}{\sim} g \overset{\text{def}}{\iff} \begin{array}{l}\text{there exists } \mathbb{I} \text{ in } \Sigma_h, \text{ and an } \mathbb{I}\text{-homotopy} \\ \text{from } f \text{ to } g.\end{array}$$

The corresponding equivalence relation R_h in $\mathrm{Fl}(M)$ is already saturated, namely stable under compositions, moreover it satisfies condition C_{12} above – thus we may view this homotopy relation as defined in terms of a homotopisms notion – thus in fact the second dotted arrow should go from 3) to 2) rather than from 3) to 1)! Now, if we look at the subset Σ_h of $\mathrm{Int}(M)$ defined in terms of R_h as in A) (namely the set of "homotopy intervals with respect to R_H"), we get

$$R_h \subset \overline{R}_h,$$

and the equality holds iff R_h satisfies (Sat 3)? At the same time, in case of arbitrary R_h, we get the construction of its saturation, $\overline{R}_h$, which may of course be described alternatively as the smallest saturated subset of $\mathrm{Int}(M)$ containing R_h.

[p. 117] We'll call *weak homotopy interval structures* on M, any set Σ_h of intervals in M, satisfying the saturation condition above. The set of all such structures on M will be denoted by $\mathrm{Hom}_3(M)$, thus we get two embeddings

$$\mathrm{Hom}_3(M) \hookrightarrow \mathrm{Hom}_2(M) \hookrightarrow \mathrm{Hom}_1(M),$$

in such a way that a weak homotopy interval structure on M may be viewed also as a particular case of a homotopism structure on M, and a fortiori as a particular case of a homotopy relation on M. Of course, the homotopy relations or homotopism structures on M we'll ultimately be interested in, are those stemming from weak homotopy interval structures on M. Recall that M admits finite products, and these structures satisfy automatically the accessory assumption, namely commutation of the canonical functor $M \to M_h$ to finite products.

It is immediate that if we start with a homotopy relation R_h, the corresponding Σ_h as defined in A) is saturated. Thus, the canonical embedding r_{13} of Hom_3 into Hom_1 admits a canonical left inverse r_{31}, the restriction of which to Hom_2 is a canonical left inverse r_{32} of the natural embedding r_{23} of Hom_3 into Hom_2.

D) Contractibility structures. (We still assume M admits finite products.) As a type of structure, it is a set of objects of M

$$C_h \subset \mathrm{Ob}(M),$$

without any "basic assumption" on C_h it seems. These objects will be called the *contractible* objects. Sorry, there *is* a basic assumption here I just overlooked, namely every X in C_h should have at least one section (thus I better assume beforehand, as in C), that M has a final object e). To get the natural saturation condition on C_h, we'll make use of the third dotted arrow in the organigram, by associating to C_h the set Σ_h of "contractible intervals", namely intervals $\mathbb{I} = (I, \delta_0, \delta_1)$ such that I is in C_h. Of course in general there is no reason that Σ_h should be saturated, never mind – it defines anyhow (as seen in C)) a homotopism notion in M, and hence (as seen in B)) a notion of contractible objects, i.e., another subset $\overline{C}_h$ of $\mathrm{Ob}(M)$. Now it occurs to me that it is by no means clear that the latter contains C_h, which brings near the necessity of a more stringent *basic assumption* on C_h, namely for the very least

$$C_h \subset \overline{C}_h$$

(this will imply that any X in C_h has indeed a section over e, as this is automatically the case for W_h-contractible objects). The saturation condition (Sat 4) will of course be equality

$$C_h = \overline{C}_h,$$

and for general C_h (satisfying the basic assumption $C_h \subset \overline{C}_h$), $\overline{C}_h$ can be viewed as the "saturation" of C_h, namely the smallest saturated subset of $\mathrm{Ob}(M)$ satisfying the basic assumption, or in other words, the smallest *contractibility structure* on M such that the objects in C_h are contractible. [p. 118]

It may be worth while to state more explicitly the basic assumption here, and the saturation condition on C_h.

(Bas 4) For any X in C_h, we can find a finite sequence of maps from X to X,

$$f_0 = \mathrm{id}_X, f_1, \dots, f_N = c_s,$$

joining the identity map of X to a constant map c_s (defined by some section s of X), in such a way that two consecutive maps f_i, f_{i+1} are C_h-homotopic in the strict sense, namely we can find Y_i in C_h and two sections δ_0^i and δ_1^i of Y_i over e, and a map

$$h_i : Y_i \times X \to X,$$

such that

$$h_i \circ (\delta_0^i \times \mathrm{id}_X) = f_i, \quad h_i \circ (\delta_1^i \times \mathrm{id}_X) = f_{i+1}.$$

(Sat 4) *Any* object X in M satisfying the condition just stated is in C_h.

The third dotted arrow can be viewed as denoting an embedding of $\mathrm{Hom}_4(M)$ (the set of all contractibility structures on M) into $\mathrm{Hom}_3(M)$, we finally get a cascade of three inclusions

$$\mathrm{Hom}_4(M) \hookrightarrow \mathrm{Hom}_3(M) \hookrightarrow \mathrm{Hom}_2(M) \hookrightarrow \mathrm{Hom}_1(M),$$

in terms of which a contractibility structure on M can be viewed a a particular case of any of the three types of homotopy structures on M considered before.

If we start with a homotopism structure W_h on M, and consider the corresponding set C_h of contractible objects of M (namely objects X

such that $X \to e$ is in W_h), it is pretty clear that C_h satisfies the saturation condition (Sat 4), but by no means clear that it satisfies the basic assumption (Bas 4), even in the special case when we assume moreover that W_h comes from a weak homotopy interval structure Σ_h on M. The trouble comes from the circumstance that there is no reason in general that the contractibility of an object X of M can be described in terms of a sequence of elementary homotopies between maps $f_i : X \to X$ (joining id_X to a constant map) involving weak homotopy intervals $\mathbb{I}_i$ *which are moreover contractible*. I doubt this is always so, and there doesn't come either any plausible extra condition on Σ_h which may ensure this,
[p. 119]
except precisely that Σ_h can be generated (through saturation) by the subset Σ_{hc} of its contractible elements, which is just another way of saying that this $\Sigma_h \in \mathrm{Hom}_3(M)$ comes already from a contractibility structure $C_h \in \mathrm{Hom}_4(M)$! Thus, definitely the uniformity of formal relationships between successively occurring notions seems broken here, namely there does not seem to be any natural retraction r_{43} of $\mathrm{Hom}_3(M)$ onto the subset $\mathrm{Hom}_4(M)$. For the least, if there is such a retraction, its definition should be presumably a somewhat more delicate one than the first that comes to mind. I will not pursue this matter any further now, as it is not clear if we'll need it later.

It is clear that for any weak homotopy interval structure Σ_h on M, Σ_h is stable under the natural notion of finite products of intervals (in the sense of the category structure of $\mathbf{Int}(M)$). We saw already that this is handy, as the consideration of products of intervals allows to show that the family of homotopy relations $\underset{\mathbb{I}}{\sim}$ in $\mathrm{Fl}(M)$, for variable $\mathbb{I}$ in Σ_h, is "filtrant décroissant", so we get the relation $\underset{h}{\sim}$ as the filtering direct limit or union of the more elementary relations $\underset{\mathbb{I}}{\sim}$. Similarly, if $C_h \subset \mathrm{Ob}(M)$ is a contractibility structure on M, C_h is stable under finite products.

Remark. From the way we've been working so far with homotopy notions, it would seem that we're only interested here in homotopy notions which stem from a structure in $\mathrm{Hom}_3(M)$, namely which can be described in terms of a notion of weak homotopy intervals. The focus on contractibility has set in only lately, and it is too soon to be sure whether we'll be working only with homotopy structures on M which can be described in terms of a contractibility notion, namely which are in $\mathrm{Hom}_4(M)$. In the cases I've had in mind so far, it turns out, it seems that the homotopy notions dealt with do come from a structure in $\mathrm{Hom}_4(M)$, i.e., from a contractibility structure.

E) Generating sets of weak homotopy intervals. Contractors.

Let Σ_h be a weak homotopy interval structure on M.[‡] A subset Σ_h^0 is called *generating*, if Σ_H is just its saturation (cf. C) above), i.e., for any $\mathbb{I}$ in Σ_h, the two endpoints can be joined by a chain as in (Sat 3), involving only intervals in Σ_h^0. This implies that all homotopy notions dealt with so far can be checked directly in terms of intervals in Σ_h^0. We've met a particular case of this before, when Σ_h^0 is reduced to just one element $\mathbb{I}$ – we then called $\mathbb{I}$ "characteristic", but "*generating weak homotopy*

[‡] This implies we assume M stable under finite products.

interval" now would seem the more appropriate expression. Even when there should not exist such a generating interval, the natural next best assumption to make is the existence of a generating set Σ_h^0 which is "small" (namely an element of the "universe" we are working in). The case of a finite generating set of intervals reduces to the case of a single one though, by just taking the product of those intervals. [p. 120]

An interesting case is when the generating set Σ_h^0 consists of contractible objects of M. Such a generating set exists iff the structure considered Σ_h comes from a contractibility structure. About the best we could hope for is the existence of a single *generating contractible weak homotopy interval* $\mathbb{I}$. If we got any interval $\mathbb{I}$ in M, this can be viewed as a generating contractible weak homotopy interval for a suitable homotopy structure on M (then necessarily unique) iff the identity map of I can be joined to a constant one by a chain of maps, such that two consecutive ones are tied by an $\mathbb{I}$-homotopy or an $\check{\mathbb{I}}$-homotopy. The most evident way to meet this condition is by a *one-step chain* from id_I to the constant map defined by one of the endpoints, δ_1 say. This brings us back to structure of a composition law

$$I \times I \to I$$

in I, having e_0 as a left unit and e_1 as a left zero element. Let's call an interval, endowed with such a composition law, a *contractor* in M. Thus starting from a contractor in M is about the nicest way to define a homotopy structure in M, as a matter of fact the strongest type of such a structure – namely a contractibility structure, admitting a *generating contractible weak homotopy interval* (and better still, admitting a *generating contractor*).

Of course, starting with the weakest kind of homotopy structure on M, namely just a homotopy relation $R_h \in \mathrm{Hom}_1(M)$, if $\mathbb{I}$ is a homotopy interval which is moreover endowed with a structure of a contractor, i.e., if it is a contractor such that the end-point sections δ_0, δ_1 are homotopic, then I is automatically contractible (never mind if it is generating or not).

It seems to me that the homotopy structures I've looked at so far (such as (Cat)) and various standard elementary modelizers $A^\wedge$) are not only contractibility structures, but they all can be defined by a single contractor each.

Besides the "*basic contractor*" $\mathbb{\Delta}_1$ in (Cat), there are two general ways I've met so far for getting contractors. One has been made explicit in these notes a number of times, namely the *Lawvere element* L_M in M if it exists, and if moreover M has a *strict* initial element, $\varnothing_M$. Recall that L_M represents the contravariant functor on M [p. 121]

$$X \mapsto \text{set of all subobjects of } X,$$

and that the "full" and "empty" subobjects of X, for variable X, define two sections δ_0 and δ_1 of L_M. I forgot to state the extra condition that in M fibered products exist (intersection of two subobjects would be enough); then the intersection law endows L_M with a structure of a contractor

$\mathbb{L}_M$, admitting δ_0 as a unit and δ_1 as a zero element. Moreover, it is clear that $\mathbb{L}_M$ as an interval is separated, i.e., $\operatorname{Ker}(\delta_0, \delta_1) = \varnothing_M$. More precisely still, $\mathbb{L}_M$ can be viewed as a *final object of the category of all separated intervals in M*, namely for any such interval, there is a *unique* map of intervals

$$\mathbb{I} \to \mathbb{L}_M.$$

This implies that if M is endowed with a homotopy structure, such that there exists a weak homotopy interval which is separated, then $\mathbb{L}_M$ is such an interval, and it is moreover contractible. It is doubtful though, even if we can find a *generating* contractor for the given homotopy structure on M, that the Lawvere contractor is generating too.

Here now is a second interesting way of getting contractors. We assume that M admits finite products (as usual). Let X be an objects, and assume the object $\operatorname{Hom}(X,X)$, representing the functor

$$Y \mapsto \operatorname{Hom}(X \times Y, X) = \operatorname{Hom}_Y(X_Y, X_Y),$$

exists in M. (NB X_Y denotes $X \times Y$, viewed as an object of $M_{/Y}$.) Composition of endomorphisms of X_Y clearly endow this functor with an associative composition law, admitting a two-sided unit, which I call e_0. Notice that sections of

$$\mathbb{I} = \mathbf{Hom}(X,X)$$

can be identified with maps $X \to X$, and the section corresponding to id_X is of course the two-sided unit. On the other hand, if X admits sections, i.e., admits "constant" endomorphisms, it is clear that the corresponding sections of $\mathbb{I}$ are left zero elements. If we choose a section of X, $\mathbb{I}$ becomes a contractor. Its interest lies in the following

§[?]

Proposition. *Assume finite products exist in M, and M endowed with a homotopy structure.*[§] *Let X be an object of M endowed with a section e_X, and suppose the object* $\mathbf{Hom}(X,X)$ *exists, hence a contractor $\mathbb{I}$ as seen above. The following two conditions are equivalent:*

a) *X is contractible.*

b) *$\mathbb{I}$ is contractible* (*or, equivalently as seen above, $\mathbb{I}$ is a weak homotopy interval, namely the two endpoints are homotopic*).

[p. 122] *Moreover, this condition implies the following two:*

c) *For any object Y in M, if* $\mathbf{Hom}(Y,X)$ *exists, it is contractible.*

d) *For any Y in M, if* $\mathbf{Hom}(X,Y)$ *exists, the natural map*

$$Y \mapsto \mathbf{Hom}(X,Y)$$

(*identifying Y to the "subobject of constant maps from X to Y"*) *is a homotopism.*

The equivalence of a) and b) is just a tautological translation of contractibility and homotopy relations in terms of weak homotopy intervals $\mathbb{J}$ (cf. cor. 2 below). That c) and d) follow comes from the fact that the monoid object $I = \mathbf{End}(X) = \mathbf{Hom}(X,X)$ operates on the left on $\mathbf{Hom}(Y,X)$, on the right on $\mathbf{Hom}(X,Y)$, and the following:

Corollary 1. *Let $\mathbb{I}$ be a weak homotopy interval, assume the underlying I "operates" on an object H, namely we are given a map*

$$h : I \times H \to H \quad (\text{“operation” of } I \text{ on } H)$$

satisfying the relations (where $h(u, f)$ is written simply $u \cdot f$):

$$e_0 \cdot f = f, \quad e_1 \cdot (e_1 \cdot f) = e_1 \cdot f,$$

namely e_0 acts as the identity and e_1 acts as an idempotent p on H (a very weak associativity assumption indeed if I is a contractor, as $e_1 \cdot e_1 = e_1$). Assume the image of p, i.e., $\mathrm{Ker}(\mathrm{id}_H, p)$ *exists, let H_0 be the corresponding subobject of H, and*

$$p_0 : H \to H_0$$

the map induced by p. Then p_0 is a homotopism (and hence the inclusion $i : H_0 \to H$, which is a section of p_0, is a homotopism too).

Because of the saturation property c') on homotopies, it is enough to check that $p = ip_0$ is a homotopism (as $p_0 i = \mathrm{id}_H$ already is one), and for this it is enough to see it is homotopic to the identity map of H. But a homotopy between the two is realized by h, qed.

The argument for equivalence of a) and b) above can be generalized as follows:

Corollary 2. *Let M be as before, and X and Y objects such that $H =$* $\mathbf{Hom}(X, Y)$ *exists in M. Let $f, g : X \rightrightarrows Y$ be two maps, which we'll identify to the corresponding sections of H. Then f and g are homotopic maps iff they give rise to homotopic sections of H.*

F) The canonical homotopy structure: preliminaries on π_0.

In order to simplify life, I will in this section make the following assumptions on M (which presumably, except for the first, could be considerably weakened, but these will be sufficient): [p. 123]

a) Finite products exist in M ("pour mémoire").

b) Arbitrary sums exist in M, they are "disjoint" and "universal" (which implies that M has a *strict* initial object).

c) Every object in M is isomorphic to a direct sum of 0-connected ones.

I recall an object is called 0-*connected* if it is a) "non-empty", i.e., non-isomorphic to $\varnothing_M$, and b) connected, i.e., any decomposition of it into a sum of two subobjects is trivial (namely, one is "empty", the other is "full"). Also, under the assumption b), well-known standard arguments show that for any object X, a decomposition of X into a direct sum of 0-connected components is essentially unique (namely the corresponding set of subobjects of X is unique), if it exists. The subobjects occurring in the sum are called the *connected components* of X, and the set of connected components of X is denoted as usual by $\pi_0(X)$. We thus get a natural functor

$$X \mapsto \pi_0(X), \quad M \to (\text{Sets}),$$

This can be equally described as a *left adjoint* to the functor

$$E \mapsto E_M, \quad (\text{Sets}) \to M,$$

associating to a set E the corresponding "*constant object*" E_M of M, sometimes also denoted by the product symbol $E \times e_M = E \times e$ (e the final object of M), namely the direct sum in M of E copies of e. The adjunction formula is

$$\mathrm{Hom}_M(X, E_M) \simeq \mathrm{Hom}_{(\text{Sets})}(\pi_0(X), E).$$

The adjunction relation implies that the functor π_0 commutes with all direct limits which exist in M, and in particular (and trivially so) to direct sums.

I'll finally assume also, to fix the ideas:

d) The final object of M is 0-connected, i.e., $\pi_0(e) =$ one-point set.

Empty objects of M, on the other hand, are of course characterized by the condition

$$\pi_0(\varnothing_M) = \emptyset.$$

Finally, I'll make in the end a very strong assumption on M, which however is satisfied more or less trivially in the cases we are interested in, when M is a would-be modelizer:

e) (Total 0-asphericity of M): the product of two 0-connected objects of M is again 0-connected.

This is clearly equivalent to the condition

e') The functor $\pi_0 : M \to (\text{Sets})$ commutes to finite products.

[p. 124] **Remarks.** The crucial assumptions here seem to be b) (which allows definition of a π_0 functor, and topological intuition tied up with connectedness to enter into play), and e), which implies that with respect to cartesian products, the usual intuitive background for connectedness, rooted in the example of $M =$ (topological spaces) is indeed valid. This condition is clearly stronger than d), which is a mere condition for convenience in itself (otherwise, a decomposition of e into connected components would mean a corresponding decomposition of M as a product category, and everything could be looked at "componentwise"). As for e) it could probably be dispensed with, by still defining $\pi_0(X)$ as a strict pro-set. In the case of modelizers anyhow, such generalization seems quite besides the point.

The condition e), however innocent-looking in terms of topological intuition, seems to me an extremely strong condition indeed. I suspect that in case M is a topos, it is equivalent to total asphericity. In any case, if M is the topos associated to a locally connected topological space, we've seen time ago that the condition e) implies X is irreducible, and hence totally aspheric. In view of this exactingness of e), I'll not use it unless specifically stated.

3.4.

52 Before pursuing the review of "pure homotopy notions" begun in yesterday's notes, I would like to correct some inaccuracies which flew in when looking at the relationship between the two first basic homotopy notions, namely the notion of a homotopy relation, and the notion of a homotopism structure. As usual, the provisional image I had in mind was still somewhat vague, while the reasonable expectations came out more clearly through the process of writing things down (including factual inaccuracies!).

Inaccuracies rectified.

The two notions clearly correspond to two kinds of ways of constructing new categories M' in terms of a given one, and a functor

$$M \to M'$$

which is bijective on objects, and has the property moreover that for any category C, the corresponding functor

$$\mathbf{Hom}(M', C) \to \mathbf{Hom}(M, C)$$

is a fully faithful embedding in the strict sense, namely injective on objects. One way is to take for M' any quotient category of M, by an equivalence relation which is the discrete one on objects – thus it corresponds just to an equivalence relation R in $\mathrm{Fl}(M)$, compatible with the source and target maps and with compositions. The other is to take as M' a localization with respect to some $W \subset \mathrm{Fl}(M)$, and if we take W strongly saturated we get a bijective correspondence between these M' and the set of strongly saturated subsets of $\mathrm{Fl}(M)$. [p. 125]

It wouldn't be any more reasonable to call an arbitrary R as above, corresponding to an arbitrary quotient category M' with the same objects as M, a "homotopy relation" on M (as I did though yesterday), as it would be to call an arbitrary strongly saturated $W \subset \mathrm{Fl}(M)$ a "homotopism structure" on M (as I nearly did yesterday, but then rectified in the stride). *The characteristic flavor of homotopy theory comes in, when we get an M' which is* both *a quotient category and a localization of M*. Thus neither approach, via R or via W, is any more contained in the other, then the converse. We should regard the *homotopy structure* on M to be embodied in the basic functor

$$M \to M',$$

which is a description where no choice yet is made between the two possible descriptions of M', either by an R, or by a W. If we describe M' in terms of R, the extra assumption to make on R for calling it a "*homotopy relation*", is that the canonical functor

$$M \to M/R$$

should be localizing. Alternatively, describing M' by W, the extra assumption to make on W (as we did yesterday) in order to call W a *homotopism structure* on M, is that the canonical localization functor

$$M \to W^{-1}M$$

be essentially a passage-to-quotient functor, namely surjective on arrows (as we know already it is bijective on objects). Thus the set of all homotopy relations on M is in one-to-one correspondence with the set of all homotopism structures on M, and if we denote these sets (in accordance with yesterday's provisional notations) $\mathrm{Hom}_1(M)$ and $\mathrm{Hom}_2(M)$, we get thus a *bijective* correspondence

$$\mathrm{Hom}_1(M) \longleftrightarrow \mathrm{Hom}_2(M).$$

The set of homotopy structures $\mathrm{Hom}(M)$ on M may be either defined as the usual quotient set defined by the previous two-member system of transitive bijections between sets, or more substantially, as a set of isomorphism classes of categories M' "under M", i.e., endowed with a functor $M \to M'$, and subject to the following two extra conditions:

[p. 126]

a) The functor $M \to M'$ is bijective on objects, surjective on arrows.

b) The functor $M \to M'$ is a localization functor (it will be so then, in view of a)), in the strict sense, namely M' will be M-isomorphic, not only M-equivalent, to a localization $W^{-1}M$).

However, the question arises whether it is possible to define such a homotopy structure on M in terms of an arbitrary R, i.e., an arbitrary quotient category having the same objects (let $Q(M) = Q$ be the set of all such R's) or in terms of an arbitrary localization of M, or what amounts to the same, in terms of an arbitrary strongly saturated $W \subset \mathrm{Fl}(M)$ (let's call $L(M) = L$ the set of all such W's). The first thing that comes to mind here, is that we got two natural maps

(1)
$$\begin{tikzcd} Q \arrow[r, bend left, "r"] & L \arrow[l, bend left, "s"] \end{tikzcd}$$

between Q and L, which are defined by the observation that whenever we have a functor $i : M \to M'$, injective on objects, it defines both an $R \in Q$ (namely $f \underset{R}{\sim} g$ iff $i(f) = i(g)$) and a $W \in L$ (namely $f \in W$ if $i(f)$ is invertible). For defining $r(R)$ resp. $s(W)$, we apply this to the case when $M' = M/R$ resp. $W^{-1}M$. Let's look a little at the two cases separately.

Start with R in Q, we get $W = r(R)$,

$$W = \big\{ f \in \mathrm{Fl}(M) \,\big|\, i(f) \text{ invertible} \big\},$$

where

$$i : M \to M/R$$

is the canonical functor, we thus get a canonical functor (compatible with the structures "under M")

(2)
$$\alpha_R : M_W \to M_R,$$

where for simplicity I write

$$M_R = M/R, \quad M_W = W^{-1}M.$$

We may define W as the largest element in L (for the natural order relation in L, namely inclusion of subsets of $\mathrm{Fl}(M)$) such that a functor (2) exists (compatible with the functors from M into both sides) – such a functor of course is unique (by the preliminaries on functors $M \to M'$ made at the beginning). In terms of (1), we can say that R is actually a homotopy relation (let's call $Q_0(M) = Q_0$ the subset of Q of all such relations) iff (2) is an isomorphism, or equivalently (as it is clearly bijective on objects, surjective on arrows) iff it is *injective on arrows*, i.e., *faithful*.

Conversely, start now with W in L, we get $R = s(W)$,

$$R = \left\{(f,g) \in \mathrm{Fl}(M) \times \mathrm{Fl}(M) \,\middle|\, i(f) = i(g)\right\},$$

where now [p. 127]

$$i : M \to W^{-1}M = M_W$$

is the canonical functor defined in terms of W; we thus get a canonical functor of categories "under M"

$$\beta = \beta_W : M_R \to M_W, \tag{3}$$

as a matter of fact, R is the largest element in Q (for the inclusion relation of subsets of $\mathrm{Fl}(M) \times \mathrm{Fl}(M)$) for which a functor (3) exists (then necessarily unique, as before). We may say that W is a homotopism structure on M, i.e., $W \in L_0$ (where L_0 is the subset of L of all homotopism structures on M) iff the functor (3) is an isomorphisms, or equivalently (as it is clearly bijective on objects, injective on arrows) iff it is *surjective on arrows*.

We may describe Q_0 and L_0 in a purely set-theoretic way, in terms of the system (r,s) of maps in (1), by the formula (which is just a translation of the definitions of Q_0 and L_0)

$$\begin{aligned} Q_0 &= \left\{q \in Q \,\middle|\, sr(q) = q\right\} \\ L_0 &= \left\{\ell \in L \,\middle|\, rs(\ell) = \ell\right\}, \end{aligned}$$

and we can describe formally the pair of subsets (Q_0, L_0) of Q, L as the largest pair of subsets, such that r and s induce between Q_0 and L_0 bijections inverse of each other. In the general set-theoretic set-up, it is by no means clear, and false in general, that r maps Q into L_0 or L into Q_0 (thus both Q_0 and L_0 may well be empty, whereas Q and L are not). Thus it is not clear at all that starting with an arbitrary $R \in Q$, the corresponding $W = i(R)$ is a homotopism structure, and it is easily seen that this is *not* so in general, contrarily to what I hastily stated in yesterday's notes. (Take M which just one object, therefore defined by a monoid G, and M_R corresponds to a quotient monoid G', we may take $G' =$ unit monoid, thus W is G itself, and M_W the enveloping group $\overline{G}$ of G – the map $\overline{G} \to G'$ need not be injective!) In the opposite direction, starting with an arbitrary W in L, the corresponding R need not be a homotopy relation. (If M is reduced to a point, this amounts to saying that if we localize a monoid G with respect to a subset $W \subset G$, by making invertible the elements in W, the corresponding map $G \to \overline{G}$ need no be surjective!)

What we may say, though, is that if we start with a pair

$$(R,W) \in Q \times L$$

such that the *two* functors (2), (3) exist, i.e., such that

$$W \le r(R) \quad \text{and} \quad R \le s(W),$$

then $(R,W) \in Q_0 \times L_0$, i.e., R is a homotopy relation and W is a ho-
[p. 128] motopism structure, and the two are associated. This comes from the fact that both compositions of the functors (2), (3) must be the identity functors (being compatible with the "under M" structure), hence α and β are isomorphisms, which shows both $R \in Q_0$ and $W \in L_0$, and that R and W are associated. In terms of the set-theoretic situation (1), this may be described by using the order relations on Q and L, and the fact that r and s are monotone maps, and satisfy moreover

$$(*) \qquad sr(q) \le q, \quad rs(\ell) \le \ell \quad (\text{any } q \in Q, \ell \in L),$$

which implies that the set C_0 of pairs (q,ℓ) of associated elements of Q_0, L_0 can be described also as

$$C_0 = \left\{ (q,\ell) \in Q \times L \;\middle|\; \ell \le r(q),\ q \le s(\ell) \right\}.$$

Thus it doesn't seem evident to get a homotopy structure on M, just starting with an $R \in Q$ or a $W \in L$, without assuming beforehand that R is a homotopy relation, or W a homotopism structure. The condition C_{12} on page 115 may be viewed as a condition on a pair $(R,W) \in Q \times L$, and it clearly implies

$$R \le s(W);$$

if we assume moreover $W = r(R)$ we get $R \le s(r(R))$ and hence, in view of the first inequality (*) above,

$$R = sr(R), \quad \text{i.e.,} \quad R \in Q_0,$$

i.e., R is a homotopy relation, as asserted somewhat quickly yesterday.

The only standard way for getting homotopy structures in a general category M which I can see by now, is in terms of an arbitrary set Σ_h of intervals in M (assuming only that M admits finite products). As soon as the focus gets upon intervals for describing homotopy structures, the situation becomes typically non-autodual – in contrast to Quillen's autodual treatment of homotopy relations (via "left" and "right" homotopies, involving respectively "cylinder" and "path" objects). This is in keeping with the highly non-autodual axiom on universal disjoint sums in M, which we finally introduced by the end of yesterday's reflection.

To come back however upon the relationships between the four basic "homotopy notions" introduced in yesterday's notes, I would now rather symbolize these relations in the following diagram of maps between
[p. 129] sets

$$\begin{tikzcd}
Q(M) \arrow[r, bend left, "r"] \arrow[dd, bend right=60] & L(M) \arrow[l, bend left, "s"] \\
Q_0 = \mathrm{Hom}_1(M) \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] & \mathrm{Hom}_2(M) = L_0 \arrow[u, hook] \\
\mathrm{Hom}_3(M) \arrow[u, hook] & \mathrm{Hom}_4(M) \arrow[u, hook] \arrow[l, hook]
\end{tikzcd}$$

("homotopy structures" on M, defined in terms of homotopy relations, or homotopism structures)

(weak homotopy interval structures) (contractibility structures) ,

Déployer le diagramme, trop tassé dans les deux dimensions. [I think I've fixed that; I also reversed the brace for clarity.]

which in terms of the preceding reflections, and yesterday's, is self-explanatory. The vertical arrow from $Q(M)$ to $\mathrm{Hom}_3(M)$ is the canonical retraction – in terms of the latter, and its composition with $s : L(M) \to Q(M)$, there *are* ways after all to associate to any R in $Q(M)$ or W in $L(M)$ a homotopy structure on M, provided only M admits finite products, by using weak homotopy intervals. If h is the homotopy structure thus defined, we get a priori a functor

$$M_h \to M_R = M/R \quad \text{resp.} \quad M_h \to M_W = W^{-1}M,$$

provided we assume R resp. W satisfy the "accessory assumption", namely that the corresponding functor $M \to M'$ (where M' is either M_R or M_W) commute with finite products.

The main fact to remember from this whole discussion, it seems to me, is that there are not really *four*, but only *three* essentially distinct types of structure (among yesterday's) we may consider upon M as "homotopy-flavored" structures, namely

$$\begin{array}{l} \text{homotopy structures} \\ \mathrm{Hom}_1(M) \leftrightarrow \mathrm{Hom}_2(M) \end{array} \Leftarrow \begin{array}{l} \text{weak homotopy} \\ \text{interval structures} \\ \mathrm{Hom}_3(M) \end{array} \Leftarrow \begin{array}{l} \text{contractibility} \\ \text{structures } \mathrm{Hom}_4(M). \end{array}$$

It would seem at present that the homotopy structures that naturally come up in our present "modelizer story" are all of the strictest type, and even describable in terms of just one *generating contractible weak homotopy interval* (I would like to drop the qualification "weak", definitely when a contractibility assumption comes in!), and even a *generating contractor*, with commutative idempotent composition law. [p. 130]

Compatibility of a functor $u : M \to N$ with a homotopy structure on M.

53 Before pursuing yesterday's reflection about the π_0-functor and its relation to homotopy structures on M, it seems more convenient to interpolate some more or less obvious "functorialities" on the homotopy notions just developed. They all seem to turn around the relationship of such notions in M with a more or less arbitrary functor

$$u : M \to N,$$

for the time being I am not making any special assumption on u. In terms of the four ways we got for describing a homotopy structure in M, we get four corresponding natural conditions of "compatibility" of u with a given homotopy structure h, namely:

(i) If $f \underset{h}{\sim} g$ in M, i.e. $(f, g) \in R_h$, then $u(f) = u(g)$.

(i') If $\mathbb{I} = (I, \delta_0, \delta_1) \in \Sigma_h$ is a weak homotopy interval, then $u(\delta_0) = u(\delta_1)$.

(ii) If $f \in W_h$, i.e., f is a homotopism, then $u(f)$ is an isomorphism.

(ii') If $X \in C_h$ is a contractible object, then $u(X)$ is a final object (NB here we assume that $u(e_M)$ is a final object in N).

(NB It is understood implicitly, whenever dealing with intervals and with contractible objects, that M admits finite products.)

The conditions (i) and (ii) are clearly equivalent, and equivalent to the requirement that u factors into

(1) $$M \to M_h \to N,$$

where $M \to M_h$ is the canonical functor of M into the corresponding *homotopy-types* category. We also have the tautological implications (i) $\Rightarrow$ (i') and (ii) $\Rightarrow$ (ii'). Moreover we have (trivially) (ii') $\Rightarrow$ (i') whenever the homotopy structure on M is a weak homotopy interval structure, and *moreover* the functor u commutes to finite products. All these implications are summarized in the diagram

$$\begin{tikzcd} \text{(i)} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{(ii)} \arrow[d, Rightarrow] \\ \text{(i')} \arrow[u, Rightarrow, bend left, "\mathrm{Hom}_3 \, *"] & \text{(ii')} \arrow[l, Rightarrow, "\mathrm{Hom}_4"'] \end{tikzcd},$$

where the symbol Hom_3 or Hom_4 indicates that the implication qualified by it is valid provided we assume that the homotopy structure on M is in Hom_3 (namely is defined in terms of weak homotopy intervals) resp. in Hom_4 (namely is a contractibility structure), and where (*) denotes the extra assumption of commutation of u with finite products.

[p. 131] We'll say the functor u is *compatible with the homotopy structure h* on M, if it satisfied the equivalent conditions (i), (ii), i.e., if it factors as in (1) above. In case u commutes with finite products, and if either the homotopy structure h can be described in terms of weak homotopy intervals, or in terms of contractible objects, the compatibility of u with h can be checked correspondingly, either by (i'), or by (ii').

Compatibility of a homotopy structure with a set W of "weak equivalences". The homotopy structure h_W.

54 An important particular case is the one when

$$N = W^{-1}M$$

is a localization on M by a set of arrows in M

$$W \subset \mathrm{Fl}(M).$$

We'll say that the homotopy structure h on M is *compatible with W*, if it is with the canonical functor $M \to W^{-1}M$. If W is *strongly saturated*, this is most readily expressed by the condition that $W_h \subset W$, i.e., any homotopism is in W (i.e., "any homotopism is a weak equivalence", if the elements of W are named "weak equivalences"); in case M admits finite products and the localization functor commutes to these (e.g., the case (M,W) is a *strict* modelizer), and if moreover h is a contractibility structure, it is sufficient to check that for any contractible X, the projection $X \to e$ is in W.

If we don't assume or know beforehand that W is strongly saturated, but just saturated say, we may still introduce a more stringent compatibility condition, by saying that the homotopy structure h and W are *strictly compatible* if $W_h \subset W$. Using the saturation condition c') on W, it is easily seen that in the case when h is a contractibility structure, then $W_h \subset W$ (strict compatibility) is equivalent to: for contractible X, the projection $X \to e$ is "universally in W", or (as we'll say) W*-aspheric*. Indeed, to deduce from this that any homotopism $f : X \to Y$ is in W, we are reduced to checking that any endomorphism of either X or Y which is homotopic to the identity map, is in W. Now this will follow from the assumption, and the following

Proposition. *Assume the homotopy structure h on M can be defined by a generating set Σ_h^0 of weak homotopy intervals $\mathbb{I} = (I, \delta_0, \delta_1)$ which are* W-aspheric (*i.e., I W-aspheric over e*)*, where $W \subset \mathrm{Fl}(M)$ is any saturated subset* (*in fact, mildly saturated is enough*)*. Then W is the inverse image by the canonical functor $M \to M_h = \overline{M}$ of a subset $\overline{W} \subset \mathrm{Fl}(\overline{M})$, i.e., if f, g are homotopic arrows in M, if one is in W, so is the other. Moreover* (*if W is saturated*) *$W_h \subset W$, i.e., h and W are strictly compatible.*

The first statement is just the "homotopy lemma" part b) (page 99), the second follows by the argument sketched above. [p. 132]

We're about back now to the context we started with three days ago (par. 48, page 98 and following), where we started with a W (viewed as a notion of "weak equivalence"), and in terms of W constructed various homotopy notions – namely those, we would now say, corresponding to the homotopy structure defined by the set of all intervals $\mathbb{I}$ in M which are W-aspheric (i.e., I is W-aspheric over e). As a matter of fact, we were a little stricter still, by restricting to intervals which are moreover "disjoint" (and which we called "homotopy intervals" relative to W), but this restriction now definitely appears as awkward and artificial. I will henceforth call *homotopy intervals* (with respect to W), any interval (not necessarily a separated one) which is W-aspheric. Let h_W be the corresponding homotopy structure on M, which is a weak homotopy interval structure admitting the set of all W-homotopy intervals as a generating set of weak homotopy intervals. (Clearly, there will be many weak homotopy intervals for this structure, which are far from being W-aspheric, i.e., far from being homotopy intervals.) Of course, as stated in the preceding proposition, h_W and W are strictly compatible, i.e.,

$$W_{h_W} \subset W,$$

i.e., any h_W-homotopism is in W (i.e., is a "weak equivalence"). As a matter of fact, the definition of homotopy notions in terms of W we gave in loc. sit. were just the widest one we could think of by that time,

11.4. By which we mean that $I \to e$ is "universally in W". The terminology used here for "W-aspheric" is highly ambiguous, cf. discussion p. 181 and following.

which would ensure the "compatibility" of these notions with W, in a sense which wasn't technically clear (not even definable at that point) as it is now, but however reasonably clear in terms of mathematical "bon sens". At present though the question arises rather naturally whether the homotopy structure h_W we selected "au flair" by that time is indeed the best one, namely the widest one, we could get. More explicitly, this means whether the homotopy structure h_W is the *widest* (in terms of the natural order relation considered in the previous paragraph) among all those which are compatible with W in the strict sense $W_{h_W} \subset W$. Now this is certainly not so, if we are not a little more specific about restricting to homotopy structure definable in terms of a weak homotopy interval structure. For instance, if we take for W a homotopism structure on M, compatible with products, corresponding to a homotopy structure h, to say h_W is the "best" would imply that W itself can be described in
[p. 133] terms of weak homotopy intervals which is not always the case. (Take for instance M to be an abelian category, say projective complexes of modules and quasi-isomorphism between these; in this case, more generally whenever M is a "zero objects" namely one which is both initial and final, any interval in M is trivial, i.e., $\delta_0 = \delta_1$, and hence any weak homotopy interval structure on M is trivial, namely W_h is reduced to isomorphisms...)

Thus the more reasonable question here is whether any homotopy structure h on M, definable in terms of a weak homotopy interval structure, and such that $W_h \subset W$, satisfies $h \leq h_W$. Clearly, for such an h, any weak homotopy interval $\mathbb{I}$ (for h) satisfies $u(\delta_0) = u(\delta_1)$, where $u : M \to W^{-1}M$ is the canonical functor (indeed, it is enough for this that $W_h \subset \overline{W}$ instead of $W_h \subset W$, where $\overline{W}$ is the strong saturation of W), and conversely, if $W = \overline{W}$ and if moreover *u commutes to finite products*. On the other hand, $h \leq h_W$ means that any $\mathbb{I} \in \Sigma_h$ is in Σ_{h_W}, which also means that its endpoint sections δ_0, δ_1 are h_W-homotopic, namely may be joined by a chain of sections, any two consecutive of which are related by some $\mathbb{J}$-homotopy, where $\mathbb{J}$ is a *W-aspheric interval*. Thus we get the:

Proposition. *Let W a saturated set of arrows in M (M stable under finite products), hence a corresponding homotopy structure h_W on M, defined in terms of W-aspheric intervals*† *in M as a generating set of weak homotopy intervals for h_W. Consider the following conditions:*

(i) *h_W is the widest of all homotopy structures h on M which are*
  (a) *strictly compatible with W, i.e., such that $W_h \subset W$ and moreover*
  (b) *definable in terms of a weak homotopy interval structure.*

(ii) *For any object I of M and two sections δ_0, δ_1 of I such that $u(\delta_0) = u(\delta_1)$ (where $u : M \to W^{-1}M$ is the canonical functor), δ_0 and δ_1 are h_W-homotopic, namely can be joined by a chain of elementary homotopies as above, involving W-*aspheric† *intervals $\mathbb{J}$.*

Then (ii) implies (i), and conversely if W is strongly saturated and moreover u commutes to finite products.

†11.4. Cf. note on preceding page.

In connection with the π_0-functor, we are going to get pretty natural conditions in terms of 0-connectedness for ensuring (ii) and hence (i), which should apply I guess to all "reasonable" modelizers (M, W). It would thus seem that in practical terms, the definition of h_W is the best, in all cases of actual interest to us. Of course, in case W is strongly saturated and u commutes with finite products, the widest h is the one whose weak homotopy intervals are triples (I, δ_0, δ_1) satisfying $u(\delta_0) = u(\delta_1)$, which we could have used instead of just W-aspheric intervals, which are also the intervals such that $I \to e$ is in W (and hence universally so in terms of the assumption made of compatibility of u with products). The trouble with working with this h, rather than with h_W as above, is twofold though: a) The assumptions of strong saturation on W and compatibility with products are not so readily verified in the cases of interest to us, and the second moreover is not always satisfied, e.g., there are test categories which are not strict, i.e., elementary modelizers which are not strict; b) the condition $u(\delta_0) = u(\delta_1)$ is not readily verified in terms of W directly, whereas the condition of W-asphericity is – still more so if W is compatible with products and hence $I \to e$ is W-aspheric just means it is in W. [p. 134]

55 Let's now look at "*morphisms*" between categories endowed with homotopy structures, (M_1, h_1) and (M_2, h_2) say. The natural definition here is to take as morphisms between these homotopy structures the functors $u : M_1 \to M_2$ that give rise to a commutative square of functors *Maps between homotopy structures.*

$$\begin{tikzcd} M_1 \arrow[r, "u"] \arrow[d] & M_2 \arrow[d] \\ \overline{M}_1 \arrow[r, "\overline{u}"] & \overline{M}_2 \end{tikzcd},$$

where the vertical arrows are the canonical functors into the respective homotopy-types categories, and $\overline{u}$ a suitable functor, necessarily unique. The existence of $\overline{u}$ can be expressed at will in terms of the Hom_1 or Hom_2 structures, namely as

(i) $f \underset{h_1}{\sim} g$ implies $u(f) \underset{h_2}{\sim} u(g)$,

or as

(ii) $f \in W_{h_1}$ implies $u(f) \in W_{h_2}$.

These conditions, when M_1 and M_2 have final objects and these are respected by u, imply that u transforms weak homotopy intervals into weak homotopy intervals, and homotopisms into homotopisms. Conversely, *if u commutes with finite products*, and if h_1 can be defined by a weak homotopy interval structure (respectively, by a contractibility structure), then for u to be a morphism of homotopy structures, it is (necessary and) sufficient that u carry weak homotopy intervals (resp. contractible objects) into same.

In case h_1 is described in terms of a generating set $\Sigma^0_{h_1}$ of weak homotopy intervals, and if *u commutes with finite products*, the most economic way often to express that u is a morphism of homotopy structures, is by [p. 135] the condition that for any $\mathbb{I}$ in $\Sigma^0_{h_1}$, $u(\mathbb{I})$ be a weak homotopy interval in M_2, namely $u(\delta_0) \underset{h_2}{\sim} u(\delta_1)$. If we assume moreover that the intervals $\mathbb{I}$ in $\Sigma^0_{h_1}$ are contractible, the previous condition is equivalent to $u(I)$ being a contractible object of M_2 for any $\mathbb{I}$ in $\Sigma^0_{h_1}$. The case I am mainly thinking of, of course, is the one when h_1 can be described by a single generating weak homotopy interval, possibly even contractible, or even by a (generating) contractor. In the latter case, because of commutation of u with finite products, $u(\mathbb{I})$ will be equally a contractor – and contractible for h_2 iff u is a morphism of homotopy structures.

In the precedent paragraphs, I forgot to mention the reduction of the corresponding compatibility conditions (of a functor $u : M \to N$, or of a saturated $W \subset \mathrm{Fl}(M)$) with a homotopy structure h, when the latter is defined in terms of just a generating set Σ^0_h of weak homotopy intervals, possibly reduced to a single one, and moreover *u or W is "compatible with finite products"*. In the first case, it is enough to check $u(\delta_0) = u(\delta_1)$ for any $\mathbb{I}$ in Σ^0_h – and if $\mathbb{I}$ is contractible, this amounts to demanding $u(\mathbb{I})$ is a final object of N. In the second case it is enough for strict compatibility, i.e., $W_h \subset W$, to check that for any $\mathbb{I}$ in Σ^0_h, $I \to e$ is in W (this condition is also necessary, if any $\mathbb{I}$ in Σ^0_h is contractible). Even if W is not supposed compatible with finite products, namely $M \to W^{-1}M$ does not commute with finite products, it is still sufficient for $W_h \subset W$ that all $\mathbb{I}$'s in Σ^0_h be *W-aspheric* (as stated in prop. (p. 131)), and this is necessary too, if the $\mathbb{I}$'s are contractible.

4.4.

56 *Another glimpse upon canonical modelizers. Provisional working plan – and recollection of some questions.*

This seemingly endless review of generalities on homotopy notions is getting a little fastidious - and still I am not quite through yet I feel. One main motivation for embarking on this review was one strong impression which grew out of the reflections of now just one week ago (paragraph 48), namely that the interesting "test functors" from a test category A into a modelizer (M, W) are those which factor through the full subcategory M_{c} of *contractible objects* of M. The presumable extra condition to put on a functor $A \to M_{\mathrm{c}}$ to correspond to an actual test functor i from A to M are strikingly weak, such as $i^*(\mathbb{I})$ should be aspheric over $e_{A^\wedge}$ under the assumption we got a contractible generating homotopy interval $\mathbb{I}$ in M. In any case, if M^0_{c} is any full subcategory of M_{c} which gives rise (by taking intervals in M^0_{c}) to a family of homotopy intervals which [p. 136] *generates* the homotopy structure h_W on M associated to W, it should suffice, if M is a "canonical modelizer", that the asphericity condition on $i^*(\mathbb{I})$ should be verified for any $\mathbb{I}$ in M^0_{c}; this will presumably turn out in due course as part of the definition (still ahead) of a "canonical" modelizer. Now, the most evident way to meet this condition, is to take for i a *fully faithful* functor *whose image contains* M^0_{c}, or what amounts essentially to the same, any full subcategory of M containing M^0_{c}! As we would like though A to be "small", this will be feasible only if M^0_{c} is small – hence the significance of the condition of a small generating family of weak homotopy intervals for h_W (which will imply that we can find such a family with *contractible* intervals, provided only the homotopy structure h_W can be described by a contractibility structure, as we did indeed assume). Just as the homotopy structure h_W of M was defined in terms of W, conversely the notion of weak equivalences W should be recoverable in terms of the homotopy structure, and more specifically in terms of the subcategory M_c and the small "generating" subcategories M_c^0 of M_c, which we may now as well denote by A, by taking the inclusion functor $i : A \to M$ of such an A, hence a functor

$$i^* : M \to A^\wedge,$$

and taking

$$W = (i^*)^{-1}(W_{A^\wedge}).$$

Of course, we'll have still to check under which general conditions upon a pair (M, W) of a category and a saturated set of arrows W, or rather, upon a pair (M, M_c) of a category endowed with a contractibility structure $M_c = C_h$ (where we think of h as an h_W), is it true that the saturated set of arrows

$$W(A) = (i^*)^{-1}(W_{A^\wedge}) \subset \mathrm{Fl}(M)$$

in M does not depend upon the choice of the full small homotopy-generating subcategory A of M_c (if restrictive conditions are needed indeed). It may be reasonable to play safe, to restrict at first to subcategories A which are stable under finite products in M, which will ensure that A is a *strict* test category, i.e., $A^\wedge$ is a *strict* elementary modelizer, namely $A^\wedge$ is totally aspheric. But such restriction – as well as to test functors which are fully faithful – should be a provisional one, as ultimately we want of course to be able to use test categories such as $\triangle$ for "testing" rather general (canonical) modelizers, whereas $\triangle$ is by no means stable under products, nor embeddable *faithfully* in modelizers such as (Spaces) say.

This expectation of W to be recoverable in terms of the corresponding homotopy structure $h = h_W$ on M takes its full meaning, when joined with another one, namely that the latter can be canonically described in terms of the category structure of M and the corresponding notion of 0-connectedness. This latter expectation is extremely strongly grounded, and I'll come back to it circumstantially very soon I think (I started on it two days ago, but then it got too late to take it to the end, and yesterday was spent on some formal digressions. . .). The two "expectations" put together, when realized by carefully cutting out the suitable notions, should imply that the structure of any "canonical modelizer" is indeed determined "canonically" in terms of its category structure alone. [p. 137]

To come back to the relationship between test categories and categories of the type M_c^0, the idea which has been lurking lately is that possibly, test categories can be viewed as no more, no less, as categories endowed with a homotopy structure (necessarily unique) which is a contractibility structure, and for which *all objects are contractible*. At any rate, there must be a very close relationship between the two notions, which I surely want to understand. But as it would be quite unreasonable to restrict the notion of a test category (and its weak and strong variants) to categories admitting finite products, this shows that for a satisfactory understanding of the above relationship, we should be able to work with contractibility structures, and presumably too with weak homotopy interval structures, in categories A where we do not make the assumption of stability under products. Thus in the outline of the last few days, I still wasn't general enough it would seem! This situation reminds me rather strongly of the early stages when developing the language of sites, and restricting to sites where fiber-products exist – this seems by then a very weak and natural assumption indeed, before it appeared (first to Giraud, I believe) that it was quite an awkward and artificial restriction indeed, which had to be overcome in order to work really at ease...

All this now gives a lot of interesting things to look up in the short run! I'll make a provisional plan of work, as follows:

a) Relation between a homotopy structure and the π_0 functor, and description of the so-called canonical homotopy structures.

b) Write down in the end the "key result" on test functors $A \to (\mathrm{Cat})$ which is overripe since the reflections of four days ago (par. 47).
[p. 138] Presumably, this will yield at the same time an axiomatic characterization of $W_{(\mathrm{Cat})}$, namely of the notion of weak equivalence for functors between categories.

c) At this point, we could go on and try and carry through the similar characterization for test functors $A \to B\hat{}$, where A and B are both test categories. There are also some generalities to develop about "morphisms" between test categories, which is ripe too for quite a while and cannot be pushed off indefinitely – here would be the right moment surely. If the expected "key result" for test functors $A \to B\hat{}$ carries through nicely it could presumably be applied at once in order to study general test functors $A \to M$, and thus get the clues for cutting out "the" natural notion of a canonical modelizer, which "was in the air" since the "naive question" of par. 46 (page 95).

d) However, there is another approach to canonical modelizers which is just appearing, via the idea (described above) of associating canonically a notion of "weak equivalence" W to a homotopy structure of type Hom_4, i.e., to a contractibility structure, subject possibly to some restrictions. This ties in, as explained above, with a closer look at the relationship between test categories, and "coarse" contractibility structures (where all objects are contractible).

It would seem unreasonable to push off a) and b) any longer now – so I'll begin with these. I am hesitant however between c) and d) – with a feeling that the later approach d) may well turn out to be technically the most expedient one. Both have to be carried through anyhow, and the two together should give a rather accurate picture of what canonical modelizers are about.

On the other hand, there are still quite a bunch of questions which have been waiting for investigation – for instance the list of questions of nearly three weeks ago (page 42). Among the six questions stated there, four have been settled, or are about to be settled through the previous program (if it works out), questions 4) and 6) remain, the first one being about $A^\wedge$ being a closed model category, and about the homotopy structure of (Cat). There are a number of more technical questions too, for instance I did not finish yet my review of the "standard" test categories and never wrote down the proof that Δ^{f} (simplices with face operations and no degeneracies) is indeed a weak test category. But for the time being, all these questions appear as somewhat marginal with respect to the strong focus the reflection has been gradually taking nearly since the very beginning – namely an investigation of modelizers and, more specifically, the gradual unraveling of a notion of "canonical modelizer". I certainly feel like carrying this to the end at once, without any digressions except when felt relevant for the main focus at present. As for choosing precedence between c) and d), it is still time to decide, when we're through with a) and b)! [p. 139]

Relation of homotopy structures to 0-connectedness and π_0. The canonical homotopy structure h_M of a category M.

57 Relation of homotopy structures to 0-connectedness and to π_0. Here we're resuming the reflection started in par. 51 F) (page 122). All we did there was to introduce some conditions on a category M, namely a) to e) (page 123), and introduce the functor

$$\pi_0 : M \to (\text{Sets}).$$

As before, we'll assume now M satisfies conditions a) to d) (the first three, or rather b) and c), are all that is needed for defining the functor π_0), and will not assume e), or the equivalent e') of π_0 commuting to finite products, unless explicitly specified.

We suppose now, moreover, M endowed with a homotopy structure h. We'll say that h is π_0-*admissible*, or simply 0-*admissible*, if h is compatible with the functor π_0, which can be expressed by either one of the following two equivalent conditions (cf. page 130):

(i) $f \underset{h}{\sim} g$ implies $\pi_0(f) = \pi_0(g)$,

for any two maps f, g in M, or

(ii) $f \in W_h$ (i.e., f a homotopism) implies $\pi_0(f)$ bijective.

Another equivalent formulation is that the functor π_0 factors through the quotient category M_h of homotopy types

$$M \to M_h \to (\text{Sets}),$$

we'll still denote by π_0 the functor $M_h \to (\text{Sets})$ obtained.

If h is 0-admissible, it satisfies (i') and (ii') below:

(i') For any weak homotopy interval $(I, \delta_0, \delta_1) \in \Sigma_h$, $\pi_0(\delta_0) = \pi_0(\delta_1)$.

As $\pi_0(e_M)$ is a one-point set by the assumption d) on M, for any section δ of an object I, $\pi_0(\delta)$ may be described as just an element of $\pi_0(I)$, which is the unique connected component of I through which factors the given section; thus (i') can be expressed by saying that for any weak homotopy interval, the two "endpoints" belong to the same connected component of I, a natural condition indeed! It is automatically satisfied if I is connected. In fact, (i') is satisfied iff Σ_h admits a generating subset Σ_h^0 made up with *connected* intervals.

[p. 140]

(ii') Any contractible object X is connected (and hence 0-connected).

Conversely (cf. page 130), if condition e) holds, i.e., π_0 commutes to products, and if moreover h is definable in terms of Σ_h, i.e., comes from a weak homotopy interval structure, then (i') implies 0-admissibility. If M admits even a generating family of contractible weak homotopy intervals, namely if h comes from a contractibility structure on M, then (ii') equally implies 0-admissibility.

Remarks. 1) We can generalize these converse statements, by dropping the condition e) on M, but demanding instead that the connected component involved (namely I itself in case (ii')) is not only 0-connected, but even "0-*connected over* e_M", which just means that its products by any 0-connected object of M is again 0-connected. In the case (ii'), this condition es equally *necessary* for admissibility. (The corresponding statements could have been made in the general context of page 130 of course...)

2) The name of 0-admissibility suggests there may exist correspondingly "higher" notions of n-admissibility for h, where n is any natural integer. I *do* see a natural candidate, namely whenever we have a functor π_n from M to n-truncated homotopy types (as is the case, say, when M is either a topos – we then rather get *pro*homotopy types – or a modelizer). But it would seem that in all cases of geometrical significance, and when moreover h is defined by a weak homotopy interval structure, that 0-admissibility implies already n-admissibility for any n.

Due to the existence of sup for an arbitrary subset, in each of the ordered sets $\mathrm{Hom}_i(M)$ ($i \in \{0,3,4\}$) of homotopy structures in M (either unqualified, or weak homotopy interval structures, or contractibility structures), it follows that for each of these three types of homotopy structures, there is a widest one h_i among all those of this type which are 0-admissible. We are interested here, because of topological motivations, by the case of Hom_3, namely weak homotopy interval structures. We call the corresponding homotopy structure $h = h_3$ the *canonical homotopy structure* of M. In case M is totally 0-connected (by which we mean condition e)), the weak homotopy intervals for this structure are just those intervals for which δ_0, δ_1 correspond to the same connected component of I – and we get a generating set of weak homotopy intervals, by just taking *all connected intervals*.

[p. 141]

Remark. If M is not totally 0-connected, we still get a description of Σ_h, as those intervals such that for any 0-connected X, the corresponding sections of $X \times I$ over X correspond to the same connected component of $X \times I$. It is enough for this that the common connected component I_0 of I for δ_0, δ_1 should be 0-connected over e – it is not clear to me whether this condition is equally necessary. Anyhow, presumably the case M totally 0-connected will be enough for all we'll have to do.

In case M is totally 0-connected, the homotopy notions in M for the canonical homotopy structure are just those which can be described in terms of "homotopies" using connected intervals – which is intuitively the first thing that comes to mind indeed, when trying to mimic most naively, in an abstract categorical context, the familiar homotopy notions for topological spaces.

Let let

$$W \subset \mathrm{Fl}(M)$$

be any saturated set of arrows in M (viewed as a notion of "weak equivalence" in M). Consider the corresponding homotopy structure h_W on M, defined in terms of W-aspheric intervals as a generating family of weak homotopy intervals. Let h_M be the canonical homotopy structure on M. Thus the condition

$$h_W \leq h_M$$

just means that h_W is 0-admissible, or equivalently, that W-aspheric objects over e which have a section, are 0-connected (for simplicity, I assume from now on M totally 0-connected). This looks like a very reasonable condition indeed, if W should correspond at all to the intuitions associated to the notion of "weak equivalence"! As a matter of fact, this condition is clearly implied by the condition that W itself should be "0-*admissible*", by which we mean that the functor π_0 is compatible with W, i.e., transforms weak equivalences into bijections, or equivalently, factors through $M \to W^{-1}M$.

What we are looking for however is conditions on W for the *equality*

$$h_W = h_M$$

to hold. When the previous condition (expressing $h_W \leq h_M$) is satisfied, all that remain is to express the opposite inequality, which is done in the standard way. We thus get:

Proposition. *Let M be a category, assume M totally* 0*-connected* (*i.e., satisfying conditions* a)b)c)e) *of page 123*). *Let $W \subset \mathrm{Fl}(M)$ be a saturated set of arrows in M, consider the associated homotopy structure h_W on M, with W-aspheric intervals as a generating family of weak homotopy intervals. Consider also the canonical homotopy structure h_M on M, with* 0*-connected intervals as a generating family of weak homotopy intervals* (*thus $h_M = h_{W_0}$, where $W_0 \subset \mathrm{Fl}(M)$ consists of all arrows made invertible by the functor $\pi_0 : M \to$* (Sets)). *In order for the equality $h_W = h_M$ to hold, it is necessary and sufficient that the following two conditions be satisfied:* [p. 142]

a) *Any object I of M which is W-aspheric over e and admits a section, is connected* (*it is enough for this that W by* 0*-admissible, i.e., $f \in W$ imply $\pi_0(f)$ bijective*)*;*

b) *For any connected object I of M and any two sections δ_0, δ_1 of I, these can be "joined" by a finite chain of sections s_i $(0 \le i \le n)$, $s_0 = \delta_0$, $s_n = \delta_1$, such that for any two consecutive ones, there exists an object J, W-aspheric over e, a map $J \to I$ and two sections of J mapped into the sections s_i, s_{i+1} of I.*

The condition a) says there are not too many weak equivalences, whereas b) says there are still enough for "testing" connectedness in terms of W-aspheric intervals. Both conditions look plausible enough!

Next step one would think of, in this context, is to give conditions on W (independently of the previous ones) which will allow to express W in terms of h_W. But for this, I should develop first a description of a notion of "weak equivalence" in terms of an arbitrary homotopy structure h on M, as contemplated in the previous paragraph. I decided however to give precedence to the "key result" still ahead.

Maybe the condition $h_W = h_M$, expressed in the previous proposition, merits a name – we'll say that the notion of weak equivalence W is "*geometric*", if it satisfies the two conditions a) and b) above, or rather the slightly stronger a') in place of a) – namely 0-admissibility of W (plus b) of course). The conditions a') and b) are the explicit ones for checking – but for using that W is geometric, the more conceptual statement $h_W = h_M$ (besides 0-admissibility) is the best. Thus, as any *contractor* in M which is connected (hence a weak homotopy interval for h_M) is h_M-contractible, it is h_W-contractible, and hence W-aspheric over e, and conversely of course. Using this, we get a converse to part
[p. 143] c) and d) of the proposition of page 121, in the present context, which we may state in a more complete form as follows:

Proposition. *Let M be a totally 0-connected category, W a geometric saturated set of arrows in M. Assume moreover that for any two objects in M, the object* $\mathbf{Hom}(X,Y)$ *in M exists. Let X be an object of M, then the following conditions a) to c") are equivalent:*

a) *For any object Y,* $\mathbf{Hom}(Y,X)$ *is 0-connected.*

a') *For any object Y,* $\mathbf{Hom}(Y,X)$ *is W-aspheric (i.e., its map to e is $\in W$).*

a") *For any object Y,* $\mathbf{Hom}(Y,X)$ *is W-aspheric over e (i.e., its map to e is "universally in W").*

a"') *For any object Y,* $\mathbf{Hom}(Y,X)$ *is h_W-contractible.*

b)b')b")b"') *Same as above, with Y replaced by X.*

c) *For any object Y, $Y \to$* $\mathbf{Hom}(X,Y)$ *is in W.*

c') *For any object Y, $Y \to$* $\mathbf{Hom}(X,Y)$ *is W-aspheric.*

c") *For any object Y, Y as a subobject of* $\mathbf{Hom}(X,Y)$ *is a deformation retract with respect to h_W.*

If moreover X has a section, these conditions are equivalent to:

d) *X is h_M-contractible.*

d') *X is h_W-contractible.*

Example. Let $M = (\mathrm{Cat})$, the assumptions on M are clearly satisfied. If we take for W the usual weak equivalences, it is clear too that W satisfies a') and b) above, i.e., W is geometric. Thus the preceding proposition is just an elaboration of the result stated after the prop. of page 97 – which was the moment when it became clear that contractibility was an important notion in the context of test categories and test functors, and which was the main motivation too for the somewhat lengthy trip through generalities on homotopy notions, which is coming now (in the long last!) to a provisional end...

To check if the notions developed in this section are handy indeed, I would still like to try them out in the case M is an elementary modelizer $A\hat{}$, corresponding to a test category A. But it's getting late this night...

5.4. [p. 144]

58 Some comments still on the canonical homotopy structure of a category M, which we assume again totally 0-connected. Two sections of an object K are homotopic iff they belong to the same connected component of X, thus we get an *injective* canonical map from homotopy-classes of sections

Case of totally 0-connected category M. The category $\overline{(\mathrm{Cat})}$ of (small) categories and homotopy classes of functors.

$$\overline{\Gamma}(X) = \overline{\mathrm{Hom}}(e, X) \to \pi_0(X).$$

We are interested in the case when this map is always a bijection, or what amounts to the same, when M satisfies the extra condition:

f) Every 0-connected objects of M has a section.

Proposition. *Let M be a category satisfying conditions a) to d) of page 123. If M satisfies moreover condition f) above, it satisfies also condition* refit:51.F.e*, i.e., M is totally 0-connected.*

Indeed, let X, Y be 0-connected objects, we must prove $X \times Y$ is 0-connected. First, it is "non-empty", because it got a section (as X and Y each have a section). All that remains to do is show that two connected components of $X \times Y$ are equal. By assumption f), each has a section, say (s_i, t_i) with $i \in \{1, 2\}$. These two sections can be joined by a two-step chain

$$(s_1, t_1), \quad (s_2, t_1), \quad (s_2, t_2)$$

the first step contained in $X \times t_1$ which is connected, the second in $s_2 \times Y$ which is connected, hence the two sections belong to the same connected component, qed.

The condition f) is *strictly* stronger than e) (when a) to d) are satisfied), as we see by taking for M the category of all sheaves on an irreducible topological space X – thus M is a totally aspheric topos, a lot better it would seem than just totally 0-connected, but condition f) is satisfied iff the topos is equivalent to the final topos or "one-point topos", i.e., iff the topology of X is the chaotic one (X and $\emptyset$ are the only open subsets). We'll say M is *strictly totally 0-connected*, if it satisfies the conditions a) to f) (where e) is a consequence of the others). Thus, if M is strictly totally 0-disconnected, we get for any object X a canonical bijection, functorial in X

$$\overline{\Gamma}(X) \stackrel{\text{def}}{=} \begin{matrix}\text{homotopy classes}\\ \text{of sections of } X\end{matrix} \xrightarrow{\sim} \pi_0(X). \tag{1}$$

Assume now, moreover, that X, Y are two objects of M such that the
[p. 145] object $\mathbf{Hom}(X, Y)$ exists. Then, using the observation of cor. 2 (p. 122), we get the familiar relationship

$$\overline{\mathrm{Hom}}(X, Y) \xrightarrow{\sim} \pi_0(\mathbf{Hom}(X, Y)), \tag{2}$$

where $\overline{\mathrm{Hom}}$ denotes homotopy classes of maps, with respect to the canonical homotopy structure of M.

The first example which I have in mind is the case $M = (\mathrm{Cat})$. I was a little short yesterday about $W_{(\mathrm{Cat})}$ being "geometric" – the condition a') of page 142 is evident indeed in terms of the (geometric!) definition of weak equivalence in terms of non-commutative cohomology (where we need to case only about zero dimensional cohomology!). But condition b) is a consequence of the fact that 0-connectedness in (Cat) can be checked using only $\mathbb{\Delta}_1$ (i.e., $\mathbb{\Delta}_1$ is a generating contractor for the canonical homotopy structure of (Cat)), and that $\mathbb{\Delta}_1$ is $W_{(\mathrm{Cat})}$-aspheric over the final category e, i.e., that for any C in (Cat), the projection

$$C \times \mathbb{\Delta}_1 \to C$$

is in $W_{(\mathrm{Cat})}$. I want to start being attentive from now on about what exactly are the formal properties of $W_{(\mathrm{Cat})}$ we are using – it really seems they boil down to very few, which we have kept using without ever having to refer to the "meaning" of weak equivalence in terms of cohomology (except for proving the formal properties we needed, in terms of the precise definition of weak equivalence we have been starting with from the very beginning).

In terms of the canonical homotopy structure in (Cat), admitting $\mathbb{\Delta}_1$ as a generating homotopy interval (a contractor, as a matter of fact), we now get a notion of two functors from a category X to a category Y being homotopic, and a corresponding notion of homotopy classes of functors from X to Y, which are in one-to-one correspondence with the connected components of the category $\mathbf{Hom}(X, Y)$ of all functors from X to Y. An elementary homotopy from a functor f to another g (with respect of course to the basic generating interval $\mathbb{\Delta}_1$, which is always understood here) is nothing else but a morphism from f into g. Thus, homotopy classes of functors are nothing but equivalence classes for the equivalence relation in $\mathrm{Hom}(X, Y)$ generated by the relation "there exists a morphism from f to g". To consider at all this equivalence relation, rather than the usual one of isomorphy between functors, is rather far from the spirit in which categories are generally being used – as is also the very notion of a functor which is a weak equivalence, which has been our starting point. Topological motivation alone, it seems, would induce anyone to introduce such barbaric looking notions into category theory!

From our point of view, the main point for paying attention to the homotopy relation for functors, is of course because homotopic functors define the same map in the localized category (Hot), which category for the time being is (together with the modelizers designed for describing it) our main focus of attention. According to the general scheme of homotopy theory as reviewed previously, homotopy classes of functors give rise to a quotient category of (Cat), which is at the same time a localization of (Cat) with respect to homotopisms, and which we'll denote by $\overline{\text{(Cat)}}$. Thus we get a factorization of the canonical functor from (Cat) to (Hot) [p. 146]

$$\text{(Cat)} \to \overline{\text{(Cat)}} \to \text{(Hot)}.$$

This is just the factorization of the localization functor with respect to $W_{\text{(Cat)}} = W$, through the "*partial localization*" with respect to the smaller set $W_h \subset W$ consisting of homotopisms only. All we have used about $W = W_{\text{(Cat)}}$ (from which the localization $\text{(Hot)} = W^{-1}\text{(Cat)}$ is deduced) for getting this factorization, was that $f \in W$ implies $\pi_0(f)$ bijective, and that the projections $C \times \triangle_1 \to C$ are in W. (Of course, we assume tacitly that W is saturated too.) We may view (Hot) as a localization of $\overline{\text{(Cat)}}$ with respect to the set of arrows $\overline{W}$ corresponding to W – as a matter of fact, W may be viewed as the inverse image of $\overline{W}$ by the canonical functor from (Cat) to its partial localization $\overline{\text{(Cat)}}$ – the category of "homotopy types" relative to the canonical homotopy structure of (Cat).

6.4. As a matter of fact, the condition about π_0 visibly isn't needed!

Well-known analogies would suggest at this point that we may well be able to describe (Hot) in terms of $(\overline{\text{(Cat)}}, \overline{W})$ by a *calculus of fractions* – right fractions presumably, or maybe either right or left. This may possibly lead to a direct proof of the notion of weak equivalence we have been working with being strongly saturated, without having to rely upon Quillen's closed model theory. But it is not yet the moment to pursue this line of thought, which would take us off the main focus at present.

Case of the "next best" modelizer (Spaces) *– and need of introducing the π_0-functor as an extra structure on a would-be modelizer M.*

59 Before pushing ahead, I would like to make still another point about the work done yesterday – a point suggested by looking at the case of the modelizer (Spaces), which after all is the next best "naive" modelizer, less close to algebra than (Cat), but still worth being taken into account! This category satisfies the conditions a) to f), *except* the condition c) – which would mean that the connected components of a space (as defined in terms of usual topology) are open subsets, which is true (for a space and its open subsets) *only* for locally connected spaces. The point is that this doesn't (or shouldn't) really matter – the way topological spaces are used as "homotopy models" in standard homotopy theory, it is *pathwise connected* components that count, and not the topological ones. In terms of these, there is still a canonical functor [p. 147]

$$\pi_0 : \text{(Spaces)} \to \text{(Sets)},$$

this functor however is no longer left adjoint of the functor in opposite direction, associating to every set E, the corresponding discrete topological space. (To get an adjunction, we should have to restrict to the category of pathwise locally connected topological spaces.) It doesn't matter visibly – all that's being used is that π_0 commutes to arbitrary sums, and takes pathwise 0-connected spaces into one-point sets.

This suggests that we should generalize the notions around the "canonical" homotopy structure on a category M, to the case of a category which need not satisfy the exacting conditions of total 0-connectedness, by introducing as an *extra structure* upon M a given functor

$$\pi_0 : M \to (\mathrm{Sets}),$$

subject possibly to suitable restrictions. The first which comes to mind here is commutation with sums – it doesn't seem though we've had to use this property so far. All we've used occasionally was existence of finite products in M, and commutation of π_0 to these.

If we think of M as a would-be modelizer, and therefore endowed with a hoped-for functor

$$M \to (\mathrm{Hot}),$$

there is a natural functor π_0 indeed on M, namely the composition

$$M \to (\mathrm{Hot}) \to (\mathrm{Sets}),$$

where the canonical functor

$$\pi_0 : (\mathrm{Hot}) \to (\mathrm{Sets})$$

is deduced from the π_0-functor $(\mathrm{Cat}) \to (\mathrm{Sets})$ considered previously, by factorization through the localized category (Hot) of (Cat). Thus, "the least we would expect" from a category M for being eligible as a modelizer is that there should be a natural functor π_0 around, corresponding to the intuition of connected components. In case of a "canonical" modelizer M (maybe we should say rather: canonical with respect to a given π_0), there is the feeling that the functor $M \to (\mathrm{Hot})$ we are after could eventually be squeezed out from just π_0, and that it could be viewed as something like a "total left derived functor" of the functor π_0. But this for the time being is still thin air...

[p. 148] What we *can* do however at present, in terms of a given functor π_0, is to introduce the corresponding notion of 0-connectedness (understood: with respect to π_0), namely objects X such that $\pi_0(X)$ is a one-point set, the notion of compatibility of a homotopy structure h on M with π_0, and the π_0-canonical (or simply, "canonical") homotopy structure on M, which now should be denoted by h_{π_0} rather than h_M (unless we write $h_{\mathbb{M}}$ where $\mathbb{M}$ denotes the pair (M, π_0)), which is the widest weak homotopy interval structure on M which is π_0-admissible, and can be described (assuming π_0 commutes with finite products) in terms of all 0-connected intervals as a generating family of weak homotopy intervals. The generalities of par. 54 about the relationship of $h_{\mathbb{M}}$ with h_W (where $W \subset \mathrm{Fl}(M)$) should carry over verbatim, as well as those of the next, provided everywhere 0-connectedness is understood relative to the given functor π_0, and "total 0-connectedness" is interpreted as just meaning that π_0 commutes to finite products. Thus, our contact with "geometry" via true honest connected components of objects was of short duration, and back we are to pure algebra with just a functor given which we call π_0, God knows why – the culprit for this change of perspective being poor modelizer (Spaces), which was supposed to represent the tie with so-called "topology"...

Case of strictly totally aspheric topos. A timid start on axiomatizing the set $\mathcal{W}$ of weak equivalences in (Cat).

60 I almost forgot I still have to check "handiness" of the notions developed yesterday, on the example of test categories or rather, the corresponding elementary modelizers $A^\wedge$. As usual, I can't resist being a little more general, so let's start with an arbitrary topos $\mathcal{A}$ first. It always satisfies conditions a) and b) of page 123. Condition c), namely that every object of $\mathcal{A}$ could be decomposed into a sum of 0-connected ones, is equivalent with saying that $\mathcal{A}$ admits a generating subcategory A made up with 0-connected objects. In this case, $\mathcal{A}$ is called *locally* 0*-connected* or simply, *locally connected* – which generalizes the notion known under this name from topological spaces to topoi. On the other hand, condition d) is expressed by saying that the topos considered is 0*-connected* – equally a generalization of the corresponding notion for spaces. Condition e), about the product of two 0-connected objects being 0-connected, is a highly unusual one in ordinary topology. For a topological space, it means that the space is irreducible (hence reduced to a point if the space is Hausdorff). In accordance with the terminology introduced yesterday, we'll say that $\mathcal{A}$ is *totally* 0*-connected* if it is locally connected, and if the product of two 0-connected objects is again 0-connected. The standard arguments show that for this, it is enough that the product of two elements in A be 0-connected. The topos is called *strictly totally* 0*-connected* if it is locally connected, and if moreover every 0-connected object admits a section – which (as we saw earlier today) implies $\mathcal{A}$ is totally 0-connected, as the wording suggests. It amounts to the same to demand that every "non-empty" object have a section – and for this it is enough that the elements in A have a section. This latter condition is trivially checked for all standard test categories I've met so far (they all have a final object, and there maps of the latter into any other object of A). A noticeable counterexample here is $(\mathbb{\Delta}^{\mathrm{f}})^\wedge$ (semisimplicial face complexes, *without degeneracies*), where the *weak* test category $\mathbb{\Delta}^{\mathrm{f}}$ hasn't got a final object ($\mathbb{\Delta}^0$ definitely isn't!) and no $\mathbb{\Delta}_n$ in $\mathbb{\Delta}^{\mathrm{f}}$ except $\mathbb{\Delta}_0$ only has got a section.

[p. 149]

Remark. I wonder, when $\mathcal{A}$ is totally 0-connected, and moreover modelizing, i.e., the Lawvere element is aspheric over e, if this implies $\mathcal{A}$ is totally aspheric, and that every element which is "non-empty" has a section (i.e., *strict* total 0-connectedness).

Next thing is to look at

$$W \subset \mathrm{Fl}(\mathcal{A}),$$

the set of weak equivalences (as defined by non-commutative cohomology of topoi), and see if it is “geometric” (page 142). Condition a’) is clearly satisfied, there remains the condition b), namely whether 0-connectedness of an object of $\mathcal{A}$ ($\mathcal{A}$ supposed totally 0-connected) can be tested, using “intervals” which are *aspheric over* e. More specifically, we want to test that two sections of I belong to the same connected component, using for “joining” them intervals that are aspheric over e. The natural idea here is to assume the generating objects in A to be aspheric over e (which implies $\mathcal{A}$ is *totally aspheric,* not only totally 0-connected), and to use these objects (endowed with suitable sections) as testing intervals. This goes through smoothly, indeed, if we assume moreover strict total zero-connectedness. Thus:

Proposition. *Let $\mathcal{A}$ be a topos which is strictly totally aspheric* (*namely totally aspheric, and every “non-empty” object has a section*). *Then the set $W \subset \mathrm{Fl}(\mathcal{A})$ of weak equivalences in $\mathcal{A}$ is “geometric”, and accordingly, the homotopy structure h_W defined in terms of aspheric homotopy intervals, is the same as the canonical homotopy structure $h_{\mathcal{A}}$ defined in terms of merely* 0*-connected homotopy intervals. Moreover, for any set $A \subset \mathrm{Ob}\,\mathcal{A}$ which is generating and whose objects are* 0*-connected, the set of* 0*-connected intervals $\mathbb{I} = (I, \delta_0, \delta_1)$ with I in A, generate the homotopy structure h_W.*

[p. 150] A topos $\mathcal{A}$ as in the proposition (namely strictly totally aspheric) need not be a modelizer, i.e., the Lawvere element L need not be aspheric, or what amounts to the same because of $h_W = h_{\mathcal{A}}$ and L being a contractor, L need not be connected: take $\mathcal{A} = (\mathrm{Sets})$! I suspect though this to be the only counterexample (up to equivalence). For $\mathcal{A}$ to be a modelizer, we need only find an object in A which has got two *distinct* sections (because then they must be disjoint, i.e., $e_0 \cap e_1 = \varnothing_{\mathcal{A}}$, because e has only the full and the “empty” subobject, as a consequence of every “non-empty” object of $\mathcal{A}$ having a section), thus getting a “homotopy interval” (more specifically, a separated and relatively aspheric one) as requested for $\mathcal{A}$ to be a modelizer. Now for any would-be test category met with so far (except precisely $\mathbb{\Delta}^{\mathrm{f}}$ and the like, which are *not* test categories but only weak ones), this condition that there are objects in A which have more than just one “point” (= section), is trivially verified.

In case of a topos of the type $A\hat{}$, the notion of weak equivalence in $A\hat{}$ can be described (independently of cohomological notions) in terms of the notion of weak equivalence in (Cat), more precisely

$$W_A \overset{\mathrm{def}}{=} W_{A\hat{}} = i_A^{-1}(\mathcal{W}),$$

where

$$i_A : A\hat{} \to (\mathrm{Cat}), \quad F \mapsto A_{/F}$$

is the canonical functor, and where

$$\mathcal{W} = W_{(\mathrm{Cat})}$$

is the set of weak equivalences in (Cat). These of course, for the time being, are defined in terms of cohomology (including a bit of non-commutative one in dimension 1...). We may however start with any

$\mathcal{W} \subset \mathrm{Fl}((\mathrm{Cat}))$ and look at which formal properties on $\mathcal{W}$ (satisfied for usual weak equivalences) allow our arguments to go through, in various circumstances. We may make a list of those which have been used today, and go on this way a little longer, with the expectation we'll finally wind up with an axiomatic characterization of weak equivalences, i.e., of $\mathcal{W}$, in terms of the category (Cat), say.

a) (Pour mémoire!) $\mathcal{W}$ is saturated (cf. page 101).

b) $\mathcal{W}$ is 0-admissible, i.e., if $f : C \to C'$ is in $\mathcal{W}$, $\pi_0(f)$ is bijective.

c) $\blacktriangle_1$ is $\mathcal{W}$-aspheric over $e = \blacktriangle_0$, i.e., for every C in (Cat), the projection $C \times \blacktriangle_1 \to C$ is in $\mathcal{W}$.

d) Any C in (Cat) which has a final element is $\mathcal{W}$-aspheric, i.e, $C \to e$ is in $\mathcal{W}$.

The condition a) will be tacitly understood throughout, when taking a $\mathcal{W}$ to replace usual weak equivalences. Conditions b) and c) then were seen to be enough to imply that $h_{\mathcal{W}} = h_{(\mathrm{Cat})}$. On the other hand, one sees at once that for the proposition over for a topos $A^\wedge$ which is strictly totally aspheric, when we define now $\mathcal{W}_A \subset \mathrm{Fl}(A^\wedge)$ as just $i_A^{-1}(\mathcal{W})$, in order to conclude $h_{\mathcal{W}_A} = h_{A^\wedge}$, all we made use of was (besides saturation of $\mathcal{W}$ of course, i.e., a)) b) and d). [p. 151]

One may object that d) isn't expressed in terms of the category structure of (Cat) only, but we could express it in terms of this structure, by the remark that C has a final object iff there exists a $\blacktriangle^{-}$-homotopy of id_C to a constant section of C (this "section" will indeed be defined necessarily by a final object of C). As was to be expected, in this formulation, as in c) too, the object $\blacktriangle_1$ of (Cat) is playing a crucial role. But at this point it occurs to me that c) implies d), by the homotopy lemma – thus for the time being all we needed was a)b)c).

61 Remembering about the promised "key result" at last!

61 We now in the long last get back to the "key result" promised time ago, and which we kept pushing off. To pay off the trouble of the long digression in between, maybe it'll come out more smoothly. It shall be concerned with the functor

$$i : A \to (\mathrm{Cat}), \tag{1}$$

where in the end A will be (or turn out to be) a strict test category, and we want to give characterizations for i to be a (weak) test functor, namely the corresponding functor

$$i^* : (\mathrm{Cat}) \to A^\wedge \tag{2}$$

to induce an equivalence between the localizations, with respect to "weak equivalences". We'll now be a little more demanding, and instead of just assuming it is the usual notion of weak equivalence either in (Cat) or in $A^\wedge$, I'll assume that the set $\mathcal{W}_A \subset \mathrm{Fl}(A^\wedge)$ is defined in terms of a saturated set $\mathcal{W} \subset \mathrm{Fl}((\mathrm{Cat}))$ of arrows in (Cat), by taking the inverse image by the functor

Plus a little more, see below (8).

$$i_A : A^\wedge \to (\mathrm{Cat}) \tag{3}$$

as above, whereas in the target category (Cat) of (2), we'll work with another saturated set $\mathcal{W}'$ – thus besides (1), the data are moreover

$$\mathcal{W}, \mathcal{W}' \subset \mathrm{Fl}((\mathrm{Cat})), \tag{4}$$

[p. 152] two saturated sets of arrows in (Cat), with no special assumption otherwise for the time being. We will introduce the properties we need on these, as well as on A and on i, stepwise as the situation will tell us. We want to derive a set of conditions ensuring that both (2) and (3) induce equivalences for the respective localizations, namely

$$\overline{i}^* : (\mathcal{W}')^{-1}(\mathrm{Cat}) \xrightarrow{\simeq} \mathcal{W}_A^{-1}A^\wedge \quad \text{and} \quad \overline{i}_A : \mathcal{W}_A^{-1}A^\wedge \xrightarrow{\simeq} \mathcal{W}^{-1}(\mathrm{Cat}). \tag{5}$$

We may assume beforehand that A is a weak test category ("with respect to $\mathcal{W}$") and hence the second functor in (5) is already an equivalence, in which case the condition that the first functor in (5) exist and be an equivalence (existence just meaning the condition

$$\mathcal{W}' \subset (i^*)^{-1}(\mathcal{W}_A), \quad \text{i.e.,} \quad i^*(\mathcal{W}') \subset \mathcal{W}_A) \tag{*}$$

is equivalent to the corresponding requirement for the composition

$$f_i : (\mathrm{Cat}) \xrightarrow{i^*} A^\wedge \xrightarrow{i_A} (\mathrm{Cat}), \tag{6}$$

namely that this induce an equivalence

$$(\mathcal{W}')^{-1}(\mathrm{Cat}) \xrightarrow{\simeq} \mathcal{W}^{-1}(\mathrm{Cat}). \tag{7}$$

As a matter of fact, we are going to be slightly more demanding (in accordance with the notion of a weak test functor as developed previously, cf. page 85), namely that the inclusion (*) be in fact an equality

$$\mathcal{W}' = (i^*)^{-1}(\mathcal{W}_A), \tag{8}$$

the similar requirement for the functor i_A (3) being satisfied by the very definition of $\mathcal{W}_A$ in terms of $\mathcal{W}$ as

$$\mathcal{W}_A = i_A^{-1}(\mathcal{W}). \tag{9}$$

In view of this, the extra requirement (8) boils down to the equivalent requirement in terms of the composition f_i (6):

$$\mathcal{W}' = f_i^{-1}(\mathcal{W}). \tag{10}$$

To sum up, we want to at least develop sufficient conditions on the data $(\mathcal{W}, \mathcal{W}', A, i)$ for (8) to hold (which allows to define the first functor in (5), whereas the second is always defined), and the functors in (5) to be equivalences; or equivalently, for (10) to hold, hence a functor (7), and for the latter to be an equivalence, and equally i_A to induce an equivalence for the localizations (i.e., the second functor (5) an equivalence). It should be noted that the latter condition depends only on $(A, \mathcal{W})$, not on i nor on $\mathcal{W}'$ – it will be satisfied automatically if we assume A to be a *weak test category* relative to $\mathcal{W}$ (namely i_A and the right adjoint functor [p. 153]

$$j_A : (\mathrm{Cat}) \to A^\wedge$$

to induce quasi-inverse equivalences for the localizations $\mathcal{W}_A^{-1}A^\wedge$ and $\mathcal{W}^{-1}(\mathrm{Cat})$). We have already developed handy n.s. conditions for this in case $\mathcal{W} = W_{(\mathrm{Cat})}$ – and it would be easy enough to look up which formal properties exactly on $W_{(\mathrm{Cat})}$ have been used in the proof, if need be. At any rate, we know beforehand that we can find $(A, \mathcal{W})$ such that A be a weak test category (and even a strict test category!) relative to $\mathcal{W}$. When $(A, \mathcal{W})$ are chosen this way beforehand, the question just amounts to finding conditions on $(\mathcal{W}', i)$ for (10) to hold and for (7) to be an equivalence of categories. If we find conditions which actually can be met, then we get as a byproduct the formula (10) precisely, which says that there is *just one* $\mathcal{W}'$ satisfying the conditions on $\mathcal{W}'$, namely $f_i^{-1}(\mathcal{W})$! Of course, taking $\mathcal{W}$ to be just $W_{(\mathrm{Cat})}$, it will follow surely that $\mathcal{W}'$ is just $W_{(\mathrm{Cat})}$ – i.e., we should get an axiomatic characterization of weak equivalences.

Let's now go to work, following the idea described in par. 57 (pages 96–98), and expressed mainly in the basic diagram of canonical maps in (Cat), associated to a given object C in (Cat):

$$(11)\qquad \begin{tikzcd} A_{/C} \arrow[r] & A_{/\!\!/C} & \\ & A \times C \arrow[u] \arrow[r] & A \end{tikzcd},$$

which will allow to compare $f_i(C) = A_{/C}$ with C. When $\mathcal{W} = W_{(\mathrm{Cat})}$, it was seen in loc. cit. that the two latter among these three arrows are in $\mathcal{W}$, provided (for the middle one) we assume that i takes its values in the subcategory of (Cat) of all contractible categories. What remained to be done, for getting the conditions for f_i to be "weakly equivalent" to the identity functor (and hence induce an equivalence for the localizations) was to write down conditions for the first functor in (11), $A_{/C} \to A_{/\!\!/C}$, to be in $\mathcal{W}$. We have moreover to be explicit on the conditions to put on a general $\mathcal{W}$, in order for the two latter maps in (11) to be in $\mathcal{W}$. For the last functor, this conditions as a matter of fact involves both A and $\mathcal{W}$, it is clearly equivalent to

W 1) A is $\mathcal{W}$-aspheric over e.

If we assume that A has a final element, this condition is satisfied provided $\mathcal{W}$ satisfies the condition (where there is no A anymore!) that any X in (Cat) with final element is $\mathcal{W}$-aspheric over e – a condition which is similar to condition d) above (page 150), but a littler stronger still (as we want $X \times C \to C$ in $\mathcal{W}$ for any C), it is a consequence however of condition c), as was seen on page 150 using the remark that X is $\mathbb{\Delta}_1$-contractible. Thus we get the handy condition [p. 154]

W 1') $\mathcal{W}$ satisfies condition c) of page 150, i.e., $\mathbb{\Delta}_1$ is $\mathcal{W}$-aspheric over e, which will allow even to handle the case of an A which is contractible (for the canonical homotopy structure of (Cat), namely $\triangle_1$-contractible), and not only when A has a final object.

To insure that the canonical map

$$(*) \qquad A \times C \to A_{/\!\!/C}$$

is in $\mathcal{W}$, using the argument on page 97, we'll add one more condition to the provisional list on page 150, namely:

e) For any cartesian functor $u : F \to G$ of two fibered categories over a third one B (everything in (Cat)), such that the induced maps on the fibers are in $\mathcal{W}$, u is in $\mathcal{W}$.

We are now ready to state the condition we need (stronger than W 1')):

W 2) $\mathcal{W}$ satisfies conditions a) to c) (page 150) and e) above.

As a matter of fact, a) to c) ensure that $\mathcal{W}$ is "geometric", i.e., essentially $h_{\mathcal{W}} = h_{(\mathrm{Cat})}$, hence the proposition page 143 applies, to imply that the maps

$$C \to \mathbf{Hom}(i(a), C)$$

are in $\mathcal{W}$ (they are even $h_{\mathcal{W}}$-homotopisms) and by condition e) this implies that (*) above is in $\mathcal{W}$. We don't even need b) (0-admissibility for $\mathcal{W}$), as all we care about is $h_{(\mathrm{Cat})} \le h_{\mathcal{W}}$ (not the reverse inequality), but surely we're going to need b) or something stronger soon enough, as $\mathcal{W} = \mathrm{Fl}((\mathrm{Cat}))$ say surely wouldn't do!

Now to the last (namely first) map of our diagram (11), namely

$$(12) \qquad A_{/C} \to A_{/\!\!/C}.$$

To give sufficient conditions for this to be in $\mathcal{W}$, we want to mimic the standard asphericity criterion for a map in (Cat), which we have used constantly before. This leads to the extra condition

[p. 155] f) Let $u : X' \to X$ be a map in (Cat) such that for any a in X, the induced category $X'_{/a}$ be $\mathcal{W}$-aspheric, i.e., $X'_{/a} \to e$ is in $\mathcal{W}$ (or what amounts to the same if we assume d), e.g., if we assume the stronger condition c), the induced map $X'_{/a} \to X_{/a}$ is in $\mathcal{W}$). Then u is in $\mathcal{W}$.

If $u : X' \to X$ satisfies the condition stated above, namely that after any base change $X_{/a} \to X$, the corresponding map $u_{/a}$ is in $\mathcal{W}$, we'll say that u is *weakly* $\mathcal{W}$-aspheric (whereas "$\mathcal{W}$-aspheric" means that after *any* base change $Y \to X$, the corresponding f_Y is in $\mathcal{W}$). Thus, condition f) can be stated as saying that *a weakly $\mathcal{W}$-aspheric map in* (Cat) *is in $\mathcal{W}$.*

For making use of this latter assumption on $\mathcal{W}$, we have to look at how the induced categories for the functor (12) look like, which functor (I recall) induced a bijection on objects. These can be described as pairs (a, p), with a in A and p a map in (Cat)

$$p : i(a) \to C.$$

An easy computation shows the

Lemma. *Let (a, p) as above. The induced category $(A_{/C})_{/(a,p)}$ (for the functor (12)) is canonically isomorphic to the induced category $A_{/G}$, where G is the fibered product in $A^\wedge$ displayed in the diagram*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & a \arrow[d] \\ i^*(\mathbf{Fl}(C)) \arrow[r] & i^*(C) \end{tikzcd} , \tag{13}$$

where

$$\mathbf{Fl}(C) \stackrel{\text{def}}{=} \mathbf{Hom}(\triangle_1, C)$$

and where the second horizontal arrow in (13) is the i^*-transform of the target map in (Cat)

$$\mathbf{Fl}(C) \xrightarrow{t} C.$$

Corollary. *In order for (12) to be weakly $\mathcal{W}$-aspheric, it is n.s. that the map*

$$i^*(t) : i^*(\mathbf{Fl}(C)) \to i^*(C) \tag{14}$$

in $A^\wedge$ be $\mathcal{W}_A$-aspheric (i.e., be "universally in $\mathcal{W}_A$").

To make the meaning of the latter condition clear, it should be noted that the condition f) on $\mathcal{W}$ guarantees precisely that for a map $u : F' \to F$ in $A^\wedge$ (A any category) to be $\mathcal{W}_A$-aspheric, it is n.s. that the corresponding map $i_A(u)$ in (Cat) be weakly $\mathcal{W}$-aspheric – the kind of thing we have been constantly using before of course, when assuming $\mathcal{W} = W_{(\text{Cat})}$. [p. 156]

It is in the form of (14) that weak $\mathcal{W}$-asphericity of (12) will actually be checked, whereas it will be *used* just by the fact that (12) is in $\mathcal{W}$.

6.4.

An embarrassing case of hasty over-axiomatization. The unexpected riches. . .

62 I finally stopped with the notes last night, by the time when I started feeling a little uncomfortable. A few minutes of reflection then were enough to convince me that definitely I hadn't done quite enough preliminary scratchwork yet on this "key result" business, and embarked overoptimistically upon a "mise en équation" of the situation, with the pressing expectation that a characterization of weak equivalences should come out at the same time. First thing that became clear, was that the introduction of two different localizing sets of arrows $\mathcal{W}, \mathcal{W}'$ in (Cat) was rather silly alas, nothing at all would come out unless supposing from the very start $\mathcal{W} = \mathcal{W}'$. Indeed, the crucial step for getting the "key result" on test functors we are out for, goes as follows.

As the target map

$$t : \mathbf{Fl}(C) = \mathbf{Hom}(\triangle_1, C) \to C$$

in (Cat) is clearly a homotopy retraction, and $i^* : (\text{Cat}) \to A^\wedge$ commutes with products, we do have a good hold on the condition (14) of the last corollary, namely that $i^*(t)$ be $\mathcal{W}_A$-aspheric – e.g., it is enough that the contractor $i^*(\Delta_1)$ in $A^\wedge$ be $\mathcal{W}_A$-aspheric over $e_{A^\wedge}$ (for instance, it is often enough it be 0-connected!). In view of the corollary and the condition f) on $\mathcal{W}$ (last page), we thus get a very good hold upon the map

$$(*) \qquad A_{/C} \to A_{/\!/C}$$

in (Cat) being in $\mathcal{W}$, and hence on all three maps in the diagram (11) (page 153) being in $\mathcal{W}$. With this in mind, the key step can be stated as follows:

Lemma. *Assume that $\mathcal{W}$ satisfies the conditions a)c)e)f) (pages 150, 154, 155), that A is $\mathcal{W}$-aspheric over e (i.e., $A \times C \to C$ is in $\mathcal{W}$ for any C, which will be satisfied if A is Δ_1-contractible in* (Cat)*, for instance if A has a final or initial object), and that the objects $i(a)$ in* (Cat) *(for any a in A) are contractible (for the canonical homotopy structure of* (Cat)*, i.e., Δ_1-contractible, or even only for the wider homotopy structure $h_{\mathcal{W}}$ based* [p. 157] *on $\mathcal{W}$-aspheric homotopy intervals). Under these conditions, the following conclusions hold:*

a) $\mathcal{W} = f_i^{-1}(\mathcal{W})$ *(where $f_i = i_A i^*$ with yesterday's notations).*

b) *The functor $\overline{f}_i$ from $\mathcal{W}^{-1}$*(Cat) *to itself induced by f_i (which is defined because of a)) is isomorphic (canonically) to the identity functor, and hence is an equivalence.*

The use we have for the three maps in the diagram (12) is completely expressed in this lemma. The pretty obvious proof below would not work at all if in a) above, we replace $\mathcal{W}$ in the left hand side by a $\mathcal{W}'$! We have to prove that for a map $C \to C'$ in (Cat), this is in $\mathcal{W}$ iff $A_{/C} \to A_{/C'}$ is. Now this is seen from an obvious diagram chasing in the diagram below, using saturation condition b) on $\mathcal{W}$:

$$\begin{tikzcd}
A_{/C} \arrow[r] \arrow[d] & A_{/\!/C} \arrow[d] & A \times C \arrow[l] \arrow[r] \arrow[d] & C \arrow[d] \\
A_{/C} \arrow[r] & A_{/\!/C} & A \times C \arrow[l] \arrow[r] & C
\end{tikzcd},$$

where all horizontal arrows are already known to be in $\mathcal{W}$ (the assumptions in the lemma were designed for just that end). At the same time, we see that the corresponding statements are equally true for the functors $C \mapsto A_{/\!/C}$ and $C \mapsto A \times C$, and that two consecutive among the four functors we got from $H_{\mathcal{W}} = \mathcal{W}^{-1}$(Cat) to itself, deduced by localization by $\mathcal{W}$, are canonically isomorphic, which proves b) by taking the composition of the three isomorphisms

$$\gamma(A_{/C}) \xrightarrow{\sim} \gamma(A_{/\!/C}) \xrightarrow{\sim} \gamma(A \times C) \xrightarrow{\sim} \gamma(C),$$

where $\gamma : (\mathrm{Cat}) \to H_{\mathcal{W}} = \mathcal{W}^{-1}(\mathrm{Cat})$ is the canonical functor.

With this lemma, we have everything needed in order to write down the full closed chain of implications, between various conditions on

$(\mathcal{W}, A, i)$, from which to read off the "key result" we're after. Before doing so, I would like still to make some preliminary comments on the role of $\mathcal{W}$, and on the nature of the conditions we have been led so far to impose upon $\mathcal{W}$.

A first feature that is striking, is that all conditions needed are in the nature either of stability conditions (if such and such maps are in $\mathcal{W}$, so are others deduced from them), or conditions stating that such and such unqualified maps (the projection $\triangle_1 \times C \to C$ for any C, say) are in $\mathcal{W}$. We did not have any use of the only condition stated so far, namely b) (if $f \in \mathcal{W}$, $\pi_0(f)$ is bijective) of a *restrictive* type on the kind of arrows allowed in $\mathcal{W}$ – which is quite contrary to my expectations. Thus, all conditions are trivially met if we take $\mathcal{W} = \mathrm{Fl}((\mathrm{Cat}))$ – *all* arrows in (Cat)! This circumstance seems tied closely to the fact that, contrarily to quite unreasonable expectations, *we definitely do* not *get an axiomatic characterization of weak equivalences*, in terms of the type of properties of $\mathcal{W}$ we have been working with so far. As soon as one stops for considering the matter without prejudice, this appears rather obvious. As a matter of fact, using still cohomological invariants of topoi and categories, there are lots of variants of the cohomological definition of "weak equivalence", which will share all formal properties of the latter we have been using so far, and presumably a few more we haven't met yet. For instance, starting with any ring k (interesting cases would be $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Q}$), we may demand on a morphism of topoi [p. 158]

$$f : X \to Y$$

to induce as isomorphism for cohomology with coefficients in k, or with coefficients in any k-module, or with any twisted coefficients which are k-modules – already three candidates for a $\mathcal{W}$, depending on a given k! We may vary still more, by taking, instead of just one k, a whole bunch (k_i) of such, or a bunch of (constant) commutative groups – we are thinking of choices such as all rings $\mathbb{Z}/n\mathbb{Z}$, with possibly n being subjected to be prime to a given set of primes, along the lines of the Artin-Mazur theory of "localization" of homotopy types. And we may combine this with an isomorphism requirement on twisted non-commutative 1-cohomology, as for the usual notion of weak equivalence. Also, in all the isomorphism requirements, we may restrict to cohomology up to a certain dimension (which will give rise to "truncated homotopy types"). The impression that goes with the evocation of all these examples, is that the theory we have been pursuing, to come to an understanding of "models for homotopy types", while we started with just usual homotopy types in mind and a corresponding tacit prejudice, is a great deal richer than what we had in mind. Yesterday's (or rather last night's) embarrassment of finding out finally I had been very silly, is a typical illustration of the embarrassment we feel, whenever a foreboding appears of our sticking to inadequate ideas; still more so if it is not just mathematics but ideas about ourselves say or about something in which we are strongly personally implies. This embarrassment then comes as a rescue, to bar the way to an unwished-for overwhelming richness dormant in ourselves, ready to wash away forever those ideas so dear to us…

[p. 159] I am definitely going to keep from now on a general $\mathcal{W}$ and work with this and the corresponding localization, which in case of ambiguity we better won't denote by (Hot) any longer (as we might be thinking of usual homotopy types in terms of usual weak equivalences), but by $H_{\mathcal{W}}$ or $(\mathrm{Hot})_{\mathcal{W}}$, including such notions as rational homotopy types, etc. (for suitable choices of $\mathcal{W}$). The idea that now comes to mind here is that possibly, the usual $W_{(\mathrm{Cat})}$ of usual weak equivalences could be characterized as the *smallest* of all $\mathcal{W}$'s, satisfying the conditions we have been working with so far (tacitly to some extent), and maybe a few others which are going to turn up in due course – i.e. that the usual notion of weak equivalence is the *strongest* of all notions, giving rise to a modelizing theory as we are developing. This would be rather satisfactory indeed, and would imply that other categories $H_{\mathcal{W}}$ we are working with *are all localizations of* (Hot), with respect to a saturated set of arrows in (Hot), satisfying some extra conditions which it may be worth while writing down explicitly, in terms of the internal structures of (Hot) directly (if at all possible). All the examples that have been flashing through my mind a few minutes ago, do correspond indeed to equivalence notions weaker than so-called "weak" equivalence, and hence to suitable localizations of the usual homotopy category (Hot). But it is quite conceivable that this is not so for all $\mathcal{W}$'s, namely that the characterization just suggested of $W_{(\mathrm{Cat})}$ is not valid. This would mean that there are refinements of the usual notion of homotopy types, which would still however give rise to a homotopy theory along the lines I have been pursuing lately. There is of course an immediate association with Whitehead's *simple homotopy types* – maybe after all they can be interpreted as elements in a suitable localization $H_{\mathcal{W}}$ of (Cat) (and correspondingly, of any one of the standard modelizers, such as semi-simplicial complexes and the like)? In any case, sooner or later one should understand what the smallest of all "reasonable" $\mathcal{W}$'s looks like, and to which geometric reality it corresponds. But all these questions are not quite in the present main line of thought, and it is unlikely I am going to really enter into it some day…

63 *Review of terminology* (*provisional*).

What I should do though immediately, is to put a little order in the list of conditions for a set $\mathcal{W}$, which came out somewhat chaotically yesterday. After the notes I still did a little scratchwork last night, which I want now to write down, before coming to a formal statement of the "key [p. 160] result" – as this will of course make use of some list of conditions on $\mathcal{W}$.

First of all, I feel a review is needed of the few basic notions which have appeared in our work, relative to a set of arrows $W \subset \mathrm{Fl}(M)$ in a general category M. We will not give to the maps in W a specific name, such as "weak equivalences", as this may be definitely misleading, in the general axiomatic set-up we want to develop; here $W_{(\mathrm{Cat})}$ is just one among many possible $\mathcal{W}$'s and correspondingly for a small category A, $W_A = W_{A^\wedge}$ is just one among the many $\mathcal{W}_A$'s, associated to the previous $\mathcal{W}$'s. When $M = (\mathrm{Cat})$, it will be understood we are working with a fixed set $\mathcal{W} \subset \mathrm{Fl}((\mathrm{Cat}))$, consisting of the basic "equivalences", on which the whole modelizer story hinges. We may call them $\mathcal{W}$-equivalences –

for the time being there will be no question of varying $\mathcal{W}$.

Coming back to a general pair $(M, W \subset \mathrm{Fl}(M))$ (not necessarily a "modelizer"), we may call the maps in W W-equivalences. If M has a final object e, we get the corresponding notion of *W-aspheric object* of M, namely an object X such that the unique map

$$X \to e$$

is in W, i.e., is a W-equivalence. We'll define a *W-aspheric map*

$$f : X \to Y$$

in M as one which is "universally in W", by which I mean that for any base-change

$$Y' \to Y,$$

the fiber-product $X' = X \times_Y Y'$ exists (i.e., f is "squarable") and the map

$$f' : X' \to Y'$$

deduced from f by base change is in W. The thing to be quite careful about is that for an object X in M, to say that X is *W-aspheric over e* (meaning that the *map* $X \to e$ is W-aspheric) implies X is W-aspheric &c., but the converse need not hold true. This causes a slight psychological uneasiness, due to the fact I guess that the notion of a W-aspheric object has been defined after all in terms of the *map* $X \to e$, and consequently may be thought of as meaning is "in W over e". Maybe we shouldn't use at all the word "W-aspheric object" here, not even by qualifying it as "weakly W-aspheric" to cause a feeling of caution, but rather refer to this notion as "X is a W-object" – and denote by $M(W)$ the set of all these objects (or the corresponding full subcategory of M, and call X W-aspheric (dropping "over e") when it is "universally" a W-object. The terminology we have been using so far was of course suggested by the case when M is a *topos* and W the usual notion of weak equivalence, but then to call X in M "W-aspheric" or simply "aspheric" does correspond to the usual (absolute) notion of asphericity for the induced topos $M_{/X}$, only in the case when the topos M itself is aspheric. This is so in the case I was most interested in (e.g., M a modelizing topos), but if we want to use it systematically in the general setting, the term I used of "W-aspheric object" is definitely misleading. Thus we better change it now than never, and use the word "W-object" instead, and the notation $M(W)$. As for the notion of W-aspheric map, in the present case of a topos with the notion of weak equivalence, it does correspond to the usual notion of asphericity for the induced morphism of topoi

But we'll see immediately that this conflicts with the standard terminology in topoi – so no good either!

[p. 161]

$$M_{/X} \to M_{/Y},$$

which is quite satisfactory.

There is still need for caution with the notion of W-aspheric maps (W-aspheric objects have disappeared in the meanwhile!), when working in $M = (\mathrm{Cat})$ (and the same thing if $M = (\mathrm{Spaces})$). Namely, when we got a map $f : A \to B$ in (Cat), this is viewed for topological intuition as corresponding to a morphism of topoi

$$A\hat{} \to B\hat{}.$$

Now, the requirement that $f : A \to B$ should be $W_{(\mathrm{Cat})}$-aspheric is a lot stronger than the asphericity of the corresponding morphism of topoi. Indeed, the latter just means that for any base-change in (Cat) of the very particular "localization" or "induction" type, namely

$$B_{/b} \to B,$$

the corresponding map deduced by base change

$$f_{/b} : A_{/b} \to B_{/b}$$

is a weak equivalence (or equivalently, that $A_{/b}$ is aspheric), whereas W-asphericity of f means that the same should hold for *any* base-change $B' \to B$ in (Cat), or equivalently, that for *any* such base-change, with B' having a final element moreover, the corresponding category $A' = A \times_B B'$ is aspheric. To keep this distinction in mind, and because the weaker notion is quite important and deserves a name definitely, I will refer to this notion by saying f is *weakly W-aspheric* (returning to the case of a general $\mathcal{W} \subset \mathrm{Fl}((\mathrm{Cat}))$) if for any base change of the particular
[p. 162] type $B_{/b} \to B$ above, the corresponding map $A_{/b} \to B_{/b}$ is in $\mathcal{W}$. We could express this in terms of the morphism of topoi $A\hat{} \to B\hat{}$ by saying that the latter is W-aspheric – being understood that the choice of an "absolute" $\mathcal{W}$ in (Cat), implies as usual a corresponding notion of $\mathcal{W}$-equivalence for arbitrary morphisms of arbitrary topoi, in terms of the corresponding morphism between the corresponding homotopy types. (This extension to topoi of notions in (Cat) should be made quite explicit sooner or later, but visibly we do not need it yet for the time being.) One relationship we have been constantly using, and which is nearly tautological, comes from the case of a map

$$u : F \to G$$

in a category $A\hat{}$, hence applying i_A a map in (Cat)

$$A_F \to A_G.$$

For this map to be weakly $\mathcal{W}$-aspheric, it is necessary and sufficient that for any base-change in $A\hat{}$ of the particular type

$$G' = A \to G \quad \text{with } a \text{ in } A,$$

the corresponding map

$$F \times_G a \to a$$

in $A\hat{}$ be in $\mathcal{W}_A$ – a condition which is satisfied of course if u is $\mathcal{W}_A$-aspheric. This last condition is also necessary, if we assume that $\mathcal{W}$ satisfies the standard property we've kept using all the time in case of (Cat), namely that a map in (Cat) that is weakly W-aspheric, is in $\mathcal{W}$ (condition f) on page 155). Thus we get the

Proposition. *Assume that any map in* (Cat) *which is weakly $\mathcal{W}$-aspheric is in $\mathcal{W}$, and let A be any small category, $u : F \to G$ a map in $A^\wedge$. Then u is $\mathcal{W}_A$-aspheric (and hence in $\mathcal{W}_A$) iff $i_A(u) : A_{/F} \to A_{/G}$ is weakly $\mathcal{W}$-aspheric.*

The assumption we just made on $\mathcal{W}$ is of such constant use, that we are counting it among those we are making once and for all upon $\mathcal{W}$ (which I still have to pass in review).

As for $\mathcal{W}$-aspheric maps in (Cat), this is a very strong notion indeed when compared with $\mathcal{W}$-equivalence or even with weak $\mathcal{W}$-asphericity. We did not have any use for it yet, but presumably this notion will be of importance when it comes to a systematic study of the internal properties of (Cat) with respect to $\mathcal{W}$ (which is still in our program!).

Coming back to a general (M, W), we have defined earlier a canonical homotopy structure h_W on M, which we may call "associated to W" – this is also the weak homotopy interval structure on M, generated by intervals in M which are W-aspheric over e. This makes sense at least, provided in M finite products exist. If moreover M satisfies the conditions b) to d) of page 123 concerning sums and connected components of objects of M, we have defined (independently of W) the *canonical* homotopy structure h_M in M, which may be viewed as the weak homotopy interval structure generated by all 0-connected intervals in M. It still seems that in all cases we are going to be interested in, we have the equality [p. 163]

$$h_W = h_M.$$

When speaking of homotopy notions in M (such as f and g being homotopic maps, written

$$f \sim g,$$

or a map being a homotopism, or an object being contractible) it will be understood (unless otherwise stated) that this refers to the homotopy structure $h_W = h_M$. In case we should not care to impose otherwise unneeded assumptions which will imply $h_W = h_M$, we'll be careful when referring to homotopy notions, to say which structure we are working with.

Review of properties of the "basic localizer" $\mathcal{W}_{(\mathrm{Cat})}$.

64 We recall that a set of arrows $W \subset \mathrm{Fl}(M)$ is called *saturated* if it satisfies the conditions:

a) Identities belong to W.

b) If f, g are maps and fg exists, then if two among f, g, gf are in W, so is the third.

c) If $f : X \to Y$ and $g : Y \to X$ are such that gf and fg are in W, so are f and g.

On the other hand, *strong saturation* for W means that W is the set of arrows made invertible by the localization functor

$$M \to M_W = W^{-1}M,$$

or equivalently, that W can be described as the set of arrows made invertible by some functor $M \to M'$. The trouble with strong saturation

is that it is a condition which often is not easy to check in concrete situations. This is so for instance for the notion of weak equivalence in (Cat), and the numerous variants defined in terms of cohomology.
[p. 164] Therefore, we surely won't impose the strong saturation condition on $\mathcal{W}$ (which we may call the "*basic localizer*" in our modelizing story), but rather be happy if we can prove strong saturation as a consequence of other formal properties of $\mathcal{W}$, which are of constant use and may be readily checked in the examples we have in mind.

Let's give finally a provisional list of those properties for a "localizer" $\mathcal{W}$.

L 1) **(Saturation)** $\mathcal{W}$ is saturated, i.e., satisfies conditions a)b)c) above.

L 2) **(Homotopy axiom)** $\mathbb{\Delta}_1$ is $\mathcal{W}$-aspheric over e, i.e., for any C in (Cat), the projection

$$\mathbb{\Delta}_1 \times C \to C$$

is in $\mathcal{W}$.

L 3) **(Final object axiom)** Any C in (Cat) which has a final object is in (Cat)($\mathcal{W}$), i.e., $C \to e$ is in $\mathcal{W}$ (or, as we will still say when working in (Cat), C is $\mathcal{W}$-aspheric).

For a justification of this terminology,see proposition on p. 167 below.

L 3') **(Interval axiom)** $\mathbb{\Delta}_1$ is $\mathcal{W}$-aspheric, i.e., $\mathbb{\Delta}_1 \to e$ is in $\mathcal{W}$.

L 4) **(Localization axiom)** Any map $u : A \to B$ in (Cat) which is weakly $\mathcal{W}$-aspheric (i.e., the induced maps $A_{/b} \to B_{/b}$ are in $\mathcal{W}$) is in $\mathcal{W}$.

L 5) **(Fibration axiom)** If $f : X \to Y$ is a map in (Cat) over an object B of (Cat), such that X and Y are fiber categories over B and f is cartesian, and if moreover for any $b \in B$, the induced map on the fibers $f_b : X_b \to Y_b$ is in $\mathcal{W}$, then so is f.

These properties are all I have used so far, it seems, in the case $\mathcal{W} = W_{(\mathrm{Cat})}$ we have been working with till now, in order to develop the theory of test categories and test functors, including "weak" and "strong" variants, and including too the generalized version of the "key result" which is still waiting for getting into the typewriter. Let's list at once the implications

$$\text{L 2)} \Rightarrow \text{L 3)} \Rightarrow \text{L 3')},$$

and

> if L 1) and L 4) hold (saturation and localization axioms), then the homotopy axiom L 2) is already implied by the final object axiom L 3).

Thus, the set of conditions L 1) to L 4) (not including the last one L 5), i.e., the fibration axiom) is equivalent to the conjunction of L 1) L 3) L 4). This set of conditions is of such a constant use, that we'll assume it throughout, whenever there is a $\mathcal{W}$ around:

[p. 165] **Definition.** A subset $\mathcal{W}$ of Fl((Cat)) is called a *basic localizer*, if it satisfies the conditions L 1), L 3), L 4) above (saturation, final object and localization axioms), and hence also the homotopy axiom L 2).

These conditions are enough, I quickly checked this night, in order to validify all results developed so far on test categories, weak test categories, strict test categories, weak test functors and test functors with values in (Cat) (cf. notably the review in par. 44, pages 79–88), provided in the case of test functors we restrict to the case of loc. cit. when each of the categories $i(a)$ has a final object. All this I believe is justification enough for the definition above.

As for the fibration axiom L 5), this we have seen to be needed (at least in the approach we got so far) for handling test functors $i : A \to (\mathrm{Cat})$, while no longer assuming the categories $i(a)$ to have final objects (which was felt to be a significant generalization to carry through, in view of being able subsequently to replace (Cat) by more general modelizers). While still in the nature of a stability requirement, this fibration axiom looks to me a great deal stronger than the other axioms. Clearly L 5), together with the very weak "interval axiom" L 3') ($\Delta_1 \to e$ is in $\mathcal{W}$) implies the homotopy axiom. It can be seen too that when joined with L 1), it implies the localization axiom L 4) (using the standard device of a mapping-cone for a functor...). Thus, a basic localizer satisfying the fibration axiom L 5) can be viewed also as a $\mathcal{W}$ satisfying the conditions

L 1) (saturation), L 3') (interval axiom), L 5) (fibration axiom).

In the next section, after we will have stated the two key facts about weak test functors and test functors, which both make use of L 5), we'll presumably, for the rest of the work ahead towards canonical modelizers, assume the fibration condition on the basic localizer $\mathcal{W}$.

There are some other properties of weak equivalence $W_{(\mathrm{Cat})}$ and its manifold variants in terms of cohomology, which have not been listed yet, and which surely will turn up still sooner or later. Maybe it's too soon to line them up in a definite order, as their significance is still somewhat vague and needs closer scrutiny. I'll just list those which come to my mind, in a provisional order.

L a) 0-admissibility of $\mathcal{W}$, namely $f \in \mathcal{W}$ implies $\pi_0(f)$ bijective.

This condition, together with the homotopy axiom L 2), will imply [p. 166]

(1) $h_{\mathcal{W}} = h_{(\mathrm{Cat})}$ the canonical homotopy structure on (Cat) defined in terms of the generating contractor Δ_1 in (Cat),

whereas the homotopy axiom alone, I mean without a), will imply only the inequality

(1') $$h_{(\mathrm{Cat})} \le h_{\mathcal{W}},$$

which is all we care for at present. The latter implies that two maps in (Cat) which are Δ_1-homotopic (i.e., belong to the same connected component of $\mathbf{Hom}(X,Y)$ have the same image in the localization $\mathcal{W}^{-1}(\mathrm{Cat}) = (\mathrm{Hot}_{\mathcal{W}})$, and that any map in (Cat) which is a Δ_1-homotopism is in $\mathcal{W}$, and even is $\mathcal{W}$-aspheric if it is a "homotopy retraction" with respect to the Δ_1-structure $h_{(\mathrm{Cat})}$. However, in practical terms, even without assuming a) expressly, we may consider (1) to be always satisfied. This amounts indeed to the still weaker condition than a)

L a') If C in (Cat) is $\mathcal{W}$-aspheric over e, it is 0-connected, i.e., non-empty and connected.

But if it were empty, it would follow that for any X in (Cat), $\varnothing \to X$ is in $\mathcal{W}$ and hence $(\mathrm{Hot}_{\mathcal{W}})$ is equivalent to the final category. If C is non-empty and disconnected, choosing two connected components and one point in each to make C into a weak homotopy interval for $h_{\mathcal{W}}$, one easily gets that any two maps $f, g : X \rightrightarrows Y$ in (Cat) are $h_{\mathcal{W}}$-homotopic, hence have the same image in $(\mathrm{Hot}_{\mathcal{W}})$, which again must be the final category. Thus we get:

Proposition. *If $\mathcal{W}$ satisfies L 1), L 2) (e.g., $\mathcal{W}$ a basic modelizer), then we have equality (1), except in the case when $(\mathrm{Hot}_{\mathcal{W}})$ equivalent to the final category.*

This latter case isn't too interesting one will agree. Thus, we would easily assume (1), i.e., a'). But the slightly stronger condition a) seems hard to discard; even if we have not made any use of it so far, one sees hardly of which use a category of localized homotopy types $(\mathrm{Hot}_{\mathcal{W}})$ could possibly be, if one is not even able to define the π_0-functor on it! Thus, presumably we'll have to add this condition, and maybe even more, in order to feel $\mathcal{W}$ deserves the name of a "basic localizer"... Among other formal properties which still need clarification, even in the case of $W_{(\mathrm{Cat})}$, there is the question of exactness properties of the canonical functor

(2) $$(\mathrm{Cat}) \to \mathcal{W}^{-1}(\mathrm{Cat}) = (\mathrm{Hot}_{\mathcal{W}}).$$

I am thinking particularly of the following

[p. 167] L b) The functor (2) commutes with finite sums,

possibly even with infinite ones, which should be closely related to property a), and

L c) The functor (2) commutes with finite products

(maybe even with infinite ones, under suitable assumptions).

The following property, due to Quillen for weak equivalences, is used in order to prove for these (and the cohomological analogs) the fibration axiom L 5) (what we get directly is the case of cofibrations, as a matter of fact – cf. prop. page 97):

L d) If $f : C \to C'$ is in $\mathcal{W}$, so is $f^{\mathrm{op}} : C^{\mathrm{op}} \to (C')^{\mathrm{op}}$ for the dual categories.

This is about all I have in mind at present, as far as further properties of a $\mathcal{W}$ is concerned. The property c) however brings to mind the natural (weaker) condition that the cartesian product of two maps in $\mathcal{W}$ should equally be in $\mathcal{W}$. The argument in the beginning of par. 40 (p. 69) carries over here, and we get:

Proposition. *Let $\mathcal{W}$ be a basic localizer, and C in (Cat) such that C is $\mathcal{W}$-aspheric, i.e., $C \to e$ is in $\mathcal{W}$, then C is even $\mathcal{W}$-aspheric over e, i.e., for any A in C, $C \times A \to A$ is in $\mathcal{W}$.*

However, the proof given for the more general statement we have in mind (of proposition on page 69) does not carry over using only the localization axiom in the form L 4) it was stated above, as far as I can see. This suggests a stronger version L 4') of L 4) which we may have to use eventually, relative to a commutative triangle in (Cat) as on page 70

$$\begin{tikzcd} P' \arrow[rr, "F"] \arrow[dr] & & P \arrow[dl] \\ & C & \end{tikzcd},$$

when assuming that the induced maps (for arbitrary c in C)

$$F_{/c} : P'_{/c} \to P_{/c}$$

are in $\mathcal{W}$, to deduce that F itself is in $\mathcal{W}$. However, this "strong localization axiom" is a consequence (as is the weaker one L 4)) of the fibration axiom L 5), which implies also directly the property we have in mind, namely

$$f, g \in \mathcal{W} \text{ implies } f \times g \in \mathcal{W}.$$

To come to an end of this long terminological and notational digression, I'll have to say a word still about test categories, modelizers, and test functors. We surely want to use freely the terminology introduced so far, while we were working with ordinary weak equivalences, in the more general setting when a basic localizer $\mathcal{W}$ is given beforehand. As long as there is only one $\mathcal{W}$ around, which will be used systematically in all our constructions, we'll just use the previous terminology, being understood that a "modelizer" say will mean a "$\mathcal{W}$-modelizer", namely a category M endowed with a saturated $W \subset \mathrm{Fl}(M)$, such that $W^{-1}M$ is equivalent some way or other to $\mathcal{W}^{-1}(\mathrm{Cat})$. The latter category, however, I dare not just designate as (Hot), as this notation has been associated to the very specific situation of just ordinary homotopy types, therefore I'll always write $(\mathrm{Hot}_{\mathcal{W}})$ instead, as a reminder of $\mathcal{W}$ after all! If at a later moment it should turn out that we'll have to work with more than one $\mathcal{W}$ (for instance, to compare the $\mathcal{W}$-theory to the ordinary $W_{(\mathrm{Cat})}$-theory), we will of course have to be careful and reintroduce $\mathcal{W}$ in our wording, to qualify all notions dependent on the choice of $\mathcal{W}$. [p. 168]

7.4.

Still another review of the test notions (relative to given basic localizer $\mathcal{W}$).

65 It has been over a week now and about eighty pages typing, since I realized the need for looking at more general test functors than before and hit upon how to handle them, that I am grinding stubbornly through generalities unending on homotopy notions. The grinding is a way of mine to become familiar with a substance, and at the same time getting aloof of it climbing up, sweatingly maybe, to earn a birds-eye view of a landscape and maybe, who knows, in the end start a-flying in it, wholly at ease. . . I am not there yet! The least however one should expect, is that the test story should now go through very smoothly. As I have been losing contact lately with test categories and test functors, I feel it'll be worth while to make still another review of these notions, leading up to the key result I have been after all that time. It will be a way both to gain perspective, and check if the grinding has been efficient indeed...

If

$$u : M \to M'$$

is a functor between categories endowed each with a saturated set of arrows, W and W' say, we'll say u is "*model preserving*" (with respect to (W, W') if it satisfies the conditions:

[p. 169] a) $W = u^{-1}(W')$ (hence the functor $\overline{u} : W^{-1}M \to (W')^{-1}M'$ exists),

b) the functor $\overline{u}$ is an equivalence.

We do not assume beforehand that (M, W), (M', W') are modelizers (with respect to a given basic localizer $\mathcal{W} \subset \mathrm{Fl}((\mathrm{Cat}))$), but in the cases I have in mind, we'll know beforehand at least one of the pairs to be a modelizer, and it will follow the other is one too.

In all what follows, a basic localizer $\mathcal{W}$ is given once and for all (see definition on page 165). For the two main results below on the mere general test functors, we'll have to assume $\mathcal{W}$ satisfies the fibration axiom L 5) (page 164). We are going to work with a fixed small category A, without any other preliminary assumptions upon A, all assumptions that may be needed later will be stated in due course. Recall that A can be considered as an object of (Cat), and we'll say A is $\mathcal{W}$-aspheric if $A \to e$ is in $\mathcal{W}$, which implies that $A \to e$ is even $\mathcal{W}$-aspheric, i.e., $A \times C \to C$ is in $\mathcal{W}$ for any C in (Cat). More generally, if F is in $A\hat{}$, we'll call F $\mathcal{W}$-aspheric if the category $A_{/F}$ is $\mathcal{W}$-aspheric. Thus, to say A is $\mathcal{W}$-aspheric just means that the final object $e_{A\hat{}}$ of $A\hat{}$ is $\mathcal{W}$-aspheric.

We'll constantly be using the canonical functor

$$\text{(1)} \qquad i_A : A\hat{} \to (\mathrm{Cat}), \quad F \mapsto A_{/F},$$

and its right adjoint

$$\text{(2)} \qquad j_A = i_A^* : (\mathrm{Cat}) \to A\hat{}, \quad C \mapsto (a \mapsto \mathrm{Hom}(A_{/a}, C)).$$

The category $A\hat{}$ will always be viewed as endowed with the saturated set of maps

$$\text{(3)} \qquad \mathcal{W}_A = i_A^{-1}(\mathcal{W}).$$

This gives rise to the notions of $\mathcal{W}_A$-aspheric map in $A\hat{}$, and of an object F of $A\hat{}$ being $\mathcal{W}_A$-aspheric over $e_{A\hat{}}$, namely $F \to e_{A\hat{}}$ being $\mathcal{W}_A$-aspheric, which means that $F \times G \to G$ is in $\mathcal{W}_A$ for any G in $A\hat{}$, which by definition of $\mathcal{W}_A$ means that for any G, the map

$$A_{/F \times G} \to A_{/F}$$

in (Cat) is in $\mathcal{W}$. Using the localization axiom on $\mathcal{W}$, one sees that it is enough to check this for G and object a of A, in which case $A_{/G} = A_{/a}$ has a final object and hence is $\mathcal{W}$-aspheric by L 3), and the condition amounts to $A_{/F \times a}$ being $\mathcal{W}$-aspheric, i.e. (with the terminology introduced above), that $F \times a$ is $\mathcal{W}$-aspheric. Thus, an object F of $A^\wedge$ is $\mathcal{W}_A$-aspheric over $e_{A^\wedge}$ iff for any a in A, $F \times a$ is $\mathcal{W}$-aspheric. We should beware that for general A, this does not imply F is $\mathcal{W}$-aspheric, nor is it implies by it. We should remember that $\mathcal{W}$-asphericity of F is an "absolute notion", namely is a property of the induced *category* $A_{/F}$ or equivalently, of the induced topos $A^\wedge{}_{/F} \simeq (A_{/F})^\wedge$, whereas $\mathcal{W}_A$-asphericity of F over $e_{A^\wedge}$ is a relative notion for the *map* of categories [p. 170]

$$A_{/F} \to A$$

or equivalently, for the map of topoi $A^\wedge{}_{/F} \to A^\wedge$ (the localization map with respect to the object F of the topos $A^\wedge$). More generally, for a map $F \to G$ in $A^\wedge$, the property for this map of being $\mathcal{W}_A$-aspheric is a property for the corresponding map in (Cat)

$$A_{/F} \to A_{/G},$$

namely the property we called *weak $\mathcal{W}$-asphericity* yesterday (page 161), as we stated then in the prop. page 162. An equivalent way of expressing this is by saying that for $F \to G$ to be $\mathcal{W}_A$-aspheric, i.e., to be "universally in $\mathcal{W}_A$", it is enough to check this for base changes $G' \to G$ with G' an object a in A, namely that the corresponding map

$$F \times_G a \to a$$

in $A^\wedge$ should be in $\mathcal{W}_A$(for any a in A and map $a \to G$), which amounts to saying that $F \times_G a$ is $\mathcal{W}$-aspheric for any a in A and map $a \to G$.

With notations and terminology quite clear in mind, we may start retelling once again the test category story!

A) Total asphericity. Before starting, just one important pre-test notion to recall, namely total asphericity, summed up in the

Proposition 1. *The following conditions on A are equivalent:*

(i) *The product in $A^\wedge$ of any two objects of A is $\mathcal{W}$-aspheric.*

(i') *Every object in A is $\mathcal{W}_A$-aspheric over the final element $e_{A^\wedge}$.*

(ii) *The product of two $\mathcal{W}$-aspheric objects is again $\mathcal{W}$-aspheric.*

(ii') *Any $\mathcal{W}$-aspheric object of $A^\wedge$ is $\mathcal{W}_A$-aspheric over $e_{A^\wedge}$.*

This is just a tautology, in terms of what was just said. Condition (i) is just the old condition (T 2) on test categories...

Definition 1. If A satisfies these conditions and moreover A is $\mathcal{W}$-aspheric, $A^\wedge$ is called *totally* $\mathcal{W}$-aspheric.

Remark. In all cases when we have met with totally aspheric $A^\wedge$, this condition (i) was checked easily, because we were in one of the two following cases: [p. 171]

a) A stable under binary products.

b) The objects of A are *contractible* for the homotopy structure $h_{\mathcal{W}_A}$ of $A^\wedge$ associated to $\mathcal{W}_A$.

In case b), in the cases we've met, for checking contractibility we even could get away with a homotopy interval $\mathbb{I} = (I, \delta_0, \delta_1)$ which is in A, namely we got $\mathbb{I}$-contractibility for all elements of A, and hence for the products. All we've got to check then, to imply $h_{\mathcal{W}_A}$-contractibility of the objects $a \times b$, and hence their $\mathcal{W}$-asphericity, is that I itself is $\mathcal{W}_A$-aspheric over $e_{A^\wedge}$, namely the products $I \times a$ are $\mathcal{W}$-aspheric. This now has to be checked indeed some way or other – I don't see any general homotopy trick to reduce the checking still more. In case when $A = \mathbb{\Delta}$ (standard simplices) say, and while still working with usual weak equivalences $W_{(\mathrm{Cat})}$, we checked asphericity of the products $\mathbb{\Delta}_1 \times \mathbb{\Delta}_n$ by using a Mayer-Vietories argument, each product being viewed as obtained by gluing together a bunch of *representable* subobjects, which are necessarily $\mathcal{W}$-aspheric therefore. The argument will go through for general $\mathcal{W}$, if we assume $\mathcal{W}$ satisfies the following condition, which we add to the provisional list made yesterday (pages 166–167) of extra conditions which we may have to introduce for a basic localizer:

L e) **(Mayer-Vietoris axiom)** Let C be in (Cat), let C', C'' be two full subcategories which are cribles (if it contains a in C and if $b \to a$, it contains b), and such that $\mathrm{Ob}\, C = \mathrm{Ob}\, C' \cup \mathrm{Ob}\, C''$. Assume C', C'' and $C' \cap C''$ are $\mathcal{W}$-aspheric, then so is C.

This condition of course is satisfied whenever $\mathcal{W}$ is described in terms of cohomological conditions, as envisioned yesterday (page 158). We could elaborate on it and develop in this direction a lot more encompassing conditions ("of Čech type" we could say), which will be satisfied by all such cohomologically defined basic localizers. It would be fun to work out a set of "minimal" conditions such as L e) above, which would be enough to imply all Čech-type conditions on a basic localizer. At first sight, it isn't even obvious that L e) say isn't a consequence of just the general conditions L 1) to L 4) on $\mathcal{W}$, plus perhaps the fibration axiom L 5) which looks very strong. As long as we don't have any other example of basic localizers than in terms of cohomology, it will be hard to tell!

B) Weak $\mathcal{W}$-test categories.

[p. 172] **Definition 2.** The category A is a weak $\mathcal{W}$-test category if it satisfies the conditions

a) A is $\mathcal{W}$-aspheric.

b) The functor $i_A^* : (\mathrm{Cat}) \to A^\wedge$ is model-preserving, i.e.,

b$_1$) $\mathcal{W} = (i_A^*)^{-1}(\mathcal{W}_A)$ $(= f_A^{-1}(\mathcal{W})$, where $f_A = i_A i_A^* : (\mathrm{Cat}) \to (\mathrm{Cat}))$,

b$_2$) the induced functor

$$\overline{i}_A^* : (\mathrm{Hot}_{\mathcal{W}}) \overset{\mathrm{def}}{=} \mathcal{W}^{-1}(\mathrm{Cat}) \to {\mathcal{W}_A}^{-1} A^\wedge$$

is an equivalence.

Proposition 2. *The following conditions on A are equivalent:*

(i) *A is a weak $\mathcal{W}$-test category.*

(ii) *The functors i_A^* and i_A are both model-preserving, the induced functors*

$$(\mathrm{Hot}_{\mathcal{W}}) \rightleftarrows \mathcal{W}_A{}^{-1}A^\wedge$$

are equivalences quasi-inverse of each other, with adjunction morphisms in $(\mathrm{Hot}_{\mathcal{W}})$ and in $(\mathrm{Hot}_A) \overset{\text{def}}{=} \mathcal{W}_A{}^{-1}A^\wedge$ deduced from the adjunction morphisms for the pair of adjoint functors i_A, i_A^*.

(iii) *The functor i_A^* transforms maps in $\mathcal{W}$ into maps in $\mathcal{W}_A$(i.e., $f_A = i_A i_A^*$ transforms maps in $\mathcal{W}$ into maps in $\mathcal{W}$), and moreover A is $\mathcal{W}$-aspheric.*

(iii') *Same as in (iii), but restricting to maps $C \to e$, where C in* (Cat) *has a final object.*

(iv) *The categories $f_A(C) = A_{/i_A^*(C)}$, where C in* (Cat) *has a final object, are $\mathcal{W}$-aspheric.*

The obvious implications are

$$\text{(ii)} \Rightarrow \text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(iii')} \Rightarrow \text{(iv)}$$

and the proof of (iv) $\Rightarrow$ (ii) follows from an easy weak asphericity argument and general non-sense on adjoint functors and localization (cf. page 35 and prop. on page 38).

Remark. In case $\mathcal{W}$ is strongly saturated, and hence A $\mathcal{W}$-aspheric just means that its image in $(\mathrm{Hot}_{\mathcal{W}})$ is a final object, the condition of $\mathcal{W}$-asphericity of A in (iii) or in def. 2 can be restated, by saying that the endomorphism $\overline{f}_A$ of $(\mathrm{Hot}_{\mathcal{W}})$ induced by f_A transforms final object into final object – which is a lot weaker than being an equivalence!

C) $\mathcal{W}$-test categories.

Definition 3. The category A is a $\mathcal{W}$-*test category* if it is a weak $\mathcal{W}$-test category, and if the localized categories $A_{/a}$ for a in A are equally weak $\mathcal{W}$-test categories. We say A is a *local $\mathcal{W}$-test category* if the localized categories $A_{/a}$ are weak $\mathcal{W}$-test categories. [p. 173]

Clearly, A is a $\mathcal{W}$-test category iff if is a local $\mathcal{W}$-test category, and moreover A is $\mathcal{W}$-aspheric (as the categories $A_{/a}$ are $\mathcal{W}$-aspheric by L 3)). Also, A is a local $\mathcal{W}$-test category iff the functors $i_{A_{/a}}^*$ (for a in A) are model preserving.

Proposition 3. *The following conditions on A are equivalent:*

(i) *A is a local $\mathcal{W}$-test category.*

(ii) *The Lawvere element*

$$L_{A^\wedge} = i_A^*(\triangle_1)$$

in $A^\wedge$ is $\mathcal{W}_A$-aspheric over $e_{A^\wedge}$, i.e., the products $a \times L_{A^\wedge}$ for a in A are all $\mathcal{W}$-aspheric.

(iii) *There exists a separated interval* $\mathbb{I} = (I, \delta_0, \delta_1)$ *in* $A^\wedge$ *(i.e., an object endowed with two sections such that* $\operatorname{Ker}(\delta_0, \delta_1) = \varnothing_{A^\wedge}$*), such that* I *be* $\mathcal{W}_A$*-aspheric over* $e_{A^\wedge}$*, i.e., all products* $a \times I$ *are* $\mathcal{W}$*-aspheric.*

The obvious implications here are

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)}.$$

on the other hand (iii) ⇒ (ii) by the homotopy interval comparison lemma (p. 60), and finally (i) ⇒ (i) by the criterion for weak $\mathcal{W}$-test categories of prop. 2 (iv), using an immediate homotopy argument (cf. page 62).

Corollary 1. *A is a* $\mathcal{W}$*-test category iff it is* $\mathcal{W}$*-aspheric and satisfies (ii) or (iii) of proposition 3 above.*

Remark. In the important case when $A^\wedge$ is totally $\mathcal{W}$-aspheric (cf. prop. 1, the asphericity condition on $L_{A^\wedge}$ or on I in prop. 3 is equivalent to just $\mathcal{W}$-asphericity of $L_{A^\wedge}$ resp. of I. In case $A^\wedge$ is even "strictly totally $\mathcal{W}$-aspheric", i.e., if moreover every "non-empty" object in $A^\wedge$ admits a section, then we've seen that $h_{\mathcal{W}_A} = h_{A^\wedge}$ (prop. page 149, which carries over to a general $\mathcal{W}$ satisfying L a') of page 166, i.e., provided ($\mathrm{Hot}_{\mathcal{W}}$) isn't equivalent to the final category, which case we may discard!), then condition (ii) just means that the contractor $L_{A^\wedge}$ is 0-connected – a condition which does not depend upon the choice of $\mathcal{W}$.

D) Strict $\mathcal{W}$-test categories.

[p. 174] **Proposition 4.** *The following conditions on A are equivalent:*

(i) *Both functors* i_A *and* i_A^* *are model preserving, moreover* i_A *commutes to finite products "modulo* $\mathcal{W}$*".*

(ii) *A is a test category and* $A^\wedge$ *is totally* $\mathcal{W}$*-aspheric.*

(ii') *A is a weak test category and* $A^\wedge$ *is totally* $\mathcal{W}$*-aspheric.*

(iii) *A satisfies conditions (T 1) (T 2) (T 3) of page 39, with "aspheric" replaced by "*$\mathcal{W}$*-aspheric".*

This is not much more than a tautology in terms of what we have seen before, as we'll get the obvious implications

$$\text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(ii')} \Rightarrow \text{(i)}.$$

Definition 4. If A satisfies the conditions above, it is called a *strict* $\mathcal{W}$*-test category*.

Remarks. 1) When we know that the canonical functor from (Cat) to the localization ($\mathrm{Hot}_{\mathcal{W}}$) commutes with binary products, then the exactness property mod $\mathcal{W}$ in (i) implies that the same holds for the canonical functor from $A^\wedge$ to its localization (Hot_A), and conversely if $\mathcal{W}$ is known to be saturated.

2) In the case $\mathcal{W} = W_{(\mathrm{Cat})}$ we've seen that condition (T 2) implies (T 1), i.e., the conditions of prop. 1 imply A is $\mathcal{W}$-aspheric, i.e., $A^\wedge$ is totally $\mathcal{W}$-aspheric. The argument works for any $\mathcal{W}$ defined by cohomological conditions of the type considered in yesterday's notes. To have it work for more general $\mathcal{W}$, we would have to introduce some Čech-type requirement on $\mathcal{W}$, compare page 171.

3) In the statement of the theorem page 46, similar to the proposition above, in (i) no assumption is made on $i_A^* = j_A$ – which I believe was an omission by hastiness – it is by no means clear to me that we could dispense with it, and get away with an assumption on i_A alone.

E) Weak $\mathcal{W}$-test functors and $\mathcal{W}$-test functors. Let [p. 175]

$$(M, W), \quad W \subset \mathrm{Fl}(M)$$

be a category endowed with a saturated set of arrows W, and

(4) $$i : A \to M$$

a functor, hence a corresponding functor

(5) $$i^* : M \to A^\wedge, \quad X \mapsto i^*(X) = (a \mapsto \mathrm{Hom}(i(a), X)).$$

Definition 5. The functor i is called a *weak $\mathcal{W}$-test functor* (with respect to the given $W \subset \mathrm{Fl}(M)$) if A is a weak $\mathcal{W}$-test category and the functor i^* is model-preserving (for W and $\mathcal{W}_A$), i.e., if A satisfies the three conditions:

a) i^* is model preserving,

b) $i_A^* : (\mathrm{Cat}) \to A^\wedge$ is model preserving,

c) A is $\mathcal{W}$-aspheric.

The conditions b) and c), namely that A be a weak $\mathcal{W}$-test category, do not depend of course upon M, and it may seem strange in the definition not to have simply asked beforehand that A satisfy this preliminary condition – i.e., reduce to the case when we start with a weak $\mathcal{W}$-test category A. The reason for not doing so is that we'll find below handy criteria for all three conditions to hold, without assuming beforehand A to be a weak $\mathcal{W}$-test category.

As b) and c) imply that

$$i_A : A^\wedge \to (\mathrm{Cat})$$

is model-preserving too, condition a) above can be replaced by the condition

a') The composition
$$f_i = i_A i^* : M \to (\mathrm{Cat})$$
is model-preserving (for the pair $W, \mathcal{W}$).

Of course, as conditions b), c) imply that $(A^\wedge, \mathcal{W}_A)$ is a modelizer (with respect to $\mathcal{W}$), the condition a) will imply (M, W) is a modelizer too.

We recall the condition for i^* to be model-preserving decomposes into two:

a$_1$) $W = (i^*)^{-1}(\mathcal{W}_A) \quad (= f_i^{-1}(\mathcal{W}))$,

[p. 176] a$_2$) The functor $\overline{i^*}$ induced by i^* on the localizations (which exists because of a$_1$))

$$W^{-1}M \to (\mathrm{Hot}_A) \overset{\mathrm{def}}{=} \mathcal{W}_A{}^{-1}A^\wedge$$

is an equivalence.

Definition 6. The functor i is called a $\mathcal{W}$-*test functor* if this functor *and* the induced functors $i_{/a} : A_{/a} \to M$ (for a in A) are weak $\mathcal{W}$-test functors.

In view of the definition 3, this amounts to the two conditions:

a) A is a $\mathcal{W}$-test category, i.e., the functors i_A^* and $i_{A_{/a}}^*$ are model-preserving and A is $\mathcal{W}$-aspheric,

b) the functors i^* and $(i_{/a})^*$ from M into the categories $A^\wedge$ and $(A_{/a})^\wedge \to A^\wedge{}_{/a}$ are model-preserving (for W and $\mathcal{W}_A$ resp. $\mathcal{W}_{A_{/a}}$).

Example. Consider the canonical functor induced by i_A

$$i_A^0 : A \to (\mathrm{Cat}), \quad (\mathrm{Cat}) \text{ endowed with } \mathcal{W},$$

this functor is a weak $\mathcal{W}$-test functor (resp. a $\mathcal{W}$-test functor) iff A is a weak $\mathcal{W}$-test category (resp. a $\mathcal{W}$-test category).

These two definitions are pretty formal indeed. Their justification is mainly in the two theorems below.

We assume from now on that the basic localizer $\mathcal{W}$ satisfies the fibration axiom L 5) of page 164. Also, we recall that an object X in (Cat) is *contractible* (for the canonical homotopy structure of (Cat)) iff X is non-empty and the category $\mathbf{Hom}(X, X)$ is connected – indeed it is enough even that id_X belong to the same connected component as some *constant* map from X into itself. This condition is satisfied for instance if X has a final or an initial object.

Theorem 1. *We assume that $M = (\mathrm{Cat})$, $W = \mathcal{W}$, i.e., we've got a functor*

$$i : A \to (\mathrm{Cat}), \quad (\mathrm{Cat}) \textit{ endowed with } \mathcal{W},$$

and we assume that for any a in A, $i(a)$ is contractible (cf. above), i.e., that i factors through the full subcategory $(\mathrm{Cat})_{\mathrm{cont}}$ of contractible objects of (Cat). The following conditions are equivalent:

(i) *i is a $\mathcal{W}$-test functor (def. 6).*

(i') *For any a in A, the induced functor $i_{/a} : A_{/a} \to (\mathrm{Cat})$ is a weak $\mathcal{W}$-test functor, and moreover A is $\mathcal{W}$-aspheric.*

(ii) *$i^*(\triangle_1)$ is $\mathcal{W}_A$-aspheric over $e_{A^\wedge}$, i.e., the products $a \times i^*(\triangle_1)$ in $A^\wedge$ are $\mathcal{W}$-aspheric, for any a in A, and A is $\mathcal{W}$-aspheric.*

The obvious implications here are [p. 177]

$$\text{(i)} \Rightarrow \text{(i')} \Rightarrow \text{(ii)},$$

for the last implication we only make use, besides A being $\mathcal{W}$-aspheric, that the functors $(i_{/a})^*$ transform the projection $\triangle_1 \to e$ in (Cat), which is in $\mathcal{W}$ by L 3'), into a map in $\mathcal{W}_{A_{/a}}$, i.e., that the corresponding map in (Cat)

$$A_{/a\times i^*(\triangle_1)} \to A_{/a}$$

be in $\mathcal{W}$, which by the final object axiom implying that $A_{/a} \to e$ is in $\mathcal{W}$, amounts to demanding that the left-hand side is $\mathcal{W}$-aspheric, i.e., $a \times i^*(\triangle_1)$ $\mathcal{W}_A$-aspheric.

So we are left with proving (ii) $\Rightarrow$ (i). By the criterion (iii) of prop. 3 we know already (assuming (ii)) that A is a local $\mathcal{W}$-test category, hence a $\mathcal{W}$-test category as A is $\mathcal{W}$-aspheric (cor. 1); indeed we can use $I = i^*(\triangle_1)$ as a $\mathcal{W}_A$-aspheric interval, using the two canonical sections deduced from the canonical sections of $\triangle_1$. The fact that these are disjoint follows from the fact that $i(a)$ non-empty for any a in A – we did not yet have to use the contractibility assumption on the categories $i(a)$. Thus, we are reduced to proving that i^* is model-preserving – the same will then hold for the functors $i_{/a}$ (as required in part b) in def. 6), as the assumption (ii) is clearly stable under restriction to the categories $A_{/a}$. As we know already that i_A is model-preserving (prop. 2 (i) $\Rightarrow$ (ii)), all we have to do is to prove the composition $f_i = i_A i^*$ is model-preserving. But this was proved yesterday in the key lemma of page 156. We're through!

Remark. The presentation will be maybe a little more elegant, if we complement the definition of a $\mathcal{W}$-test functor by the definition of a *local $\mathcal{W}$-test functor*, by which we mean that the induced functors $i_{/a} : A_{/a} \to M$ are weak test functors, period – which means also that the following conditions hold:

a) A is a local $\mathcal{W}$-test category (def. 3), i.e., the functors $(i_{/a})^* : M \to (A_{/a})^\wedge$ are all model-preserving;

b) the functors $(i_{/a})^*$ (for a in A) are model-preserving.

Thus, it is clear that if i is a $\mathcal{W}$-test functor, it is a local $\mathcal{W}$-test functor such that moreover A is $\mathcal{W}$-aspheric. The converse isn't clear in general, because it isn't clear that if A is a $\mathcal{W}$-test category and moreover all functors $(i_{/a})^*$ are modelizing, then i^* is equally modelizing. The criterion (i') of theorem 1 shows however that this is so in the case when $(M, W) = ((\text{Cat}), \mathcal{W})$, and when we assume moreover the objects $i(a)$ contractible. We could now reformulate the theorem as a twofold statement: [p. 178]

Corollary. *Under the assumptions of theorem 1, i is a* local $\mathcal{W}$-test functor *(i.e., all functors $i_{/a} : A_{/a} \to$* (Cat) *are weak $\mathcal{W}$-test functors, or* *equivalently the functors* $(i_{/a})^*$ *and* $(i_{A_{/a}})^*$ *from* $(\mathrm{Cat}) \to (A_{/a})^\wedge$ *are all model-preserving) iff* $i^*(\Delta_1)$ *is* $\mathcal{W}_A$*-aspheric over* $e_{A^\wedge}$*, i.e., the products* $a \times i^*(\Delta_1)$ *in* $A^\wedge$ *are* $\mathcal{W}$*-aspheric. When this condition is satisfied, in order for* i *to be a* $\mathcal{W}$*-test functor, namely for* i^* *to be equally model-preserving, it is n.s. that* A *be* $\mathcal{W}$*-aspheric.*

F) $\mathcal{W}$**-test functors** $A \to (\mathrm{Cat})$ **of strict** $\mathcal{W}$**-test categories.** Let again

$$i : A \to (\mathrm{Cat}), \quad (\mathrm{Cat}) \text{ endowed with } \mathcal{W},$$

be a functor such that the objects $i(a)$ be contractible, we assume now moreover that $A^\wedge$ is *totally* $\mathcal{W}$*-aspheric* (def. 1), which implies A is $\mathcal{W}$-aspheric. Thus, by the corollary above i is a test functor iff it is a local $\mathcal{W}$-test functor, and by the criterion (iv) of prop. 2 (with $C = \Delta_1$) we see it amounts to the same that i be a weak $\mathcal{W}$-test functor. (Here we use the assumption of total $\mathcal{W}$-asphericity, which implies that if $i^*(\Delta_1)$ is $\mathcal{W}$-aspheric, it is even $\mathcal{W}_A$-aspheric over $e_{A^\wedge}$.) Thus, the three variants of the test-functor notion coincide in the present case. With this in mind, we can now state what seems to me the main result of our reflections so far, at any rate the most suggestive reformulation of theorem 1 in the present case:

Theorem 2. *With the assumptions above* ($A^\wedge$ *totally* $\mathcal{W}$*-aspheric and the objects* $i(a)$ *in* (Cat) *contractible), the following conditions on the functor* $i : A \to (\mathrm{Cat})$ *are equivalent:*

(i) *i is a $\mathcal{W}$-test functor.*

(ii) $i^* : (\mathrm{Cat}) \to A^\wedge$ *is model-preserving, i.e., for any map* f *in* (Cat)*,* f *is in* $\mathcal{W}$ *iff* $i^*(f)$ *is in* $\mathcal{W}_A$ *(i.e., iff* $i_A i^*(f)$ *is in* $\mathcal{W}$*), and moreover the induced functor*

$$\overline{i^*} : (\mathrm{Hot}_{\mathcal{W}}) \to (\mathrm{Hot}_A) \stackrel{\mathrm{def}}{=} {\mathcal{W}_A}^{-1} A^\wedge$$

is an equivalence.

(iii) *The functor above exists, i.e.,* f *in* $\mathcal{W}$ *implies* $i^*(f)$ *in* $\mathcal{W}_A$*, i.e.,* $i_A i^*(f)$ *in* $\mathcal{W}$*.*

(iv) *The functor* i^* *transforms* $\mathcal{W}$*-aspheric objects into* $\mathcal{W}_A$*-objects (i.e., the condition in* (iii) *is satisfied for maps* $C \to e$ *in* $\mathcal{W}$*).*

[p. 179] (v) *The functor* i^* *transforms contractible objects of* (Cat) *into objects of* $A^\wedge$*, contractible for the homotopy structure* $h_{\mathcal{W}_A}$ *associated to* $\mathcal{W}_A$ *– or equivalently,* i^* *is a morphism of homotopy structures (cf. definition on page 134).*

(vi) *The functor* i^* *transforms the projection* $\Delta_1 \to e$ *into a map in* $\mathcal{W}$*, or equivalently (as* A *is* $\mathcal{W}$*-aspheric)* $i^*(\Delta_1)$ *is* $\mathcal{W}$*-aspheric.*

We have the trivial implications

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(vi)} \Leftarrow \text{(v)},$$

where the implication (v) $\Rightarrow$ (vi) is in fact an equivalence, due to the fact that the contractibility structure on (Cat) is defined in terms of Δ_1 as a generating contractor, and that the assumption $i^*(\Delta_1)$ $\mathcal{W}$-aspheric implies already that it is $\mathcal{W}_A$-aspheric over $e_{A^\wedge}$ (because A is totally $\mathcal{W}_A$-aspheric), and hence contractible as it is a contractor and $h_{\mathcal{W}_A}$ is defined in terms of "weak homotopy intervals" which are $\mathcal{W}_A$-aspheric over $e_{A^\wedge}$. Thus, the only delicate implication is (vi) $\Rightarrow$ (i), which however follows from theorem 1 (ii) $\Rightarrow$ (i).

We got the longed-for "key result" in the end!

9.4. [p. 180]

66 After writing down nicely, in the end, that long promised key result, I thought the next thing would be to pull myself up by my bootstraps getting the similar result first for test functors $A \to B^\wedge$ with values in an elementary modelizer $B^\wedge$, and then for general "canonical" modelizers (M, W). So I did a little scratchwork pondering along those lines, before resuming the typewriter-engined work. What then turned out, it seems, is that there wasn't any need at all to pass through the particular case $M = $ (Cat) and the somewhat painstaking analysis of our three-step diagram on page 96. Finally, the most useful result of all the eighty pages grinding, since that point, is by no means the so-called key result, as I anticipated – the day after I finally wrote it down, it was already looking rather "étriqué" – why all this fuss about the special case of test functors with values in (Cat)! The main result has been finally more psychological than technical – namely drawing attention, in the long last, to the key role of contractible objects and, more specifically, of the contractibility structure associated to modelizers (M, W), suggesting that the localizer W should, conversely, be describable in terms of the homotopy structure h_W. This was point d) in the "provisional plan of work" contemplated earlier this week (page 138) – by then I had already the feeling this approach via d) would turn out to be the most "expedient" one – but it was by then next to impossible for me to keep pushing off still more the approach b) via test functors with values in (Cat), which I finally carried through. One point which wasn't wholly clear yet that day, as it is now, is the crucial role played by the circumstance that for the really nice modelizers (surely for those I'm going to call "canonical" in the end), the associated homotopy structure is indeed a *contractibility* structure. Here, as so often in mathematics (and even outside of mathematics...), the main thing to dig out and discover is where the emphasis belongs – which are the really essential facts or notions or features within a given context, and which are accessory, namely, which will follow suit by themselves. It took a while before I would listen to what the things I was in were insistently telling me. It finally got through I feel, and I believe that from this point on the whole modelizing story is going to go through extremely smoothly.

Revising (and fixing?) terminology again.

["étriqué" can be translated as "narrow-minded" here, I think.]

Before starting with the work, just some retrospective, somewhat more technical comments, afterthoughts rather I should say. First of all, I am not so happy after all with the terminological review a few [p. 181] days ago (pages 159–163), and notably the use of the word "aspheric" in the generalization "W-aspheric" map (in a category M endowed with a saturated set of arrows W) – which then practically obliged me, when working in (Cat), to call "weakly aspheric" a functor $C' \to C$ which spontaneously I surely would like to call simply "aspheric" – and as a matter of fact, it turned out I couldn't force myself to add a "weakly" before as I decreed I should – or if I did, it was against a very strong feeling of inappropriateness. That decree precisely is an excellent illustration of loosing view of where the main emphasis belongs, which I would like now to make very clear to myself.

In all this work the underlying motivation or inspiration is geometrico-topological, and expressed technically quite accurately by the notion of a topos and of maps (or "morphisms") of topoi, and the wealth of geometric and algebraic intuitions which have developed around these. One main point here is that topoi may be viewed as *the* natural common generalization of both topological spaces (the conventional support for so-called topological intuition), and of (small) categories, where the latter may be viewed as the ideal purely algebraic objects carrying topological information, including all the conventional homology and homotopy invariants. This being so, in a context where working with small categories as "spaces", the main emphasis in choice of terminology should surely be in stressing throughout, through the very wording, the essential identity between situations involving categories, and corresponding situations involving topological spaces or topoi. Thus, it has been about twenty years now that the needs for developing étale cohomology have told me a handful of basic asphericity and acyclicity properties for a map of topoi (which apparently have not yet been assimilated by topologists, in the context of maps of topological spaces...), including the condition for such a map to be aspheric. This was recalled earlier (page 37), and the corresponding notion for a functor $f : C' \to C$ was introduced. The name "*aspheric map*" of topoi, or of topological spaces, or of categories, is here a perfectly suggestive one. As the notion itself is visibly a basic one, there should be no question whatever to change the name and replace it say by "weakly aspheric", whereas the notion is surely quite a strong one, and doesn't deserve such minimizing qualification! There is indeed a stronger notion, which in the context of topological spaces or étale cohomology of schemes reduces to the previous one in the particular case of a map which is supposed *proper*. This condition could be expressed by saying that for *any* base-change
[p. 182] $Y' \to Y$ for the map $f : X \to Y$ (at least any base-change within the given context, namely either spaces or schemes with étale topology), the corresponding map

$$f' : X' = X \times_Y Y' \to Y'$$

is a weak equivalence, or what amounts to the same, that for any Y' the corresponding map is aspheric. This property, if a name is needed indeed, would properly be called "*universally aspheric*". Thus, in (Cat) a map $f : X \to Y$ will be called either aspheric, or universally aspheric, when for any base-change of the special type $Y_{/y} \to Y$, namely "localization" in the first case, or any base-change whatever $Y' \to Y$ in the second case, the corresponding map f' is a weak equivalence. On the other hand, if Y is just the final object e of (Cat), it turns out the two notions for X (being "aspheric over e" and being "universally aspheric over e") coincide, and just mean that $X \to e$ is a weak equivalence. In accordance with the use which has been prevalent for a long time in the context of spaces, such an object will be call simply an *aspheric object* – which means that the corresponding topos is aspheric (namely has "trivial" cohomology invariants, and hence trivial homotopy invariants of any kind...).

In case the notion of weak equivalence is replaced by a basic localizer $\mathcal{W} \subset \mathrm{Fl}((\mathrm{Cat}))$, there is no reason whatever to change anything in this terminology – except that, if need, we will add the qualifying $\mathcal{W}$, and speak of *$\mathcal{W}$-aspheric* or *universally $\mathcal{W}$-aspheric maps* in (Cat), as well of *$\mathcal{W}$-aspheric objects* of (Cat).

What about terminology for maps and objects within a category $A^\wedge$? Here the emphasis should be of course perfect coherence with the terminology just used in (Cat). An object F of $A^\wedge$ should always be sensed in terms of the induced topos $A^\wedge{}_{/F} \simeq (A_{/F})^\wedge$, or what amounts to the same, in terms of the corresponding object $A_{/F}$ in (Cat), which will imply that "*F is aspheric*" cannot possibly mean anything else but $A_{/F}$ is aspheric as an object of (Cat); the same if qualifying by a $\mathcal{W}$ – F is called *$\mathcal{W}$-aspheric* if $A_{/F}$ is a $\mathcal{W}$-aspheric object of (Cat). Similarly for maps – thus $f : F \to G$ will be called a weak equivalence, if the corresponding map for the induced topoi is a weak equivalence, or equivalently, if the corresponding map in (Cat)

$$A_{/F} \to A_{/G}$$

is a weak equivalence. When a $\mathcal{W}$ is given, we would say instead (if confusion may arise) that f is a *$\mathcal{W}$-equivalence*. The map will be called [p. 183]
aspheric, or *$\mathcal{W}$-aspheric*, if the corresponding map in (Cat) is. It turns out that (because of the localization axiom on $\mathcal{W}$) this is equivalent with f being "universally a $\mathcal{W}$-equivalence", i.e., f being "universally in $\mathcal{W}_A$", namely that for any base-change $G' \to G$ in $A^\wedge$, the corresponding map in $A^\wedge$

$$f' : F \times_G G' \to G'$$

be in $\mathcal{W}_A$, i.e., be a $\mathcal{W}$-equivalence. Of course, when this condition is satisfied, then for any base change, f' will be, not only a $\mathcal{W}$-equivalence, but even $\mathcal{W}$-aspheric – thus we can say that f is "universally $\mathcal{W}$-aspheric" – where "universally" refers to *base change in $A^\wedge$*. This of course does not mean (and here one has to be slightly cautious) that the corresponding map in (Cat) is universally $\mathcal{W}$-aspheric (which refers to arbitrary *base change in* (Cat)). But this apparent incoherence is of no practical importance as far as terminology goes, as the work "$\mathcal{W}$-aspheric map in $A^\wedge$" is wholly adequate and sufficient for naming the notion, without any need to replace it by the more complicated and ambiguous name "universally $\mathcal{W}$-aspheric", which therefore will never be used. We even could rule out the formal incoherence, by using the words $\mathcal{W}_A$-equivalence, $\mathcal{W}_A$-aspheric maps (which are even universally $\mathcal{W}_A$-aspheric maps, without any ambiguity any longer), as well as $\mathcal{W}_A$-aspheric objects – replacing throughout $\mathcal{W}$ by $\mathcal{W}_A$. In practical terms, I think that when working consistently with a single given $\mathcal{W}$, we'll soon enough drop anyhow both $\mathcal{W}$ and $\mathcal{W}_A$ in the terminology and notations!

A last point which deserves some caution, is that for general A, there is no implication between the two asphericity properties of an object F of $A\hat{}$, namely of F being $\mathcal{W}$-aspheric (i.e., the object $A_{/F}$ of (Cat) being $\mathcal{W}$-aspheric, i.e., the map

$$(*) \qquad A_{/F} \to e$$

in (Cat) being in $\mathcal{W}$), and the property that $F \to e$ be $\mathcal{W}$-aspheric, namely that map

$$(**) \qquad A_{/F} \to A_{/e} = A$$

in (Cat) being aspheric (which also means that the products $F \times a$ for a in A are $\mathcal{W}$-aspheric objects of $A\hat{}$, i.e., the categories $A_{/F\times a}$ are $\mathcal{W}$-aspheric, i.e., the maps

$$A_{/F\times a} \to e$$

in (Cat) are $\mathcal{W}$-aspheric. A third related notion, weaker than the last
[p. 184] one is the property that $F \to e$ be a $\mathcal{W}$-equivalence, which also means that the map (**) in (Cat) is a $\mathcal{W}$-equivalence, i.e., is in $\mathcal{W}$. If A is $\mathcal{W}$-aspheric, this third notion however reduces to the first one, namely F to be $\mathcal{W}$-aspheric.

These terminological conventions, in the all-important cases of (Cat) *and categories of the type $A\hat{}$, should be viewed as the basic ones* and there should be no question whatever to requestion them, because say of the need we are in to devise a terminology, applicable to the general case of a category M endowed with any saturated set of maps $W \subset \mathrm{Fl}(M)$ (which are being thought of as still more general substitutes of "weak equivalences"). This shows at once that we will have to renounce to the name of "W-aspheric" which we have used so far, in order to designate maps which are "universally in W"; indeed, this contradicts the use we are making of this word, in the case of (Cat). The whole trouble came from this inappropriate terminology, which slipped in while thinking of the $A\hat{}$ analogy, and forgetting about the still more basic (Cat)! The mistake is a course one indeed, and quite easy to correct – *we better refrain altogether from using the word "aspheric" in the context of a general pair* (M, W), and rather speak of maps which are "*universally in W*" or "*universal W-equivalences*", which is indeed more suggestive, and does not carry any ambiguity. The notion of "W-aspheric map" should be reserved to the case when, among all possible base-change maps $Y' \to Y$ in M, we can sort out some which we may view as "localizing maps" – all maps I'd think in cases M is a topos, and pretty few ones when $M = $ (Cat). As for qualifying objects of M, we'll just be specific in stating properties of the projection $X \to e_M$ – such as being a W-equivalence, or a universal W-equivalence, or a homotopism for h_W (in which case the name "contractible object" is adequate indeed). It may be convenient, when we got a W, to denote by

$$\mathrm{U}W \subset W \subset \mathrm{Fl}(M)$$

the corresponding set of maps which are universally in W, a property which then can be abbreviated into the simple notation

$$f \in \mathrm{U}W$$

or "f is in $\mathrm{U}W$. It should be noted that $\mathrm{U}W$ contains all invertible maps and is stable by composition, but *it need not be saturated*, thus f and fg may be in $\mathrm{U}W$ without g being so.

This terminological digression was of a more essential nature, as a matter of fact, than merely technical. There is still another correction I want to make with terminology introduced earlier, namely with the name of a "contractor" I used for intervals endowed with a suitable composition law (page 120). The name in itself seems to me quite appropriate, however I have now a notion in reserve which seems to me a lot more important still, a reinforcement it turns out of the notion of a strict test category – and which I really would like to call a *contractor*. I couldn't think of any more appropriate name – thus I better change the previous terminology – sorry! – and call those nice intervals "multiplicative intervals", thus referring to the composition law as a "multiplication" (with left unit and left zero element). The name which first slipped into the typewriter, when it occurred that a name was desirable, was not "contractor" by the way but "intersector", as I was thinking of the examples I had met so far, when composition laws were defined in terms of intersections and were idempotent. But this doesn't square too well with the example of contractors $\mathbf{Hom}(X,X)$, when X is an object which has a section – and this example turns out as equally significant. [p. 185]

One last comment is about the "Čech type" condition L e) on the basic localizer $\mathcal{W}$, introduced two days ago (page 171). As giving a "crible" in a category amounts to the same as giving a map

$$C \to \mathbb{\Delta}_1$$

(by taking the inverse image of the source-object $\{0\}$ of $\mathbb{\Delta}_1$), and therefore giving two such amounts to a map from C into $\mathbb{\Delta}_1 \times \mathbb{\Delta}_1$, we see that the situation when C is the union of two cribles is expressed equivalently by giving a functor from C into the subcategory

$$C_0 = \left(\begin{tikzcd}[row sep=small] & b \\ a \arrow[ur] \arrow[dr] & \\ & c \end{tikzcd} \right)$$

of $\mathbb{\Delta}_1 \times \mathbb{\Delta}_1$ (dual to the barycentric subdivision of $\mathbb{\Delta}_1$). The asphericity conditions on C', C'' and $C' \cap C''$ then just mean that *this functor is*

$\mathcal{W}$-*aspheric*, which by the localization axiom L 4) implies that the functor itself is a weak equivalence. Thus (by saturation), the conclusion that C should be $\mathcal{W}$-aspheric, just amounts to the following condition, which is in a way the "universal" special case when $C = C_0$, and C', C'' are the two copies of $\blacktriangle_1$ contained in C_0:

L e') The category C_0 above is $\mathcal{W}$-aspheric.

[p. 186] If we now look upon the projection map of C_0 upon one factor $\blacktriangle_1$ (carrying a and b into $\{0\}$ and c into $\{1\}$), we get a functor which is fibering, and whose fibers are $\blacktriangle_1$ and $\blacktriangle_0$, which are $\mathcal{W}$-aspheric. Hence *the fibration axiom* L 5) *on* $\mathcal{W}$ *implies the Mayer-Vietoris axiom* L e) (page 171). This argument rather convinces me that the fibration axiom should be strong enough to imply all Čech-type $\mathcal{W}$-asphericity criteria which one may devise (provided of course they are reasonable, namely hold for ordinary weak equivalences!). More and more, it seems that the basic requirements to make upon a basic localizer, which will imply maybe all others, are L 1) (saturation), L 3') (the "standard interval axiom", namely $\blacktriangle_1$ is $\mathcal{W}$-aspheric), and the powerful fibration axiom L 5). This brings to my mind though the condition L a) of page 165, namely that $f \in \mathcal{W}$ implies $\pi_0(f)$ bijective, which wasn't needed really for the famous "key result" I was then after, and which for this reason I then was looking at almost as something accessory! I now do feel though that it is quite an essential requirement, even though we made no formal use of it (except very incidentally, in the slightly weaker form L a'), which just means that $(\mathrm{Hot}_{\mathcal{W}})$ isn't equivalent to the final category). I would therefore add it now to the list of really basic requirements on a "basic localizer", and rebaptize it therefore as L 6), namely:

L 6) **(Connectedness axiom)** $f \in \mathcal{W}$ implies $\pi_0(f)$ bijective, i.e., the functor $\pi_0 : (\mathrm{Cat}) \to (\mathrm{Sets})$ factors through $\mathcal{W}^{-1}(\mathrm{Cat}) = (\mathrm{Hot}_{\mathcal{W}})$ to give rise to a functor

$$\pi_0 : (\mathrm{Hot}_{\mathcal{W}}) \to (\mathrm{Sets}).$$

This, as was recalled on page 166, is more than needed to imply

$$h_{\mathcal{W}} = h_{(\mathrm{Cat})},$$

namely the homotopy structure in (Cat) associated to $\mathcal{W}$ (in terms of $\mathcal{W}$-aspheric intervals) is just the canonical homotopy structure (defined in terms of 0-connected intervals), which is also the homotopy structure defined by the single "basic" (multiplicative) interval $\blacktriangle_1$.

11.4.

I forget to clear up still another point of terminology – namely about "weak homotopy intervals" – it turns out finally we never quite came around defining what "homotopy intervals" which aren't weak should be! The situation is very silly indeed - so henceforth I'll just drop the qualificative "weak" – thus from now on a "*homotopy interval*" (with respect to a given homotopy structure h in a category M) is just an interval whose end-point sections δ_0, δ_1 are homotopic. In case $h = h_W$, where W is a given saturated set of arrows in M, the notion we get is a lot wider than the notion of a homotopy interval (with respect to W) introduced earlier (page 132, and which we scarcely ever used it seems, so much so that I even forgot till this very minute it had been introduced formally), where we were restricting to intervals for which $I \to e$ is universally in W, i.e., is in $\mathrm{U}W$ (we may call such objects simply $\mathrm{U}W$-objects). Anyhow, it seems that so far, the only property of such intervals we kept using from the beginning is the one shared with all homotopy intervals in the wider sense I am now advocating. There is just one noteworthy extra property which is sometimes of importance, especially in the characterization of test categories, namely the property $\mathrm{Ker}(\delta_0, \delta_1) = \varnothing_M$; this was referred to earlier by the name "separated interval" – which however may lead to confusion when for objects of M we have (independently of homotopy notions) a notion of separation. Therefore, we better speak about *separating intervals* as those for which $\mathrm{Ker}(\delta_0, \delta_1) = \varnothing_M$ (initial object in M), hence a notion of *separating homotopy interval* (with respect to a given homotopy structure h, or with respect to a given saturated W, giving rise to h_W). [p. 187]

Part IV

Asphericity structures and canonical modelizers

11.4. [p. 188]

Setting out for the asphericity game again: variance of the category (Hot_A)*, for arbitrary small category* A *and aspheric functors.*

67 A little more pondering and scribbling finally seems to show that the real key for an understanding of modelizers isn't really the notion of contractibility, but rather the notion of aspheric objects (besides, of course, the notion of weak equivalence). At the same time it appears that the notion of an *aspheric map* in (Cat), more specifically of a $\mathcal{W}$-aspheric "map" (i.e., a functor between small categories) is a lot more important than being just a highly expedient technical convenience, as it has been so far – it is indeed one of the basic notions of the theory of modelizers we got into. As a matter of fact, I should have known this for a number of weeks already, ever since I did some scribbling about the plausible notion of "morphism" between test-categories (as well as their weak and strong variants), and readily convinced myself that the natural "morphisms" here were nothing else but the aspheric functors between those categories. I kind of forgot about this, as it didn't seem too urgent to start moving around the category I was working with. If I had been a little more systematic in grinding through the usual functorialities, as soon as a significant notion (such as the various test notions) appears, I presumably would have hit upon the crucial point about modelizers and so-called "asphericity structures" a lot sooner, without going through the long-winded detour of homotopy structures, and the still extremely special types of test functors suggested by the contractibility assumptions. However, I believe that most of the work I went through, although irrelevant for the "asphericity story" itself, will still be useful, especially when it comes to pinpointing the so-called "canonical" modelizers, whose modelizing structure is intrinsically determined by the category structure.

First thing now which we've to do is to have a closer look at the meaning of asphericity for a functor between small categories. There is no reason whatever to put any restrictions on these categories besides smallness (namely the cardinals of the sets of objects and arrows being in the "universe" we are implicitly working in, or even more stringently still, A and B to be in (Cat) namely to be objects of that universe). Thus, we will not assume A and B to be test categories or the like. We will be led to consider, for *any* small category A, the localization of $A^\wedge$ with respect to $\mathcal{W}$-equivalences, which I'll denote by $(\mathrm{Hot}_A^{\mathcal{W}})$ or simply (Hot_A):

$$(1) \qquad (\mathrm{Hot}_A^{\mathcal{W}}) = (\mathrm{Hot}_A) = \mathcal{W}^{-1}A^\wedge.$$

[p. 189] These categories, I suspect, are quite interesting in themselves, and they merit to be understood. Thus, one of Quillen's results asserts that (at least for $\mathcal{W} = W_{(\mathrm{Cat})}$ = ordinary weak equivalences, but presumably his arguments will carry over to an arbitrary $\mathcal{W}$) in case A is a product category $\Delta \times A_0$, where A_0 is any small category and Δ the category of standard, ordered simplices, then $A^\wedge$ is a closed model category admitting as weak equivalences the set $\mathcal{W}_A$; and hence (Hot_A), the corresponding "homotopy category", admits familiar homotopy constructions, including the two types of Dold-Puppe exact sequences, tied up with loop- and suspension functors. It is very hard to believe that this should be a special feature of the category Δ as the multiplying factor – surely any test category or strict test category instead should do as well. As we'll check below, the product of a local test category with any category A_0 is again a local test category, hence a test-category if both factors are ($\mathcal{W}$-)aspheric. Thus suggests that maybe for any local test category A, the corresponding $A^\wedge$ is a closed model category – but it isn't even clear yet if the same doesn't hold for *any* small category A whatever! It's surely something worth looking at.

3.5. This, I quickly became aware, is a misrepresentation of Quillen's result – that "weak equivalences" he introduced are a lot stronger than $\mathcal{W}_A$. I'll have to come back upon this soon enough!

As we'll see presently, it is a tautology more or less that a functor

$$(1) \qquad i : A \to B,$$

giving rise to a functor

$$(2) \qquad i^* : B^\wedge \to A^\wedge$$

(commuting to all types of direct and inverse limits), induces a functor on the localizations

$$\overline{i^*} : (\mathrm{Hot}_B) \to (\mathrm{Hot}_A),$$

provided i is $\mathcal{W}$-aspheric. Thus,

$$A \mapsto (\mathrm{Hot}_A)$$

can be viewed as a functor with respect to A, provided we take as "morphisms" between "objects" A the *aspheric* functors only – i.e., it is a functor on the subcategory $(\mathrm{Cat})_{\mathcal{W}\text{-asph}}$ of (Cat), having the same objects as (Cat), but with maps restricted to be $\mathcal{W}$-aspheric ones.

We'll denote by

$$\mathrm{Hot}(\mathcal{W}) = \mathcal{W}^{-1}(\mathrm{Cat})$$

the homotopy category defined in terms of the basic localizer $\mathcal{W}$. For [p. 190] any small category A, we get a commutative diagram

(3)
$$\begin{tikzcd} A^\wedge \arrow[r, "i_A"] \arrow[d, "\gamma_A"'] & (\mathrm{Cat}) \arrow[d, "\gamma_{\mathcal{W}} \text{ or } \gamma"] \\ (\mathrm{Hot}_A) \arrow[r, "\overline{i_A}"] & (\mathrm{Hot}(\mathcal{W})) \end{tikzcd} \quad ,$$

we denote by
$$\varphi_A = \gamma i_A = \overline{i_A}\gamma_A : A^\wedge \to (\mathrm{Hot}(\mathcal{W}))$$
the corresponding composition.

Coming back to the case of a functor i and corresponding i^* ((1) and (2)), the functors i^*, i_A, i_B do not give rise to a commutative triangle, but to a triangle *with commutation morphism* λ_i:

$$\begin{tikzcd} B^\wedge \arrow[rr, "i^*"] \arrow[dr, "i_B"'] & \arrow[d, phantom, "\overset{\lambda_i}{\longleftarrow}"] & A^\wedge \arrow[dl, "i_A"] \\ & (\mathrm{Cat}) & \end{tikzcd} \quad ,$$

i.e., for any F in $B^\wedge$, we get a map

(4) $$\lambda_i(F) : i_A i^*(F) = A_{/i^*(F)} \to i_B(F) = B_{/F},$$

the first hand side of (4), also written simply $A_{/F}$ when there is no ambiguity for i, can be interpreted as the category of pairs
$$(a, p), \quad a \in \mathrm{Ob}\,A, \quad p : i(a) \to F,$$
where p is a map in $B^\wedge$, B identified as usual to a full subcategory of $B^\wedge$ (hence $i(a)$ identified with an object of $B^\wedge$). The map $\lambda_i(F)$ for fixed F is of course the functor
$$(a, p) \mapsto (i(a), p).$$
The topological significance of course is clear: interpreting i as defining a "map" or morphisms of the corresponding topoi $A^\wedge$ and $B^\wedge$, having i^* as inverse image functor, an object F of $B^\wedge$ gives rise to an *induced topos* $B^\wedge{}_{/F} \xrightarrow{\simeq} (B_{/F})^\wedge$, and the restriction of the "topos above" $A^\wedge$ to the induced $B^\wedge{}_{/F}$, or equivalently the result of base change $B^\wedge{}_{/F} \to B^\wedge$, gives rise to the induced morphism of topoi $A^\wedge{}_{/i^*(F)} \to B^\wedge{}_{/F}$, represented precisely by the map $\lambda_i(F)$ in (Cat).

The condition of $\mathcal{W}$-asphericity on i may be expressed in manifold ways, as properties of either one of the three aspects
$$\lambda_i, \quad i^*, \quad i_A i^*$$
of the situation created by i, with respect to the localizing sets $\mathcal{W}$, $\mathcal{W}_A$, $\mathcal{W}_B$, or to the notion of aspheric object. As $\mathcal{W}_A$ is defined in terms of $\mathcal{W}$ as just the inverse image of the latter by i_A, and the same for aspheric objects, it turns out that each of the conditions we are led to express [p. 191]
on i^*, can be formulated equivalently in terms of the composition $i_A i^*$. I'll restrict to formulate these in terms of i^* only, which will be the form most adapted to the use we are going to make later of the notion of a $\mathcal{W}$-aspheric map, when introducing the so-called "asphericity structures" and corresponding "testing functors".

27.5.

68 *Digression on a "new continent".* It has been over six weeks now that I didn't write down any notes. The reason for this is that I felt the story of asphericity structures and canonical modelizers was going to come now without any problem, almost as a matter of routine to write it down with some care – therefore, I started doing some scratchwork on a few questions which had been around but kept in the background since the beginning, and which were a lot less clear in my mind. Some reflection was needed anyhow, before it would make much sense to start writing down anything on these. Finally, it took longer than expected, as usual – partly because (as usual too!), a few surprises would turn up on my way. Also, I finally allowed myself to become distracted by some reflection on the "Lego-Teichmüller construction game", and pretty much so during last week. The occasion was a series of informal talks Y. has been giving in Molino's seminar, [Y. = Yves Ladegaillerie] on Thurston's hyperbolic geometry game and his compactification of Teichmüller space. Y. was getting interested again in mathematics after a five year's interruption. He must have heard about my seminar last year on "anabelian algebraic geometry" and the "Teichmüller tower", and suggested I might drop in to get an idea about Thurston's work. This work indeed appears as closely related in various respects to my sporadic reflections of the last two years, just with a diametrically opposed emphasis – mine being on the algebro-geometric and arithmetic aspects of "moduli" of algebraic curves, his on hyperbolic riemannian geometry and the simply connected transcendental Teichmüller spaces (rather than the algebraic modular varieties). The main intersection appears to be interest in surface surgery and the relation of this to the Teichmüller modular group. I took the occasion to try and recollect about the Lego-Teichmüller game, which I had thought of last year as a plausible, very concrete way for modelizing and visualizing the whole tower of Teichmüller groupoids $T_{g,\nu}$ and the main operations among these, especially the "cutting" and "gluing" operations. The very informal talk I gave was mainly intended for Y. as a matter of fact, and it was [p. 192] an agreeable surprise to notice that the message this time was getting through. For the fiver or six years since my attention became attracted by the fascinating melting-pot of key structures in geometry, topology, arithmetic, discrete and algebraic groups, intertwining tightly in a kind of very basic Galois-Teichmüller theory, Y. has been the very first person I met so far to have a feeling for (a not yet dulled instinct I might say, for sensing) the extraordinary riches opening up here for investigation. The series of talks I had given in a tentative seminar last year had turned short, by lack of any active interest and participation of anyone among the handful of mere listeners. And the two or three occasions I had the years before to tell about the matter of two-dimensional maps ("cartes") and their amazing algebro-arithmetic implications, to a few highbrow colleagues with incomparably wider background and know-how than anyone around here, I met with polite interest, or polite indifference which is the same. As there was nobody around anyhow to take any interest the these juicy greenlands, nobody would care to see, because there was no text-book nor any official seminar notes to prove they existed, after a few years I finally set off myself for a preliminary voyage.

I thought it was going to take me a week or two to tour it and kind of recense resources. It took me five months instead of intensive work, and two impressive heaps of notes (baptized "La Longue Marche à travers la théorie de Galois"), to get a first, approximative grasp of some of the main structures and relationships involved. The main emphasis was (still is) on an understanding of the action of profinite Galois-groups (foremost among which $\mathrm{Gal}_{\overline{\mathbb{Q}}/\mathbb{Q}}$ and the subgroups of finite index) on non-commutative profinite fundamental groups, and primarily on fundamental groups of algebraic curves – increasingly too on those of modular varieties (more accurately, modular *multiplicities*) for such curves – the profinite completions of the Teichmüller group. The voyage was the most rewarding and exciting I had in mathematics so far – and still it became very clear that it was just like a first glimpse upon a wholly new landscape – one landscape surely among countless others of a continent unknown, eager to be discovered.

["recense" could be translated as "survey"]

This was in the first half of the year 1981 – just two years ago, it turns out, but it look almost infinitely remote, because such a lot of things took place since. Looking back, it turns out there have been since roughly four main alternating periods of reflection, one period of reflection on personal matters alternating with one on mathematics. The next mathematical reflection started with a long digression on tame topology and the "déployment" ("unfolding") of stratified structures, as a leading [p. 193]
thread towards a heuristic understanding of the natural stratification of the Mumford-Deligne compactifications of modular multiplicities $M_{g,\nu}$ (for curves of genus g endowed with ν points). This then led to the "anabelian seminar" which turned short, last Spring. Then a month or two sicknees, intensive meditation for three or four months, a few more months for settling some important personal matter; and now, since February, another mathematical reflection started.

I am unable to tell the meaning of this alternation of periods of meditation on personal matters and periods of mathematical reflection, which has been going through my life for the last seven years, more and more, very much like the unceasing up and down of waves, or like a steady breathing going through my life, without any attempt any longer of controlling it one way or the other. One common moving force surely is the inborn curiosity – a thirst for getting acquainted with the juicy things of the inexhaustible world, whether they be the breathing body of the beloved, or the evasive substance of one's own life, or the much less evasive substance of mathematical structure and their delicate interplay. This thirst in itself is of a nature quite different of the ego's – it is the thirst of life to know about itself, a primal creative force which, one suspects, has been around forever, long before a human ego – a bundle mainly of fears, of inhibitions and self-deceptions – came into being. Still, I am aware that the ego is strongly involved in the particular way in which the creative force expresses itself, in my own life or anyone else's (when this force is allowed to come into play at all...). The motivations behind any strong energy investment, and more particularly so when it is an activity attached with any kind of social status or prestige, are a lot more complex and ego-driven than one generally cares to admit. True, ambition by itself is powerless for discovering or understanding or perceiving anything substantial whatever, neither a mathematical relationship nor the perfume of a flower. In moments of work and of discovery, in any creative moments in life, as a matter of fact, ambition is absent; the Artisan is a keen interest, which is just one of the manifold aspect of love. What however pushes us so relentlessly to work again, and so often causes our life and passion gradually to dry off and become insensitive, even to the kindred passion of a follow-being, or to the unsuspected beauties and mysteries of the very field we are supposed to be plowing – this force is neither love nor keen interest for the things and beings in this world. It is interesting enough though, and surely deserves a close look!

[p. 194] I thought I was starting a retrospective of six weeks of scratchwork on homotopical algebra – and it turns out to be a (very) short retrospective rather of the up-and-down movement of my mathematical interests and investments during the last years. Doubtless, the very strongest attraction, the greatest fascination goes with the "new world" of anabelian algebraic geometry. It may seem strange that instead, I am indulging in this lengthy digression on homotopical algebra, which is almost wholly irrelevant I feel for the Galois-Teichmüller story. The reason is surely an inner reluctance, an unreadiness to embark upon a long-term voyage, well knowing that it is so enticing that I may well be caught in this game for a number of years – not doing anything else day and night than making love with mathematics, and maybe sleeping and eating now and then. I have gone through this a number of times, and at times I thought I was through. Finally, I came to admit and to accept, two years ago, I was not through yet – this was during the months of meditation after the "long march through Galois theory" – which had been, too, a wholly unexpected fit of mathematical passion, not to say frenzy. And during the last weeks, just reflecting a little here and there upon the Teichmüller-Lego game and its arithmetical implications, I let myself be caught again by this fascination – it is becoming kind of clear now that I am going to finish writing up those notes on algebra, almost like some homework that has got to be done (anyhow I like to finish when I started something) – and as soon as I'm through with the notes, back to geometry in the long last! Also, the idea is in the air for the last few months – since I decided to publish these informal notes on stacks or whatever it'll turn out to be – that I may well go on the same way, writing up and publishing informal notes on other topics, including tame topology and anabelian algebraic geometry. In contrast to the present notes, I got heaps of scratchwork done on these in the years before – in this respect time is even riper for me to ramble "publicly" than on stacks and homotopy theory!

From Y. who looked through a lot of literature on the subject, it strikes me (agreeably of course) that nobody yet hit upon "the" natural presentation of the Teichmüller groupoids, which kind of imposes itself quite forcibly in the set-up I let myself be guided by. Technically speaking (and this will rejoice Ronnie Brown I'm sure!), I suspect one main reason why this is so, is that people are accustomed to working with fundamental groups and generators and relations for these and stick to it, even in contexts when this is wholly inadequate, namely when you get a clear description by generators and relations only when working [p. 195] simultaneously with a whole bunch of base-points chosen with case – or equivalently, working in the algebraic context of *groupoids*, rather than groups. Choosing paths for connecting the base-points natural to the situation to just one among them, and reducing the groupoid to a single group, will then hopelessly destroy the structure and inner symmetries of the situation, and result in a mess of generators and relations no-one dares to write down, because everyone feels they won't be of any use whatever, and just confuse the picture rather than clarify it. I have known such perplexity myself a long time ago, namely in van Kampen-type situations, whose only understandable formulation is in terms of (amalgamated sums of) groupoids. Still, standing habits of thought are very strong, and during the long march through Galois theory, two years ago, it took me weeks and months trying to formulate everything in terms of groups or "exterior groups" (i.e., groups "up to inner automorphism"), and finally learning the lesson and letting myself be convinced progressively, not to say reluctantly, that groupoids only would fit nicely. Another "technical point" of course is the basic fact (and the wealth of intuitions accompanying it) that the Teichmüller groups are fundamental groups indeed – a fact ignored it seems by most geometers, because the natural "spaces" they are fundamental groups of are not topological spaces, but the modular "multiplicities" $M_{g,\nu}$ – namely topoi! The "points" of these "spaces" are just the structures being investigated (namely algebraic curves of type (g,ν)), and the (finite) automorphism groups of these "points" enter into the picture in a very crucial way. They can be adequately chosen as part of the system of basic generators for the Teichmüller groupoid $T_{g,\nu}$. The latter of course is essentially (up to suitable restriction of base-points) just the fundamental groupoid of $M_{g,\nu}$. It is through this interpretation of the Teichmüller groups or groupoids that it becomes clear that the profinite Galois group $\mathrm{Gal}_{\overline{\mathbb{Q}}/\mathbb{Q}}$ operates on the profinite completion of these and of their various variants, and this (it turns out) in a way respecting the manifold structures and relationships tying them tightly together.

29.5. [p. 196]

69 Before resuming more technical work again, I would like to have a short retrospective of the last six weeks' scratchwork, now lying on my desk as a thickly bunch of scratchnotes, nobody but I could possibly make any sense of.

Digression on six weeks' scratchwork: derivators*, and integration of homotopy types.*

The first thing I had on my mind has been there now for nearly twenty years – ever since it had become clear, in the SGA 5 seminar on L-functions and apropos the formalism of traces in terms of derived categories, that Verdier's set-up of derived categories was insufficient for formulating adequately some rather evident situations and relationships, such as the addition formula for traces, or the multiplicative formula for determinants. It then became apparent that the derived category of an abelian category (say) was too coarse an object various respects, that it had to be complemented by similar "triangulated categories" (such as the derived category of a suitable category of "triangles" of complexes, or the whole bunch of derived categories of categories of filtered complexes of order n with variable n), closely connected to it. Deligne and Illusie had both set out, independently, to work out some set-up meeting the most urgent requirements (Illusie's treatment in terms of filtered complexes was written down and published in his thesis six years later (Springer Lecture Notes Nº 239)). While adequate for the main tasks then at hand, neither treatment was really wholly satisfactory to my taste. One main feature I believe making me feel uncomfortable, was that the extra categories which had to be introduced, to round up somewhat a stripped-and-naked triangulated category, were triangulated categories in their own right, in Verdier's sense, but remaining nearly as stripped by themselves as the initial triangulated category they were intended to provide clothing for. In other words, there was a lack of inner stability in the formalism, making it appear as very much provisional still. Also, while interested in associating to an abelian category a handy sequence of "filtered derived categories", Illusie made no attempt to pin down what exactly the inner structure of the object he had arrived at was – unlike Verdier, who had introduced, alongside with the notion of a derived category of an abelian category, a general notion of triangulated categories, into which these derived categories would fit. The obvious idea which was in my head by then for avoiding such shortcomings, was that an abelian category $\mathcal{A}$ gave rise, not only to the single usual derived category $D(\mathcal{A})$ of Verdier, but also, for every type of diagrams, to the derived category of the abelian category of all $\mathcal{A}$-valued diagrams of this type. In precise terms, for any small category I, we get the category $D(\mathbf{Hom}(I,\mathcal{A}))$, depending functorially in a contravariant way on I. Rewriting this category $D_{\mathcal{A}}(I)$ say, the idea was to consider

[Illusie 1971, 1972]

[p. 197]

$$(*)\qquad I \mapsto D_{\mathcal{A}}(I),$$

possibly with I suitably restricted (for instance to finite categories, or to finite ordered sets, corresponding to finite *commutative* diagrams), as embodying the "full" triangulated structure defined by $\mathcal{A}$. This of course at once raises a number of questions, such as recovering the usual triangulated structure of $D(\mathcal{A}) = D_{\mathcal{A}}(e)$ (e the final object of (Cat)) in terms of (*), and pinning down too the relevant formal properties (and possibly even extra structure) one had to assume on (*). I had never so far taken the time to sit down and play around some and see how this goes through, expecting that surely someone else would do it some day and I would be informed – but apparently in the last eighteen years nobody ever was interested. Also, it had been rather clear from the start that Verdier's constructions could be adapted and did make sense for non-commutative homotopy set-ups, which was also apparent in between the lines in Gabriel-Zisman's book on the foundations of homotopy theory, and a lot more explicitly in Quillen's

[Gabriel and Zisman 1967]

axiomatization of homotopical algebra. This axiomatization I found very appealing indeed – and right now still his little book is my most congenial and main source of information on foundational matters of [Quillen 1967] homotopical algebra. I remember though my being a little disappointed at Quillen's not caring either to pursue the matter of what exactly a "non-commutative triangulated category structure" (of the type he was getting from his model categories) was, just contenting himself to mumble a few words about existence of "higher structure" (then just the Dold-Puppe sequences), which (he implies) need to be understood. I felt of course that presumably the variance formalism (*) should furnish any kind of "higher" structure one was looking for, but it wasn't really my business to check.

It still isn't, however I did some homework on (*) – it was the first thing indeed I looked at in these six weeks, and some main features came out very readily indeed. It turns out that the main formal variance property to demand on (*), presumably even the only one, is that for a given map $f : I \to J$ on the indexing categories of diagram-types I and J, the corresponding functor

$$f^* : D(J) \to D(I)$$

should have both a left and a right adjoint, say $f_!$ and f_*. In case $J = e$, the two functors we get from $D(I)$ to $D(e) = \mathcal{D}$ (the "stripped" triangulated [p. 198] category) can be viewed as a *substitute for taking, respectively, direct and inverse limits in* $\mathcal{D}$ (for a system of objects indexed by I), *which in the usual sense don't generally exist in* $\mathcal{D}$ (except just finite sums and products). These operations admit as important special cases, when I is either one of the two mutually dual categories

$$(**) \qquad \begin{tikzcd}[row sep=small, column sep=small] & b \\ a \arrow[ur] \arrow[dr] & \\ & c \end{tikzcd} \quad \text{or} \quad \begin{tikzcd}[row sep=small, column sep=small] & b \arrow[dl] \\ a & \\ & c \arrow[ul] \end{tikzcd} \quad ,$$

the operation of (binary) amalgamated sums or fibered products, and hence also of taking "cofibers" and "fibers" of maps, in the sense introduced by Cartan-Serre in homotopy theory about thirty years ago. I also checked that the two mutually dual Dold-Puppe sequences follow quite formally from the set-up. One just has to fit in a suitable extra axiom to ensure the usual exactness properties for these sequences.

Except in the commutative case when starting with an abelian category as above, I did not check however that there is indeed such "higher variance structure" in the usual cases, when a typical "triangulated category" in some sense or other turns up, for instance from a model category in Quillen's sense. What I did check though in this last case, under a mild additional assumption which seems verified in all practical cases is the existence of the operation $f_! = \int_I$ ("integration") and $f_* = \prod_I$ (cointegration) for the special case $f : I \to e$, when I is either of the two categories (**) above. I expect that working some more, one should get under the same assumptions at least the existence of $f_!$ and f_* for any map $f : I \to J$ between finite ordered sets.

My main interest of course at present is in the category (Hot) itself, more generally in $\mathrm{Hot}(\mathcal{W}) = \mathcal{W}^{-1}(\mathrm{Cat})$, where $\mathcal{W}$ is a "basic localizer". More generally, if (M, W) is any modelizer say, the natural thing to do, paraphrasing (*), is for any indexing category I to endow $\mathbf{Hom}(I, M)$ with the set of arrows W_I defined by componentwise belonging to W, and to define

$$D_{(M,W)}(I) = D(I) \stackrel{\text{def}}{=} W_I^{-1}\,\mathbf{Hom}(I, M),$$

with the obvious contravariant dependence on I, denoted by f^* for $f : I \to J$. The question then arises as to the existence of left and right adjoints, $f_!$ and f_*. In case we take $M = (\mathrm{Cat})$, the existence of $f_!$ goes through with amazing smoothness: interpreting a "model" object of $\mathbf{Hom}(I, (\mathrm{Cat}))$, namely a functor

$$I \to (\mathrm{Cat})$$

[p. 199] in terms of a cofibered category X over I

$$p : X \to I,$$

and assuming for simplicity f cofibering too, $f_!(X)$ is just X itself, the total category of the cofibering, viewed as a (cofibered) category over J by using the functor $g = f \circ p$! This applies for instance when J is the final category, and yields the operation of "integration of homotopy types" $\int_I$, in terms of the total category of a cofibered category over I. If we want to rid ourselves from any extra assumption on f, we can describe $D(I)$ (up to equivalence) in terms of the category $(\mathrm{Cat})_{/I}$ of categories X over I (not necessarily cofibered over I), W_I being replaced by the corresponding notion of "*$\mathcal{W}$-equivalences relative to I*" for maps $u : X \to Y$ of objects of (Cat) over I, by which we mean a map u such that the localized maps

$$u_{/i} : X_{/i} \to Y_{/i}$$

are in $\mathcal{W}$, for any i in I. Regarding now any category X over I as a category over J by means of $f \circ p$, this is clearly compatible with the relative weak equivalences $\mathcal{W}_I$ and $\mathcal{W}_J$, and yields by localization the looked-for functor $f_!$.

This amazingly simple construction and interpretation of the basic $f_!$ and $\int_I$ operations is one main reward, it appears, for working with the "basic localizer" (Cat), which in this occurrence, as in the whole test- and asphericity story, quite evidently deserves its name. It has turned out since that in some other respects – for instance, paradoxically when it comes to the question of the relationship between this lofty integration operation, and true honest amalgamated sums – the modelizers $A\hat{}$ associated to test categories A (namely the so-called "elementary modelizers") are more convenient tools than (Cat). Thus, it appears very doubtful still that (Cat) is a "model category" in Quillen's sense, in any reasonable way (with $\mathcal{W}$ of course as the set of "weak equivalences"). I finally got the feeling that a good mastery of the basic aspects of homotopy types and of basic relationships among these, will require mainly great "aisance" in playing around with a number of available descriptions of homotopy types by models, no one among which (not even by models in (Cat), and surely still less by semisimplicial structures) being adequate for replacing all others.

["aisance" here could be "ease" or "fluency"]

As for the f_* and cointegration $\prod_I$ operations among the categories $D(I)$, except in the very special case noted above (corresponding to fibered products), I did not hit upon any ready-to-use candidate for it, and I doubt there is any. I do believe the operations exist indeed, and I even have in mind a rather general condition on a pair (M,W) with $W \subset \mathrm{Fl}(M)$, for both basic operations $f_!$ and f_* to exist between the corresponding categories $D_{(M,W)}(I)$ – but to establish this expectation may require a good amount of work. I'll come back upon these matters in due course.

[p. 200]

There arises of course the question of giving a suitable name to the structure $I \to D(I)$ I arrived at, which seems to embody at least some main features of a satisfactory notion of a "triangulated category" (not necessarily commutative), gradually emerging from darkness. I have thought of calling such a structure a "*derivator*", with the implication that its main function is to furnish us with a somehow "complete" bunch (in terms of a rounded-up self-contained formalism) of categories $D(I)$, which are being looked at as "*derived categories*" in some sense or other. The only way I know of for constructing such a derivator, is as above in terms of a pair (M,W), submitted to suitable conditions for ensuring existence of $f_!$ and f_*, at least when f is any map between finite ordered sets. We may look upon $D(I)$ as a refinement and substitute for the notion of family of objects of $D(e) = D_0$ indexed by I, and the integration and cointegration operations from $D(I)$ to D_0 as substitutes (in terms of these finer objects) of direct and inverse limits in D_0. When tempted to think of these latter operations (with values in D_0) as the basic structures involved, one cannot help though looking for the same kind of structure on any one of these subsidiary categories $D(I)$, as these are being thought of as derived categories in their own right. It then appears at once that the "more refined substitutes" for J-indexed systems of objects of $D(I)$ are just the objects of $D(I \times J)$, and the corresponding integration and cointegration operations

$$D(I \times J) \to D(I)$$

are nothing but $p_!$ and p_*, where $p : I \times J \to I$ is the projection. Thus, one is inevitably conducted to look at operations $f_!$ and f_* instead of merely integration and cointegration – thus providing for the "inner stability" of the structure described, as I had been looking for from the very start.

The notion of integration of homotopy types appears here as a natural by-product of an attempt to grasp the "full structure" of a triangulated category. However, I had been feeling the need for such a notion of

integration of homotopy types for about one or two years already (with-
[p. 201] out any clear idea yet that this operation should be one out of two main ingredients of a (by then still very misty) notion of a triangulated category of sorts). This feeling arose from my ponderings on stratified structures and the "screwing together" of such structures in terms of simple building blocks (essentially, various types of "tubes" associated to such structures, related to each other by various proper maps which are either inclusion or – in the equisingular case at any rate – fiber maps). This "screwing together operation" could be expressed as being a direct limit of a certain finite system of spaces. In the cases I was most interested in (namely the Mumford-Deligne compactifications $M_{g,\nu}^{\wedge}$ of the modular multiplicities $M_{g,\nu}$), these spaces or "tubes" have exceedingly simple homotopy types – they are just $K(\pi,1)$-spaces, where each π is a Teichmüller-type discrete group (practically, a product of usual Teichmüller groups). It then occurred to me that the whole homotopy type of $M_{g,\nu}^{\wedge}$, or of any locally closed union of strata, or (more generally still) of "the" tubular neighborhood of such a union in any larger one, etc. – that all these homotopy types should be expressible in terms of the given system of spaces, and more accurately still, just in terms of the corresponding system of fundamental groupoids (embodying their homotopy types). In this situation, what I was mainly out for, was precisely an accurate and workable description of this direct system of groupoids (which could be viewed as just one section of the whole "Teichmüller tower" of Teichmüller groupoids...). Thus, it was a rewarding extra feature of the situation (by then just an expectation, as a matter of fact), that such a description should at the same time yield a "purely algebraic" description of the homotopy types of all the spaces (rather, multiplicities, to be wholly accurate) which I could think of in terms of the natural stratification of $M_{g,\nu}^{\wedge}$. There was an awareness that this operation on homotopy types could not be described simply in terms of a functor $I \to (\mathrm{Hot})$, where I is the indexing category, that a functor $i \mapsto X_i : I \to M$ (where M is some model category such as (Spaces) or (Cat)) should be available in order to define an "integrated" homotopy type $\int X_i$. This justified feeling got somewhat blurred lately, for a little while, by the definitely unreasonable expectation that finite limits should exist in (Hot) after all, why not! It's enough to have a look though (which probably I did years ago and then forgot in the meanwhile) to make sure they don't...

Whether or not this notion of "integration of homotopy types" is more or less well known already under some name or other, isn't quite clear to me. It isn't familiar to Ronnie Brown visibly, but it seems he heard
[p. 202] about such a kind of thing, without his being specific about it. It was the episodic correspondence with him which finally pushed me last January to sit down for an afternoon and try to figure out what there actually was, in a lengthy and somewhat rambling letter to Illusie (who doesn't seem to have heard at all about such operations). This preliminary reflection proved quite useful lately, I'll have to come back anyhow upon some of the specific features of integration of homotopy types later, and

there is not much point dwelling on it any longer at present.

Digression on scratchwork (2): cohomological properties of maps in (Cat) and in $A^\wedge$. Does any topos admit a "dual" topos? Kan fibrations rehabilitated.

70 This was a (finally somewhat length!) review of ponderings which didn't take me more than just a few days, because it was about things some of which were on my mind for a long time indeed. It took a lot more work to try to carry through the standard homotopy constructions, giving rise to the Dold-Puppe sequences, within the basic modelizer (Cat). Most of the work arose, it now seems to me, out of a block I got (I couldn't tell why) against the Kan-type condition on complexes, so I tried hard to get along without anything of the sort. I kind of fooled myself into believing that I was forced to do so, because I was working in an axiomatic set-up dependent upon the "basic localizer" $\mathcal{W}$, so the Kan condition wouldn't be relevant anyhow. The main point was to get, for any map $f : X \to Y$ in (Cat), a factorization

$$X \xrightarrow{i} Z \xrightarrow{p} Y,$$

where p has the property that base-change by p transforms weak equivalences into weak equivalences (visibly a Serre-fibration type condition), and i satisfies the dual condition with respect to co-base change; and moreover where either p, or i can be assumed to be in $\mathcal{W}$, i.e., to be a weak equivalence. (This, by the way, is the extra condition on a pair (M,W) I have been referring to above (page 200, for (hopefully) getting $f_!$ and f_* operations.) At present, I do not yet know whether such factorization always exists for a map in (Cat), without even demanding that either i or p should be in $\mathcal{W}$.

I first devoted a lot of attention to Serre-type conditions on maps in (Cat), which turned out quite rewarding – with the impression of arriving at a coherent and nicely auto-dual picture of cohomology properties of functors, i.e., maps in (Cat), as far as these were concerned with *base change behavior* (and *not* co-base change). Here I was guided by work done long ago to get the étale cohomology theory off the ground, and where the two main theorems achieving this aim were precisely two theorems of commutation of "higher direct images" $\mathrm{R}^i g_*$ with respect to base-change by a map h – namely, it is OK when either g is *proper*, or h is *smooth*. It was rather natural then to introduce the notion of [p. 203] (cohomological) *smoothness* and (coh.) *properness* of a map in (Cat), by the obvious base-change properties. It turned out that these could be readily characterized by suitable asphericity conditions, which are formally quite similar to the well-known valuative criterion for a map of finite type between noetherian schemes to be "universally open" (which can be viewed as a "purely topological" variant of the notion of smoothness), resp. "universally closed", or rather, more stringently, proper. These conditions, moreover, are trivially satisfied when $f : X \to Y$ turns X into a category over Y which is fibered over Y (i.e., definable in terms of a contravariant pseudo-functor $Y^{\mathrm{op}} \to (\mathrm{Cat})$), resp. cofibered over Y (namely, definable in terms of a pseudo-functor $Y \to (\mathrm{Cat})$). If we call such maps in (Cat) "fibrations" and "cofibrations" (very much in conflict, alas, with Quillen's neat set-up of fibrations-cofibrations!), it turns out that *fibrations are smooth*, *cofibrations are proper*. This is all the more remarkable, as the notions of fibered and cofibered categories were introduced with a view upon "large categories", in order to pin down some standard properties met in all situations of "base change" (and later the dual situation of co-base change) – the main motivation for this being the need to formulate with a minimum of precision a set-up for "descent techniques" in algebraic geometry. (These techniques, as well as the cohomological base change theorems, make visibly a sense too in the context of analytic spaces say, or of topological spaces, but they don't seem to have been assimilated yet by geometers outside of algebraic geometry.) That these typically "general nonsense" notions should have such precise topological implications came as a complete surprise! As a consequence, a bifibration (namely a map which is both a fibration and a cofibration) is smooth and proper and hence a (cohomological) "Serre fibration", for instance the sheaves $\mathrm{R}^i f_*(F)$, when F is a constant abelian sheaf on X (i.e., a constant abelian group object in $X^\wedge$), namely $y \mapsto \mathrm{H}^i(X_{/y}, F)$, are *local systems* on Y, i.e., factor through the fundamental groupoid of Y.

A greater surprise still was the *duality between the notions of properness and smoothness:* just as a map $f : X \to Y$ in (Cat) is a cofibration iff the "dual" map $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is a fibration, it turns out that f *is proper iff* f^{op} *is smooth*. This was a really startling fact, and it caused me to wonder, in the context of more general topoi than just those of the type $X^\wedge$, whether there wasn't *a notion of duality generalizing the relationship between two topoi* $X^\wedge$ *and* $X^\vee \overset{\mathrm{def}}{=} (X^{\mathrm{op}})^\wedge$. Indeed, these two categories of sheaves can be described intrinsically, one in terms of the other, up

[p. 204] to equivalence, by a natural pairing

$$X^\wedge \times X^\vee \to (\mathrm{Sets})$$

commuting componentwise with (small) direct limits, and inducing an equivalence between either factor with the category of "co-sheaves" on the other, namely covariant functors to (Sets) commuting with direct limits. But it isn't at all clear, starting with an arbitrary topos $\mathcal{A}$ say, whether the category $\mathcal{A}'$ of all cosheaves on $\mathcal{A}$ is again a topos, and still less whether $\mathcal{A}$ can be recovered (up to equivalence) in terms of $\mathcal{A}'$ as the category of all cosheaves on $\mathcal{A}'$.

To come back though to the factorization problem raised above (p. 202), the main trouble here is that, except the case of an isomorphism $i : X \xrightarrow{\sim} Z$, I was unable to pin down a single case of a map i in (Cat) such that co-base change by i transforms weak equivalences (in the usual sense say) into weak equivalences. One candidate I had in mind, the so-called "*open immersions*", namely functors $i : X \to Z$ inducing an isomorphism between X and a "sieve" (or "crible") in Z (corresponding to an open sub-topos of $Z^\wedge$), and dually the "closed immersions", finally have turned out delusive – a disappointment maybe, but still more a big relief to find out at last how the score was! Almost immediately in the wake of this negative result in (Cat), and in close connection with the fairly well understood $\int$ substitute for amalgamated sums in (Cat), came the big compensation valid in any category $A^\wedge$, namely the fact that for a cocartesian square

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "i'"] & X' \end{tikzcd}$$

in $A^\wedge$, where i is a monomorphism, if i (resp. g) is in $\mathcal{W}_A$, so is i' (resp. f). This implies that *co-base change by a monomorphism in $A^\wedge$ transform weak equivalences into weak equivalences*. The common main fact behind these statements is that for a diagram as above (without assuming i nor g to be in $\mathcal{W}_A$), X' can be interpreted up to weak equivalence as the "integral" of the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \\ Y' & \end{tikzcd},$$

more accurately, the natural map in (Cat)

$$\int \begin{tikzcd} A_{/Y} \arrow[r] \arrow[d] & A_{/X} \\ A_{/Y'} & \end{tikzcd} \longrightarrow A_{/X'}$$

is in $\mathcal{W}$.

The corresponding statement in (Cat) itself, even if i is supposed to be an open immersion in (Cat) say (or a closed immersion, which amounts to the same by duality), is definitely false, in other words: while the modelizer (Cat) allows for a remarkably simple description of integration of homotopy types, as seen in the previous section, in the basic case however of an integration [p. 205]

[this integral is set inline in the typescript]

$$\int \begin{tikzcd} A_{/Y} \arrow[r, "i"] \arrow[d, "g"] & A_{/X} \\ A_{/Y'} & \end{tikzcd}$$

(corresponding to "amalgamated sums"), and even when $i : Y \to X$ is an open or closed immersion, this operation does definitely *not* correspond to the operation of taking the amalgamated sum X' in (Cat). It does though when $X \to X'$ is smooth resp. proper, for instance if it is a fibration resp. a cofibration, and this in fact implies the positive result in $A^\wedge$ noted above. This condition moreover is satisfied if $g : Y \to Y'$ is equally an open resp. closed immersion, in which case the situation is just the one of an ambient X', and two open resp. closed subobjects X and Y', with intersection Y. This is a useful result, but wholly insufficient for the factorization problem we were after in (Cat), with a view of performing the standard homotopy constructions in (Cat) itself. It may be true still that if $i : Y \to X$ is not only an open or closed immersion, but a weak equivalence as well, that then $i' : Y' \to X'$ is equally a weak equivalence, or what amounts to the same, that X' can be identified up to weak equivalence with the homotopy integral (which indeed, up to weak equivalence, just reduces to Y'); but I have been unable so far to clear up this matter. If true, this would be quite a useful result, but still insufficient it seems in order to carry out the standard homotopy constructions in (Cat) itself.

To sum up, the main drawback of (Cat) as a modelizer is that, except in very special cases which are just too restricted, amalgamated sums in (Cat) don't have a reasonable meaning in terms of homotopy operations – whereas in a category $A^\wedge$, a topos indeed where therefore amalgamated sums have as good exactness properties as if working in (Sets), these amalgamated sums (for a two-arrow diagram with one arrow a monomorphism) *do* have a homotopy-theoretic meaning. This finally seems to force us, in order to develop some of the basic structure in Hot($\mathcal{W}$), to leave the haven of the basic modelizer (Cat), and work in an elementary modelizer $A^\wedge$ instead, where A is some $\mathcal{W}$-test category. This then brought me back finally to the question whether these modelizers are *closed model categories* in Quillen's sense, when we take of course for "weak equivalences" $\mathcal{W}_A$, and moreover as "cofibrations" (in the sense of Quillen's set-up) just the monomorphisms. Relying heavily upon the result on monomorphisms in $A^\wedge$ stated above, it seems to come out that we do get a closed model category indeed – and even a simplicial model category, if we are out for this. There is still a cardinality question to
[p. 206] be settles to get the Quillen factorizations in the general case, but this should not be too serious a difficulty I feel. What however makes me still feel a little unhappy in all this, is rather that I did not get a direct proof for an elementary modelizer being a closed model category – I finally have to make a reduction to the known case of semisimplicial complexes, settles by Quillen in his notes. This detour looks rather artificial – it is the first instance, and presumably the last one, where the theory I am digging out seems to depend on semi-simplicial techniques, techniques which moreover I don't really know and am not really eager to swallow. It's just a prejudice maybe, a block maybe against the semi-simplicial approach which I never really liked nor assimilated – but I do have the feeling that the more refined and specific semi-simplicial techniques and notions (such as minimal fibrations, used in Quillen's proof, alas!) are irrelevant for an understanding of the main structures featuring homotopy theory and homotopical algebra. As for the notion of a Kan complex or a Kan fibration – namely just a "fibration" in Quillen's axiomatic set-up, which I was finally glad to find, "ready for use" – I came to convince myself at last that it was a basic notion indeed, and it was no use trying to bypass it at all price. Thus, I took to the opposite, and tried to pin down a Quillen-type factorization theorem, and his characteristic seesaw game between right and left lifting properties, in as great generality as I could manage.

30.5.

71 The scratchwork done since last month has of course considerably cleared up the prospects of my present pre-stacks reflection on homotopy models, on which I unsuspectingly embarked three months ago. I would like to sketch a provisional working program for the notes still ahead.

Working program and rambling questions (group objects as models, Dold-Puppe theorem...).

A) Write down at last the story of asphericity structures and canonical modelizers, as I was about to when I interrupted the notes to do my scratchwork.

B) Study the basic modelizer (Cat), and the common properties of elementary modelizers $A^\wedge$, with a main emphasis upon base change and co-base change properties, and upon Quillen-type factorization questions. Here it will be useful to dwell somewhat on the "homotopy integral" variant of taking amalgamated sums in (Cat), on the analogous constructions for topoi, and how these compare to the usual amalgamated sums, including the interesting case of topological spaces. It turns out that the homotopy integral variant for amalgamated sums is essentially characterized by a Mayer-Vietoris type long exact sequence for cohomology, and the cases when the homotopy construction turns out to be equivalent to usual amalgamated sums, are just those when such a Mayer-Vietoris sequence exists for the latter. An interesting and typical case is for topological spaces, taking the amalgamated sum for a diagram [p. 207]

$$\begin{array}{ccc} Y & \overset{i}{\hookrightarrow} & X \\ {\scriptstyle g}\downarrow & & \\ Y' & & \end{array},$$

when i is a closed immersion and g is proper (which is also the basic type of amalgamations which occur in the "unfolding" of stratified structures).

In the course of the last weeks' reflections, there has taken place also a substantial clarification concerning the relevant properties of a basic localizer $\mathcal{W}$ and how most of these, including strong saturation of $\mathcal{W}$, follow from just the first three (a question which kept turning up like a nuisance throughout the notes!). This should be among the very first things to write down in this part of the reflections, as $\mathcal{W}$ after all is *the* one axiomatic data upon which the whole set-up depends.

C) A reflection on the main common features of the various contexts met with so far having a "homotopy theory" flavor, with a hope to work out at least some of the main features of an all-encompassing new structure, along the lines of Verdier's (commutative) theory of derived categories and triangulated categories. The basic idea here, for the time being, seems to be the notion of a *derivator*, which should account for all the kind of structure dealt with in Verdier's set-up, as well as in Deligne's and Illusie's later elaborations. There seems to be however some important extra features which are not accounted for by the mere derivator, such as external Hom's with values in (Hot) or some closely related category, and the formalism of basic invariants (such as π_i, H_i

or H^i), with values in suitable categories (often abelian ones), which among others allow to check weak equivalence. Such features seem to be invariably around in all cases I know of, and they need to be understood I feel.

Coming back to (Hot) itself and to modelizers (M, W) giving rise to it, there is the puzzling question about when exactly can we assert that taking group objects of M and weak equivalences between these (namely group-object homomorphisms which are also in W), we get by localization a category equivalent to the category of pointed 0-connected homotopy types. This is a well-known basic fact when we take as models semisimplicial complexes or (I guess) topological spaces – a fact [p. 208] closely connected to the game of associating to any topological group its "*classifying space*", defined "up to homotopy". I suspect the same should hold in any elementary modelizer $A^\wedge$, A a test category, at least in the "strict" case, namely when $A^\wedge$ is totally aspheric, i.e., the canonical functor $A^\wedge \to$ (Hot) commutes to finite products. The corresponding statement for the basic localizer (Cat) itself is definitely false. Group objects in (Cat) are indeed very interesting and well-known beings (introduced, I understand from Ronnie Brown, by Henry Whitehead long time ago, under the somewhat misleading name of "crossed modules"), yet they embody not arbitrary pointed 0-connected homotopy types X, but merely those for which $\pi_i(X) = 0$ for $i > 2$. Thus, we get only 2-truncated homotopy types – and presumably, starting with n—Cat instead of (Cat) as a modelizer, we then should get $(n+1)$-truncated homotopy types. This ties in with the observation that taking group objects either in (Cat), or in the full subcategory (Groupoids) of the latter, amounts to the same – and similarly surely for (n—Cat); on the other hand it has been kind of clear from the very beginning of this reflection that at any *finite* level, groupoids and n-groupoids (n finite) will only yield *truncated* homotopy types.

A related intriguing question is when exactly does a modelizer (M, W) give rise to a *Dold-Puppe theorem* – namely when do we get an actual *equivalence* between the category of *abelian* group objects of M, and the category of chain complexes of abelian groups? The original statement was in case $M = \mathbb{\Delta}^\wedge =$ semisimplicial complexes, and doubtlessly it was one main impetus for the sudden invasion of homotopy and cohomology by semisimplicial calculus – so much so it seems that for many people, "homotopy" has become synonymous to "semisimplicial algebra". The impression that semi-simplicial complexes is the God-given ground for doing homotopy and even cohomology, comes out rather strong also in Quillen's foundational notes, and in Illusie's thesis. Still, there are too some cubical theory chaps I heard, who surely must have noticed long ago that the Dold-Puppe theorem is valid equally for cubical complexes (I could hardly imagine that it possibly couldn't). Now it turns out that semisimplicial and cubical complexes are part of a trilogy, together with so-called "*hemispherical complexes*", which look at lot simpler still, with just two boundary operators and one degeneracy in each dimension. [I guess today we call these "globular sets"] They can be viewed as embodying the "primitive structure" of an ∞-groupoid, the boundary operators being the "source" and "target" maps, and the degeneracy the map associating to any i-object the corresponding "identity". I hit upon this structure among the first examples of test categories and elementary modelizers, and have been told since by Ronnie Brown that he has already known for a while these models, under the similar name of "globular complexes"). Roughly speaking, [p. 209] it can be said that the three types of complexes correspond to three "series" of *regular cellular subdivisions* of all spheres S_n where the two-dimensional pieces are respectively (by increasing order of "intricacy") bigons, triangles, and squares. It shouldn't be hard to show these are the only series of regular cellular subdivisions of all spheres (one in each dimension) such that for any cell of such subdivision, the induced subdivision of the bounding sphere should still be in the series (up to isomorphism). The existence moreover of suitable "degeneracy" maps, which merit a careful general definition in this context of cellular subdivisions of spheres, is an important common extra feature of the three basic contexts, whose exact significance I have not quite understood still. To come back to Dold-Puppe, sure enough it is still valid in the hemispherical context. Writing down the equivalence of categories in explicit terms comes out with baffling simplicity. I wrote it down without even looking for it, in a letter to Ronnie Brown, while explaining in a PS the "yoga" of associating to a chain complex of abelian groups an ∞-groupoid with additive structure (a so-called "Picard category" but within the context of ∞-categories or ∞-groupoids, rather than usual categories).

On the other hand, taking multicomplexes instead of simple ones, which still can be interpreted as working in a category $A^\wedge$ for a suitable test category A (namely, a product category of categories of the types ◫, ◬, and ◎), it is clear that Dold-Puppe's theorem as originally stated is no longer true in these: in such case the category of abelian group objects of $A^\wedge$ is equivalent to a category of *multiple* chain complexes. This shows that definitely, among all possible elementary modelizers, the three in our trilogy are distinguished indeed, as giving rise to a Dold-Puppe theorem. A thorough understanding of this theorem would imply, I feel, an understanding of which exactly are the modelizers, or elementary modelizers at any rate giving rise to such a theorem. I wouldn't be too surprised if it turned out that the three we got are the only ones, up to equivalence.

Another common feature of these modelizers, is that they allow for a sweeping computational description of cohomology (or homology) invariants in terms of the so-called "boundary operations". This is visibly connected to the (strongly intuitive) tie between these kinds of models, and cellular subdivisions of spheres. However, at this level (unlike the Dold-Puppe story) the regularity feature of the cellular subdivisions we got, and the fact that we allow for just one (up to isomorphism) in each dimension, seems to be irrelevant. It might be worth while [p. 210] to write down with care what exactly is needed, in order to define, in terms of a bunch of cellular structures of spheres, a corresponding test-category (hopefully even a strict one), and a way of computing in terms of boundary operators the cohomology of the corresponding models.

The more delicate point here may be that if we really want to get an actual test category, not just a weak one, we should have "enough" degeneracy maps between our cellular structures, which might well prove an extremely stringent requirement. Again, it will be interesting if we can meet it otherwise than just sticking to our trilogy.

D) "Back to topoi". They have been my main intuitive leading thread in the reflections so far, but have remained somewhat implicit most times. When working with categories X as models for homotopy types, we have been thinking in reality of the associated topos $X^\wedge$. In the same way, when relativizing over an object I of (Cat) the construction of (Hot) as a derived category, namely working with categories over I as models for a derived category $D(I)$, the leading intuition again has been to look at X over I as the topos $X^\wedge$ over $I^\wedge$. This gives strong suggestions as to defining $D(I)$ not only in terms of a given category I, but also in terms of an arbitrary topos T as well, standing for $I^\wedge$, and look to what extent the $f^*, f_!, f^*$ formalism of derivators extends to the case when f is a "map" between topoi. The definition of some $D(T)$ for a topos T can be given in a number of ways it seems. As far as I know, the only one which has been written and used so far consists in taking a suitable derived category of the category of semisimplicial objects of T. this is done in Illusie's thesis, where there is no mention though of $f_!$ and f_* functors – which, maybe, should rather be written $\mathrm{L}f_!$ and $\mathrm{R}f_*$, suggesting they are respectively left and right derived functors of the more familiar $f_!$ and f_* functors for sheaves (the left and right adjoints of the inverse image functor f^* for sheaves of sets). One would expect, at best, $\mathrm{L}f_1$ to exist when $f_!$ itself does, namely when f^* commutes not only with finite inverse limits, but with infinite products as well. As for $\mathrm{R}f_*$, whereas there is no problem for the existence of f_* itself, already in the case of a morphism of topoi coming from a map in (Cat), a map say between finite ordered sets, the existence of $\mathrm{R}f_*$ has still to be established. Thus, presumably I'll content myself with writing down and comparing a few tentative definitions of $D(T)$ and make some reasonable guesses as to its variances. Intuitively, the objects of $D(T)$ may be viewed as "sheaves of homotopy types over T", or "relative homotopy types over T", or "non-commutative chain complexes over T up to quasi-isomorphism".
[p. 211] As in the case when T is the final (one point) topos, $D(T)$ is just the homotopy category (Hot), there must be of course a vast variety of ways of defining $D(T)$ in terms of model categories, and I would like to review some which seem significant. In fact, one cannot help but looking at two mutually dual groups of models categories, giving rise to (at least) two definitely non-equivalent derived categories $D(T)$ and $D'(T)$ say, which, in case $T = I^\wedge$, would correspond to $D(I)$ and $D(I^{\mathrm{op}})$. A typical model category for the former is made up with Illusie's semisimplicial sheaves; a typical model category for $D'(T)$, on the other hand, should be made up with 1-stacks on T (in Giraud's sense), for a suitable notion of weak equivalence between these.

Maybe the most natural models of all, in this context, for "relative homotopy types over the topos T", should be *topoi* X over T (generalizing the categories over an "indexing category" I). The only trouble with this point of view though is that the best we can hope for, in term's of Illusie's $D(T)$ say, is that a topos X over T gives rise to a pro-object of $D(T)$, which needs not come from an object of $D(T)$ itself, i.e., is not necessarily "essentially constant". This brings us back to the simpler and still more basic question of associating a pro-homotopy type, namely a pro-object of (Hot), to any topos X – namely back to the Čech-Verdier-Artin-Mazur construction. This has been handled so far using the semisimplicial models for (Hot), I suspect though that using (Cat) as a modelizer will give a more elegant treatment, as already suggested earlier. Whichever way we choose to get the basic functor

$$\text{Topoi} \to \text{Pro}\,(\text{Hot}),$$

this functor will allow us, given a basic localizer $\mathcal{W}$, to define $\mathcal{W}$-equivalences between topoi as maps which become isomorphisms under the composition of the basic functor above, and the canonical functor

$$\text{Pro}\,(\text{Hot}) \to \text{Pro}(\text{Hot}(\mathcal{W}))$$

deduced from the localization functor

$$(\text{Hot}) \to (\text{Hot}(\mathcal{W})).$$

(It turns out, using Quillen's theory, that usual weak equivalences is indeed the finest of all possible basic localizers $\mathcal{W}$, hence Hot($\mathcal{W}$) is indeed a localization of (Hot).)

Maybe it is not too unreasonable to expect that all, or most homotopy constructions, involving a topos T, can be expressed replacing T by its image in Pro (Hot), or in Pro(Hot($\mathcal{W}$)) if the construction are relative to a given basic localizer $\mathcal{W}$. The very first example one would like to look up in this respect, is surely $D(T)$, defined say à la Illusie. [p. 212]

In the present context, the main point of the property of *local asphericity* for a topos (cf. section 35) is that the corresponding prohomotopy type is essentially constant, i.e., the topos defines an actual homotopy type. Thus, locally aspheric topoi and weak equivalences between these should be eligible models for homotopy types (more accurately, make up a modelizer), just as the basic modelizer (Cat) contained in it. The corresponding statements, when introducing a basic localizer $\mathcal{W}$, should be equally valid. One might expect, too, a relative variant for the notion of local asphericity or $\mathcal{W}$-asphericity, in case of a topos X over a given base topos T (which we may have to suppose already locally aspheric), with the implication that the corresponding object of Pro($D(T)$) should be again essentially constant.

A last question I would like to mention here is about the meaning of the notion of a so-called "modelizing topos", introduced in a somewhat formal way in section 35, as a locally aspheric and aspheric topos T such that the Lawvere element L_T of T is aspheric over the final object. (We assumed at first, moreover, that T be even totally aspheric, but soon after the point of view and terminology shifted a little and the totally aspheric case was referred to as a *strictly* modelizing topos, cf. page 68.

As made clear there, the expectation suggested by the terminology is of course that such a topos should indeed be a modelizer, when endowed with the usual notion of weak equivalence. This sill makes sense and seem plausible enough, when usual weak equivalence is replaced by the weaker notion defined in terms of an arbitrary basic localizer $\mathcal{W}$. A related question is to get a feeling for how restrictive the conditions put on a modelizing topos are. It is clear that nearly all topoi met with in practice, including those associated to the more common topological spaces (such as locally contractible ones) are locally aspheric – but what about the condition on the Lawvere element? For instance, taking a topological space admitting a finite triangulation and which is aspheric, i.e., contractible, is the corresponding topos modelizing?

[p. 213] 10.6.

Back to asphericity: criteria for a map in (Cat).

72 In the long last, we'll come back now to the "asphericity game"! Let's take up the exposition at the point where we stopped two months ago (section 67, p. 188). We were then about to reformulate in various ways the property of asphericity, more specifically $\mathcal{W}$-asphericity, for a given map

$$i : A \to B$$

in (Cat). To this end, we introduced the corresponding diagram of maps in (Cat) with "commutation morphism" λ_i:

$$\begin{tikzcd} B^\wedge \arrow[rr, "i^*"] \arrow[dr, "i_B"'] & \arrow[d, phantom, "\overset{\lambda_i}{\Leftarrow}"] & A^\wedge \arrow[dl, "i_A"] \\ & (\mathrm{Cat}) & \end{tikzcd} \quad .$$

As already stated, the asphericity condition on i can be expressed in a variety of ways, as a condition on either of the three "aspects"

$$\lambda_i, \quad i^*, \quad i_A i^*$$

of the situation created by i, with respect to the localizing sets $\mathcal{W}$, $\mathcal{W}_A$, $\mathcal{W}_B$ in the three categories under consideration, or with respect to the notion of $\mathcal{W}$-aspheric objects in these. As $\mathcal{W}_A$, and aspheric objects in $A^\wedge$, are defined respectively in terms of $\mathcal{W}$ and aspheric objects in (Cat) via the functor i_A, it turns out that the formulations in terms of properties of $i_A i^*$ reduce trivially to the corresponding formulations in terms of i^*, which will be the most directly useful for our purpose for later work – hence well's omit them, and focus attention instead on λ_i and i^*. Notations are those of loc. sit., in particular, we are working with the categories

$$\mathrm{Hot}_A^{\mathcal{W}} \quad \text{or} \quad \mathrm{Hot}_A \overset{\text{def}}{=} \mathcal{W}_A{}^{-1} A^\wedge$$

and

$$\mathrm{Hot}(\mathcal{W}) = \mathcal{W}^{-1}(\mathrm{Cat}),$$

where $\mathcal{W}$ is a given "basic localizer", namely a set of arrows in (Cat) satisfying certain conditions.

It seems worthwhile here to be careful to state which exactly are the properties of $\mathcal{W}$ we are going to use in the "asphericity game" – therefore I'm going to list them again here, using a labelling which, hopefully, will not have to be changed again:

Loc 1) "Mild saturation" (cf. page 59),

Loc 2) "Homotopy condition": $\Delta_1 \times X \to X$ is in $\mathcal{W}$ for any X in (Cat),

Loc 3) "Localization condition": If X, Y are objects in (Cat) over an object A, and $u : X \to Y$ an A-morphism such that for any a in A, the induced $u_{/a} : X_{/a} \to Y_{/a}$ is in $\mathcal{W}$, then so is u.

[p. 214]

[AG seems to have replaced A with $\mathcal{S}$ here, but it's not clear…]

It should be noted that the "mild saturation" condition is slightly weaker than the saturation condition introduced later (p. 101, conditions a')b')c')), namely in condition c') (if $f : X \to Y$ and $g : Y \to X$ are such that $gf, fg \in \mathcal{W}$, then $f, g \in \mathcal{W}$) we restrict to the case when gf is the identity, namely f an inclusion and g a retraction upon the corresponding subobject. On the other hand, in what follows we are going to use Loc 3) only in case $Y = A$ and $Y \to A$ is the identity. I stated the condition in greater generality than needed for the time being, in view of later convenience – as later it will have to be used in full strength. It is not clear whether the weaker form of Loc 3) (plus of course Loc 1) and Loc 2)) implies already the stronger. We'll see later a number of nice further properties of $\mathcal{W}$ implied by these we are going to work with for the time being – including *strong* saturation of $\mathcal{W}$, namely that $\mathcal{W}$ is the set of arrows made invertible by the localization functor

$$(\mathrm{Cat}) \to \mathcal{W}^{-1}(\mathrm{Cat}) = \mathrm{Hot}(\mathcal{W}).$$

Proposition 1. *Let as above $i : A \to B$ be a map in* (Cat). *Consider the following conditions on i:*

(i) *For any F in $B^\wedge$, $\lambda_i(F) : A_{/F} \to B_{/F}$ is in $\mathcal{W}$.*

(i') *For any F as above, $\lambda_i(F)$ is $\mathcal{W}$-aspheric (i.e., satisfies the assumption on u in Loc 3) above, when $Y \to A$ is an identity).*

(i") *Same as (i), but restricting to $F = b$ in B.*

(ii) *For any F in $B^\wedge$, F $\mathcal{W}$-aspheric $\Rightarrow i^*(F)$ $\mathcal{W}$-aspheric.*

(ii') *For any F in $B^\wedge$, F $\mathcal{W}$-aspheric $\Longleftrightarrow i^*(F)$ $\mathcal{W}$-aspheric.*

(iii) *For any b in B, $A_{/b} (\overset{\mathrm{def}}{=} A_{/i^*(b)})$ is $\mathcal{W}$-aspheric, i.e., i is $\mathcal{W}$-aspheric.*

(iv) *For any map f in $B^\wedge$, $f \in \mathcal{W}_B \Rightarrow i^*(f) \in \mathcal{W}_A$.*

(iv') *For any map f as above, $f \in \mathcal{W}_B \Longleftrightarrow i^*(f) \in \mathcal{W}_A$.*

(v) *Condition (iv) holds, i.e., i^* induces a functor*

$$\overline{i^*} : \mathrm{Hot}_B \to \mathrm{Hot}_A,$$

and moreover the latter is an equivalence.

The conditions (i) up to (iii) are all equivalent, call this set of conditions (As) ($\mathcal{W}$*-asphericity*)*. We moreover have the following implications between (As) and the remaining conditions (iv) to (v):*

$$
(*)\qquad
\begin{tikzcd}[arrows=Rightarrow]
\text{(As)} \arrow[rr, dashed, "if\ {}_{A,B}\ ps.test"] \arrow[dr] & & \text{(v)} \arrow[ddl] \\
& \text{(iv')} \arrow[d] & \\
& \text{(iv)} \arrow[uul, dashed, bend left, "if\ {}_{A,B}\ \mathcal{W}\text{-}asph."] &
\end{tikzcd}
,
$$

[p. 215] *where the implication (iv) $\Rightarrow$ (As) is subject to A, B begin $\mathcal{W}$-aspheric, and (As) $\Rightarrow$ (v) to A, B being "pseudo-test categories" (for $\mathcal{W}$), namely the canonical functors*

$$\overline{i_A} : \mathrm{Hot}_A \to \mathrm{Hot}(\mathcal{W}), \quad \overline{i_B} : \mathrm{Hot}_B \to \mathrm{Hot}(\mathcal{W})$$

being equivalence.

Corollary. *Assume that A and B are pseudo-test categories, and are $\mathcal{W}$-aspheric. Then all conditions (i) to (v) of the proposition above are equivalent.*

Remark 1. If we admit strong saturation of $\mathcal{W}$ (which will be proved later), it follows at once that a pseudo-test category is necessarily $\mathcal{W}$-aspheric – hence the conclusion of the corollary holds assuming only A and B are pseudo-test categories. Of course, it holds a fortiori if A and B are weak test categories, or even test categories. Also, strong saturation implies that in (*) above, we have even the implication: (v) $\Rightarrow$ (iv'), stronger than (v) $\Rightarrow$ (iv).

Proof of proposition. It is purely formal – for the first part, it follows from the diagram of tautological implications

$$
\begin{tikzcd}[arrows=Rightarrow]
& \text{(i')} \arrow[d] & & \\
& \text{(i)} \arrow[dl] \arrow[dr] & & \\
\text{(ii')} \arrow[d] & & \text{(i'')} \arrow[d, Leftrightarrow] & \\
\text{(ii)} \arrow[rr] & & \text{(iii)} \arrow[r] & \text{(i')}
\end{tikzcd}
,
$$

where the implication (i') $\Rightarrow$ (i) is contained in the assumption Loc 3 on $\mathcal{W}$. The implications of the diagram (*) are about as formal – there is no point I guess writing it out here.

Remark 2. Using the canonical functors

$$\varphi_A : A^\wedge \to \mathrm{Hot}(\mathcal{W}), \quad \varphi_B : B^\wedge \to \mathrm{Hot}(\mathcal{W}),$$

there are two other amusing versions still of (i), which look a lot weaker still, and are equivalent however to (i), i.e., to $\mathcal{W}$-asphericity of i, namely:

(vi) For any F in $B^\wedge$, $\varphi_B(F) = \gamma(B_{/F})$ and $\varphi_A(i^*(F)) = \gamma(A_{/F})$ are isomorphic objects of $\mathrm{Hot}(\mathcal{W})$.

(vi') Same as (vi), with F restricted to be object b in B. [p. 216]

Indeed, we have of course (i) $\Rightarrow$ (vi) $\Rightarrow$ (vi'), but also (vi') $\Rightarrow$ (iii) if we admit strong saturation of $\mathcal{W}$, which implies that an object X of (Cat) is $\mathcal{W}$-aspheric iff its image in $\mathrm{Hot}(\mathcal{W})$ is a final object.

Remark 3. If we don't assume A and B to be $\mathcal{W}$-aspheric, the implications

$$\text{(As)} \Rightarrow \text{(iv')} \Rightarrow \text{(iv)}$$

are both strict. As an illustration of this point, take for B a discrete category (defined in terms of a set of indices $I = \mathrm{Ob}\,B$), thus a category A over B is essentially the same as a family $(A_b)_{b\in I}$ of objects of (Cat) indexed by I. In terms of this family, we see at once that the three conditions above on $i : A \to B$ mean respectively (a) that all categories A_b are $\mathcal{W}$-aspheric, (b) that all categories A_b are non-empty, and (c) condition vacuous. This example brings to mind that *condition (iv) is a Serre-fibration type condition*, we'll come back upon this condition when studying homotopy properties of maps in (Cat), with special emphasis on base change questions. Likewise, condition (iv') appears as a strengthening of such Serre-type condition, to the effect that the restriction of A over any connected component of B should be moreover non-empty. As an example, we may take the projection $B \times C \to C$, where C is any non-empty object in (Cat).

I would like now to dwell still a little on the case, of special interest of course for the modelizing story, when A and B are test categories or something of the kind. The formulation (i) of the asphericity condition for the functor i can be expressed by stating that the functor $\overline{i^*}$ between the localizations Hot_B and Hot_A exists, and gives rise (via λ_i) to a commutation morphism which is an *isomorphism*

$$\begin{tikzcd} \mathrm{Hot}_B \arrow[rr, "\overline{i^*}"] \arrow[dr, "\overline{i_B}"'] & \arrow[d, phantom, "\overline{\lambda_i}\ \sim"] & \mathrm{Hot}_A \arrow[dl, "\overline{i_A}"] \\ & (\mathrm{Cat}) & \end{tikzcd}.$$

This shows, when A and B are pseudo-test categories for $\mathcal{W}$, i.e., the functors $\overline{i_A}$ and $\overline{i_B}$ are equivalences, that the functor $\overline{i^*}$ deduced from the $\mathcal{W}$-aspheric map $i : A \to B$, *does not depend* (up to canonical isomorphism) on the choice of i, and can be described as the composition of $\overline{i_B}$ followed by a quasi-inverse for $\overline{i_A}$. This of course is very much in keeping

with the “inspiring assumption” (section 28), which just means that up
[p. 217] to unique isomorphism, there is indeed but *one* equivalence from Hot_B to Hot_A (both categories being equivalent to $\mathrm{Hot}(\mathcal{W})$). Here we are thinking of course of the extension of the “assumption” contemplated earlier, when usual weak equivalence is replaced by a basic localizer $\mathcal{W}$ as above. It seems plausible that the assumption holds true in all cases – anyhow we didn't have at any moment to make explicit use of it (besides at a moment drawing inspiration from it...).

Proposition 2. *Let $i : A \to B$ be a map in* (Cat).

a) *If i is $\mathcal{W}$-aspheric, and $A^\wedge$ is totally $\mathcal{W}$-aspheric, then $B^\wedge$ is totally $\mathcal{W}$-aspheric too.*

b) *Assume A $\mathcal{W}$-aspheric, and that B is a $\mathcal{W}$-test category, admitting the separating $\mathcal{W}_B$-homotopy interval $\mathbb{I} = (I, \delta_0, \delta_1)$, satisfying the homotopy condition (T H 1) of page 50. Consider the following conditions:*

   1) *$i^*(I)$ is $\mathcal{W}$-aspheric over $e_{A^\wedge}$,*
   2) *i is $\mathcal{W}$-aspheric,*
   3) *$i^*(I)$ is $\mathcal{W}$-aspheric.*

   We have the implications

$$1) \Rightarrow 2) \Rightarrow 3),$$

   hence, if A is totally $\mathcal{W}$-aspheric (*hence* 3) $\Rightarrow$ 1)), *all three conditions are equivalent.*

Proof. a) We have to prove that if b, b' are in B, then their product in $B^\wedge$ is $\mathcal{W}$-aspheric, but by assumption on i we know that the images of b, b' by i^* are $\mathcal{W}$-aspheric, hence (as $A^\wedge$ is totally aspheric) their product

$$i^*(b) \times i^*(b') = i^*(b \times b')$$

is $\mathcal{W}$-aspheric too, hence so is $b \times b'$ by criterion (ii') of prop. above.

b) The homotopy condition (T H 1) referred to means that all objects of B are $\mathbb{I}$-contractible. As i^* commutes with finite products, it follows that the objects $i^*(b)$ of $A^\wedge$, for b in B, are $i^*(\mathbb{I})$-contractible. When $i^*(I)$ is $\mathcal{W}$-aspheric over $e_{A^\wedge}$, this implies that so are the objects $i^*(b)$, a fortiori they are $\mathcal{W}$-aspheric (as by assumption $e_{A^\wedge}$ is $\mathcal{W}$-aspheric). Thus 1) $\Rightarrow$ 2), and 2) $\Rightarrow$ 3) is trivial.

Corollary. *Let $i : A \to B$ be a $\mathcal{W}$-aspheric map in* (Cat)*, assume that A is totally $\mathcal{W}$-aspheric, and B is a local $\mathcal{W}$-test category. Then both A and B are strict $\mathcal{W}$-test categories.*

Indeed, by a) above we see that B is totally $\mathcal{W}$-aspheric, hence B is a strict $\mathcal{W}$-test category. In order to prove that so is A, we only have to show
[p. 218] that $A^\wedge$ admits a separating homotopy interval for $\mathcal{W}_A$. By assumption on B, there is a separating $\mathcal{W}_B$-homotopy interval $\mathbb{I} = (I, \delta_0, \delta_1)$ in $B^\wedge$. The exactness properties of i^* imply that $i^*(\mathbb{I})$ is a separating interval in $A^\wedge$,

the asphericity condition on i implies that moreover $i^*(I)$ is $\mathcal{W}$-aspheric, hence $\mathcal{W}$-aspheric over $e_{A\hat{}}$ as $A\hat{}$ is totally $\mathcal{W}$-aspheric, qed.

To finish with the more formal properties of the notion of $\mathcal{W}$-aspheric maps in (Cat), let's give a list of the standard stability conditions for this notion, with respect notably to composition, base change, and cartesian products:

Proposition 3. a) *Consider two maps*

$$A \xrightarrow{i} B \xrightarrow{j} C$$

in (Cat). *Then if i and j are $\mathcal{W}$-aspheric, so is ji. If ji and i are $\mathcal{W}$-aspheric, so is j. Any isomorphism in* (Cat) *is $\mathcal{W}$-aspheric.*

b) *Let*

$$\begin{tikzcd} A \arrow[d, "i"'] & A' \arrow[l, "f"'] \arrow[d, "i'"] \\ B & B' \arrow[l, "g"'] \end{tikzcd}$$

be a cartesian square in (Cat)*, assume i is $\mathcal{W}$-aspheric and g is fibering* (*for instance an induction functor $B_{/F} \to B$, with F in $B\hat{}$*)*. Then i' is $\mathcal{W}$-aspheric* (*and, of course, f is equally fibering*).

c) *Let*

$$i : A \to B, \quad i' : A' \to B'$$

be two $\mathcal{W}$-aspheric maps in (Cat)*, then*

$$i \times i' : A \times A' \to B \times B'$$

is $\mathcal{W}$-aspheric.

Proof. Property a) is formal, in terms of criterion (ii') of prop. 1. Property c) follows formally from the criterion (iii), and the canonical isomorphism

$$(A \times A')_{/b \times b'} \simeq (A_{/b}) \times (A'_{/b'}),$$

and the fact that a product of two $\mathcal{W}$-aspheric objects of (Cat) is again $\mathcal{W}$-aspheric (cf. prop. of page 167, making use of the localization condition Loc 3) on $\mathcal{W}$ in its full generality). Another proof goes via a) and b), by reducing first (using a)) to the case when either i or i' are identities, and using the fact that a projection map $C \times B \to B$ in (Cat) is a fibration.

We are left with proving property b). For this, we note that the asphericity condition (iii) on a map $i : A \to B$ just means that for any base change of the type [p. 219]

$$B_{/b} \to B,$$

where b is in B, the corresponding map

$$i_{/b} : A \times_B B_{/b} \simeq A_{/b} \to B_{/b}$$

is in $\mathcal{W}$. Applying this to the case of $i' : A' \to B'$, and an object b' in B', and denoting by b its image in B, using transitivity of base change, we get a cartesian square

$$\begin{tikzcd} A_{/b} \arrow[d, "i_{/b}"] & A'_{/b'} \arrow[l] \arrow[d, "i'_{/b'}"] \\ B_{/b} & B'_{/b'} \arrow[l] \end{tikzcd} ,$$

and we got to prove $i'_{/b'}$ is in $\mathcal{W}$, i.e., $A'_{/b'}$ is $\mathcal{W}$-aspheric, using the fact that we know the same holds for $i_{/b}$, i.e., $A_{/b}$ is $\mathcal{W}$-aspheric. Thus, all we have to prove is that the first horizontal arrow is in $\mathcal{W}$. But we check at once that the condition that $B' \to B$ is fibering implies that the induced functor

$$B'_{/b'} \to B_{/b}$$

is fibering too, and moreover has fibers which have final objects. Hence by base change, the same properties hold for

$$A'_{/b'} \to A_{/b}.$$

This reminds us of the "fibration condition" L 5 (page 164), which should ensure that a fibration with $\mathcal{W}$-aspheric fibers is in $\mathcal{W}$. We did not include this axiom among the assumptions (recalled above) we want to make on $\mathcal{W}$. However, it turns out that the assumptions we do make here imply already the fibration condition, as well as the dual condition on cofibrations. We'll give a proof later – in order not to diverge at present from our main purpose. There will not be any vicious circle, as all we're going to use of prop. 3 for the formalism of asphericity structures and canonical modelizers is the first part a). I included b) and b) for the sake of completeness, and because c) is useful for dealing with products of two categories, notably of two test categories – a theme which has been long pending, and on which I would like to digress next, before getting involved with asphericity structures.

Remark 4. In part a) of the proposition, if j and ji are $\mathcal{W}$-aspheric, we cannot conclude that i is. If C is the final object of (Cat), this means that a map between $\mathcal{W}$-aspheric objects in (Cat) need not be $\mathcal{W}$-aspheric.

[p. 220] 11.6.

73 *Asphericity criteria (continued).*

I am not quite through yet with generalities on asphericity criteria for a map in (Cat), it turns out – it was just getting prohibitively late last night to go on!

From now on, I'll drop the qualifying $\mathcal{W}$ when speaking of asphericity, test categories, modelizers and the like, as by now it is well understood, I guess, there is a given $\mathcal{W}$ around in all we are doing. It'll be enough to be specific in those (presumably rare) instances when working with more than one basic localizer.

Coming back to the last remark in yesterday's notes, a good illustration is the case of a functor $i : A \to B$ of aspheric objects of (Cat), when A is a final object in (Cat), i.e., a one-point discrete category. Then *i is aspheric iff $i(a)$ is an initial object of B* – an extremely stringent extra condition indeed!

In part a) of proposition 3 above, stating that isomorphisms in (Cat) are aspheric, it would have been timely to be more generous – indeed, *any equivalence of categories in* (Cat) *is aspheric*. This associates immediately with a map of topoi which is an equivalence being (trivially so) aspheric (in the case, say, when $\mathcal{W}$ is the usual notion of weak equivalence, the only one for the time being when the notion is extended from maps in (Cat) to maps between topoi). In the context of (Cat), the basic modelizer, we can give a still more general case of aspheric maps, both instructive and useful:

Proposition 4. *Let*

$$A \underset{g}{\overset{f}{\rightleftarrows}} BB$$

be a pair of adjoint functors between the objects A, B in (Cat)*, with f left and g right adjoint. Then f is aspheric.*

Indeed, by the adjunction formula we immediately get

$$A_{/b} \simeq A_{/g(b)}$$

(as a matter of fact, $f^*(b) = g(b)$), hence this category has a final object and hence is aspheric, qed.

Remark 5. The conclusion of prop. 4 is mute about g, which will rightly strike as unfair. Dualizing, we could say that $(g^{\mathrm{op}}, f^{\mathrm{op}})$ is a pair of adjoint functors between A^{op} and B^{op}, and therefore g^{op} is aspheric. We will express this fact (by lack of a more suggestive name) by saying that g is a *coaspheric* map. For a functor $i : A \to B$, in terms of the usual criterion for i^{op}, it just means that for any b in B, the category [p. 221]

$$_{b\backslash}A \overset{\text{def}}{=} A \times_B ({}_{b\backslash}B) = \text{category of pairs } (x, p), \text{ with } x \text{ in } A$$
$$\text{and } p : b \to i(x)$$

is aspheric. (NB In the case of the functor g from B to A, the corresponding categories $_{a\backslash}B$ even have *initial objects*.) This conditions comes in here rather formally, we'll see later though that it has a quite remarkable interpretation, in terms of a very strong property of cohomological "cofinality" of the functor i, implying the usual notion of i being a "cofinal" functor, or A being "cofinal" in B (namely i giving rise to isomorphisms $\varinjlim_A \to \varinjlim_B$ for any direct system $B \to M$ with values in a category M admitting direct limits), as the "dimension zero" shadow of this "all dimensions" property. To make an analogy which will acquire more precise meaning later, the asphericity property for a map i in (Cat) can be viewed as a (slightly weakened) version of a *proper map with aspheric fibers*, whereas the coasphericity property appears as the corresponding version of a *smooth map with aspheric fibers*. The qualification "slightly weakened" reflects notably in the fact that the notions of asphericity and coasphericity are *not* stable under arbitrary base change – but rather, asphericity for a map is stable under base change by *fibration functors* (more generally, by smooth maps in (Cat)), whereas coasphericity is stable under base change by *cofibration functors* (more generally, by proper maps in (Cat)). Thus, whereas a functor $i : A \to B$ which is an equivalence of categories is clearly both aspheric and coaspheric, this property is not preserved by arbitrary base change, e.g., passage to fibers: e.g., some fibers may be empty, and therefore are neither aspheric nor coaspheric!

The most comprehensive property for a map in (Cat), implying both asphericity and coasphericity, is to be in UW, i.e., a "*universal weak equivalence*" – namely it is in $\mathcal{W}$ and remains so after any base change. These maps deserve the name of "*trivial Serre fibrations*". They include all maps with aspheric fibers which are either "proper" or "smooth", for instance those which are either cofibering or fibering functors (in the usual sense of category theory). We'll come back upon these notions in a systematic way in the next part of the reflections.

74 *Application to products of test categories.*

[p. 222]

As announced yesterday, I would like still to make an overdue digression on products of test categories, before embarking on the notion of asphericity structure. It will be useful to begin with some generalities on presheaves on a product category $A \times B$, where for the time begin A, B are any two objects of (Cat). The following notation is often useful, for a pair of presheaves

$$F \in \mathrm{Ob}\, A^\wedge, \quad G \in \mathrm{Ob}\, B^\wedge,$$

introducing an "*external product*"

$$F \boxtimes G \in \mathrm{Ob}(A \times B)^\wedge$$

by the formula

$$F \boxtimes G(a, b) = F(a) \times G(b).$$

Introducing the two projections

$$p_1 : A \times B \to A, \quad p_2 : A \times B \to B,$$

and the corresponding inverse image functors p_i^*, we have

$$F \boxtimes G = p_1^*(F) \times p_2^*(G).$$

Of course, $F \boxtimes G$ depends bifunctorially on the pair (F, G), and it is easily checked, by the way, that the corresponding functor

$$A^\wedge \times B^\wedge \to (A \times B)^\wedge, \quad (F, G) \mapsto F \boxtimes G,$$

is fully faithful.

We have a tautological canonical isomorphism

$$(A \times B)_{/F \boxtimes G} \xrightarrow{\sim} A_{/F} \times B_{/G},$$

and hence the

Proposition 1. a) *If F and G are aspheric objects in $A^\wedge$ and $B^\wedge$ respectively, then $F \boxtimes G$ is an aspheric object of $(A \times B)^\wedge$.*

b) *If $u : F \to F'$ and $v : G \to G'$ are aspheric maps in $A^\wedge$ and $B^\wedge$ respectively, then*

$$u \boxtimes v : F \boxtimes G \to F' \boxtimes G'$$

is an aspheric map in $(A \times B)^\wedge$.

Indeed, this follows respectively from the fact that a product of two aspheric objects (resp. aspheric maps) in (Cat) is again aspheric.

Corollary 1. *Let F in $A^\wedge$ be aspheric over the final object $e_{A^\wedge}$, then $F \boxtimes e_{B^\wedge} = p_1^*(F)$ is aspheric over the final object of $(A \times B)^\wedge$.*

Corollary 2. *Assume A and B are totally aspheric, then so is $A \times B$.*

We have to check that the product elements

$$(a, b) \times (a', b') = (a \times a') \boxtimes (b \times b')$$

are aspheric, which follows from the assumption (namely $a \times a'$ and $b \times b'$ aspheric) and part a) of the proposition.

Assume now A is a local test category, i.e., $A^\wedge$ admits a separating interval [p. 223]

$$\mathbb{I} = (I, \delta_0, \delta_1)$$

such that I is aspheric over the final object of $A^\wedge$. Then $p_1^*(\mathbb{I})$ is of course a separating interval in $(A \times B)^\wedge$, which by cor. 1 is aspheric over the final object. Hence

Proposition 2. *If A is a local test category, so is $A \times B$ for any B in* (Cat)*. If A is a test category, then so is $A \times B$ for any aspheric B.*

The second assertion follows, remembering that a test category is just an *aspheric* local test category. Using cor. 2, we get:

Corollary. *If A is a strict test category, and B totally aspheric, then $A \times B$ is a strict test category. In particular, if A and B are strict test categories, so is their product.*

We need only remember that strict test categories are just test categories that are totally aspheric.

As an illustration, we get the fact that the categories of multicomplexes of various kinds, which we can even take mixed (semisimplicial in some variables, cubical in others, and hemispherical say in others still) are strict modelizers (as generally granted), which corresponds to the fact that a finite product of standard semisimplicial, cubical and hemispherical (strict) test categories $\mathbb{\Delta}$, $\mathbb{\Box}$ and $\mathbb{O}$, is again a strict test category.

I would like now to dwell a little upon the comparison of "homotopy models", using respectively two test categories A and B, namely working in $A^\wedge$ and $B^\wedge$ respectively. More specifically, we have two description of Hot (short for Hot($\mathcal{W}$) here), namely as

$$\mathrm{Hot}_A = {\mathcal{W}_A}^{-1}A^\wedge \quad \text{and} \quad \mathrm{Hot}_B = {\mathcal{W}_A}^{-1}B^\wedge,$$

and we want to describe conveniently the tautological equivalence between these two categories (this equivalence being defined up to unique isomorphism). The most "tautological" way indeed is to use the basic modelizer (Cat) and its localization (Hot) as the intermediary, i.e., using the diagram of equivalences of categories

(1)
$$\begin{tikzcd} \mathrm{Hot}_A \arrow[dr, "\approx", "\overline{i_A}"'] & & \mathrm{Hot}_B \arrow[dl, "\approx"', "\overline{i_B}"] \\ & (\mathrm{Hot}) & \end{tikzcd} .$$

[p. 224] Remember we have a handy quasi-inverse $\overline{j_B}$ to $\overline{i_B}$, using the canonical functor

[In this display, AG originally put the definition of i_B instead j_B, cf., e.g., (1) and (2) in §65 to recall the definitions.]

$$j_B = i_B^* : (\mathrm{Cat}) \to B^\wedge, \quad X \mapsto (b \mapsto \mathrm{Hom}(B_{/b}, X)).$$

Thus, we get a description of an equivalence

(1') $$\overline{j_B}\,\overline{i_A} : \mathrm{Hot}_A \xrightarrow{\approx} \mathrm{Hot}_B,$$

whose quasi-inverse of course is just the similar $\overline{j_A}\,\overline{i_B}$. We can vary a little this description, admittedly cumbersome in practice, by replacing $\overline{j_B}$ by the isomorphic functor $\overline{i^*}$, where $i : B \to (\mathrm{Cat})$ is any test functor from B to (Cat) (while we have to keep however $\overline{i_A}$ as it is, without the possibility of replacing i_A by a more amenable test functor).

Another way for comparing Hot_A and Hot_B arises, as we saw yesterday, whenever we have an *aspheric* functor

(2) $$i : A \to B,$$

by just taking

(2') $$\overline{i^*} : \mathrm{Hot}_B \xrightarrow{\approx} \mathrm{Hot}_A.$$

This of course is about the simplest way imaginable, all the more as the functor $i^* : B^\wedge \to A^\wedge$ commutes to arbitrary direct and inverse limits, just perfect for comparing constructions in $B^\wedge$ and constructions in $A^\wedge$ – whereas the functor $j_B i_A : A^\wedge \to B^\wedge$, giving rise to (1) above commutes just to sums and fibered products, not to amalgamated sums nor to products, sadly enough! We could add here that if we got *two* aspheric functors from A to B, namely i plus

$$i' : A \to B,$$

then (as immediately checked) any map between these functors

$$u : i \to i'$$

gives rise to an isomorphism between the corresponding equivalences

$$\begin{tikzcd} \mathrm{Hot}_B \arrow[r, bend left, "\overline{i^*}"] \arrow[r, bend right, "\overline{(i')^*}"'] \arrow[r, phantom, "\Uparrow\overline{u}"] & \mathrm{Hot}_A \end{tikzcd},$$

which is nothing but the canonical isomorphism referred to yesterday (both functors being canonically isomorphic to (1')), and yields the most evident way for "computing" the latter. Thus, it turns out that the isomorphism $\overline{u}$ does not depend upon the choice of u. If we want to ignore this fact and look at the situation sternly, in a wholly computational spirit, we could present things by stating that we get a contravariant functor, from aspheric maps $i : A \to B$ to equivalences $\mathrm{Hot}_B \to \mathrm{Hot}_A$: [p. 225]

$$\mathbf{Asph}(A,B)^{\mathrm{op}} \to \mathbf{Hom}(\mathrm{Hot}_B, \mathrm{Hot}_A), \quad i \mapsto \overline{i^*},$$

where **Asph** denotes the full subcategory of the functor category **Hom** made up with aspheric functors. This functor transforms arbitrary arrows from the left hand side into isomorphisms on the right, and therefore, it factors through the fundamental groupoid (i.e., localization of the left hand category with respect to the set of *all* its arrows):

$$\big(\Pi_1(\mathbf{Asph}(A,B))\big)^{\mathrm{op}} \to \mathbf{Hom}(\mathrm{Hot}_B, \mathrm{Hot}_A).$$

If we now remember that we had assumed A and B to be test categories (otherwise the functors just written would be still defined, but their values would not necessarily be *equivalences* between Hot_B and Hot_A, but merely functors between these), we may hope that this might imply that the category $\mathbf{Asph}(A,B)$ of aspheric functors from A to B, whenever non-empty, to be 1-connected. Whenever this is so, in any case, we get "a priori" (namely without any reference to (Hot) itself) a transitive system of isomorphisms between the functors $\overline{i^*}$, for i in $\mathbf{Asph}(A,B)$, hence a *canonical functor* $\mathrm{Hot}_B \to \mathrm{Hot}_A$, *defined up to unique isomorphism* (and which, in case A and B are test categories, or more generally pseudo-test categories, is an equivalence, and the one precisely stemming from the diagram (1)).

Remark. Here the reflection slipped, almost against will, into a related one, about comparison of Hot_A and Hot_B for arbitrary A and B (not necessarily test categories nor *even* pseudo test categories), using aspheric functors $i : A \to B$ to get $\overline{i^*} : \mathrm{Hot}_B \to \mathrm{Hot}_A$ (not necessarily an equivalence). As seen above, this functor depends rather loosely on the choice of i, and we could develop comprehensive conditions on A and B, not at all of a test-condition nature, implying that just using the isomorphisms $\overline{u}$ between these functors, we get a canonical *transitive* system of isomorphisms between them, hence a *canonical* functor $\mathrm{Hot}_B \to \mathrm{Hot}_A$, not depending on the particular choice of any aspheric functor $i : A \to B$. As we are mainly interested in the modelizing case though, I don't think I should dwell on this much longer here. Anyhow, in case both A and B are pseudo-test categories, and provided only $\mathbf{Asph}(A,B)$ is 0-connected (not necessarily 1-connected), it follows from comparison with the diagram (1) above that the isomorphisms $\overline{u}$ do give rise indeed to a *transitive* system of isomorphisms between the equivalences $\overline{i^*}$.

[p. 226] In most cases though, such as $A = \triangle$ and $B = \square$ say (the test categories of standard simplices and standard cubes respectively), we do not have any aspheric functor $A \to B$ at hand, and presumably we may well have that $\mathbf{Asph}(A,B)$ is empty, poor it! We now assume again that A and B are test categories, hence $A \times B$ is a test category, and the natural idea for comparison of Hot_A and Hot_B is to use the diagram

$$\begin{tikzcd} & A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A & & B \end{tikzcd},$$

where now p_1 and p_2 are aspheric (because B and A are aspheric). Thus, we get a corresponding diagram on the corresponding modelizers

[in the typescript this equation is tagged (2)]

$$\begin{tikzcd} & \mathrm{Hot}_{A\times B} & \\ \mathrm{Hot}_A \arrow[ur, "\overline{p_1^*}", "\approx"'] & & \mathrm{Hot}_B \arrow[ul, "\overline{p_2^*}"', "\approx"] \end{tikzcd}. \tag{3}$$

These equivalences are compatible with the canonical equivalences with Hot itself, and hence they give rise to an equivalence $\mathrm{Hot}_A \xrightarrow{\approx} \mathrm{Hot}_B$ which (up to canonical isomorphism) is indeed the canonical one. This way for comparing Hot_A and Hot_B looks a lot more convenient than the first one, as the functors p_1^* and p_2^* which serve as intermediaries have all desirable exactness properties, and their very description is the simplest imaginable.

We are very close here to an Eilenberg-Zilber situation, which will arise more specifically when A and B are both *strict* test categories, i.e., in the corresponding modelizers $A\hat{}\,, B\hat{}$ products of models do correspond to products of the corresponding homotopy types. As seen above, this implies the same for $A \times B$. Thus, if F is in $A\hat{}$, G in $B\hat{}$, the objects $F \boxtimes G$ in $(A \times B)\hat{}$ is just a $(A \times B)$-model for the homotopy type described respectively by F (in terms of A) and G (in terms of B). As a matter of fact, the relation

$$(A \times B)_{/F\boxtimes G} \simeq A_{/F} \times B_{/G},$$

already noticed before, implies that this interpretation holds, independently of strictness. In the classical statement of Eilenberg-Zilber (as I
[p. 227] recall it), we got $A = B$ (both categories being just $\triangle$); on the one hand the $A \times B$-model is used in order to get readily the Künneth type relations for homology and cohomology, whereas we are interested really in the A-model $F \times G$. Using the diagonal map

$$\delta : A \to A \times A,$$

we have of course

$$F \times G \simeq \delta^*(F \boxtimes G),$$

more generally it is expected that the functor

$$\delta^* : (A \times A)\hat{} \to A\hat{},$$

is modelizing, i.e., gives a means of passing from $(A \times A)$-models to A-models. This essentially translates into δ being an aspheric map – something which will not be true for arbitrary A in (Cat). We get in this respect:

Proposition 3. *Let A be an object in* (Cat)*. Then the diagonal map $\delta : A \to A \times A$ is aspheric iff A is totally aspheric.*

This is just a tautology – one among many which tell us that the conceptual set-up is OK indeed! A related tautology:

Corollary. *Let $i : A \to B$, $i' : A \to B'$ be two aspheric functors with same source A, then the corresponding functor*

$$(i, i') : A \to B \times B'$$

is aspheric, provided A is totally aspheric.

This can be seen either directly, or as a corollary of the proposition, by viewing (i, i') as a composition

$$A \xrightarrow{\delta} A \times A \xrightarrow{i \times i'} B \times B'.$$

To sum up: when we got a bunch of strict test categories, and a (possibly empty) bunch of aspheric functors between some of these, using finite products we get a (substantially larger!) bunch of strict test categories, giving rise to corresponding strict modelizers for homotopy types; and using projections, diagonal maps, and the given aspheric functors, we get an impressive lot of aspheric maps between all these, namely as many ways to "commute" from one type of "homotopy models" to others. The simplest example: start with just one strict test category A, taking products

$$A^I$$

where I is any finite set, and the "simplicial" maps between these, expressing contravariance of A^I with respect to I. The case most commonly used is $A = \triangle$, giving rise to the formalism of semisimplicial multicomplexes.

To finish these generalities on aspheric functors, I would like still to make some comments on $\mathbf{Asph}(A, B)$, in case B admits binary products, with A and B otherwise arbitrary in (Cat). We are interested, for two functors [p. 228]

$$i, i' : A \rightrightarrows B,$$

in the functor

$$i \times i' : a \mapsto i(a) \times i'(a) : A \to B,$$

which can be viewed indeed as a product object in $\mathbf{Hom}(A,B)$ of i and i'. Our interest here is mainly to give conditions ensuring that $i \times i'$ is aspheric. It turns out that there is no point to this end assume *both* i and i' to be aspheric, what counts is that one, say i, should be aspheric. The point is made very clearly in the following

Proposition 4. *Let $i : A \to B$ be an aspheric map in* (Cat)*, we assume for simplicity that in B binary products exist.*

a) *Let $b_0 \in \operatorname{Ob} B$, $i_{b_0} : A \to B$ be the constant functor with value b_0, consider the product functor*

$$i \times i_{b_0} : a \mapsto i(a) \times b_0 : A \to B.$$

In order for this functor to be aspheric, it is necessary and sufficient that for any object y in B, the object $\mathbf{Hom}(b_0, y)$ in $B^\wedge$ be aspheric (a condition which depends only on b_0, not upon i nor even upon A).

b) *Assume this condition satisfied for any b_0 in B, assume moreover A totally aspheric. Then for* any *functor $i' : A \to B$, $i \times i'$ is aspheric.*

Proof. a) Asphericity of $i \times i_{b_0}$ means that for any object y in B, the corresponding presheaf on A

$$(i \times i_{b_0})^*(y) = (a \mapsto \operatorname{Hom}(i(a) \times b_0, y))$$

is aspheric. Now the formula defining the presheaf $\mathbf{Hom}(b_0, y)$ on B yields

$$\operatorname{Hom}(i(a) \times b_0, y) \simeq \operatorname{Hom}(i(a), \mathbf{Hom}(b_0, y)),$$

hence the presheaf we get on A is nothing but

$$i^*(\mathbf{Hom}(b_0, y)).$$

As i is aspheric, the criterion (ii') of prop. 1 (page 214) implies that this presheaf is aspheric iff $\mathbf{Hom}(b_0, y)$ is, qed.

b) We may view $i \times i'$ as a composition

$$A \xrightarrow{\delta} A \times A \xrightarrow{i \boxtimes i'} B,$$

[p. 229] where the "external product" $i \boxtimes i' : A \times A \to B$ is defined by

$$i \boxtimes i'(a, a') \overset{\text{def}}{=} i(a) \times i'(a').$$

By prop. 3 we know that the diagonal map for A is aspheric, thus we are left with proving that $i \boxtimes i'$ is aspheric. More generally, we get:

Corollary 1. *Let B in* (Cat) *satisfy the conditions of a) and b) above, and let A, A' be two objects in* (Cat)*, and*

$$i : A \to B, \quad i' : A' \to B$$

two maps with target B, hence a map

$$i \boxtimes i' : (a, a') \mapsto i(a) \times i'(a') : A \times A' \to B.$$

If i is aspheric, then $i \boxtimes i'$ is aspheric iff A' is aspheric.

We have to express that for any object b in B, the presheaf

$$(a,a') \mapsto \mathrm{Hom}(i(a) \times i'(a'), b) \simeq \mathrm{Hom}(i(a), \mathbf{Hom}(i'(a'), b))$$

on $A \times A'$ is aspheric. For a' fixed in A', the corresponding presheaf on A is aspheric, as we saw in a). The conclusion now follows from the useful

Lemma. *Let F be a presheaf on a product category $A \times A'$, with A, A' in* (Cat). *Assume that for any a' in A', the corresponding presheaf $a \mapsto F(a,a')$ on A be aspheric. Then the composition*

$$(A \times A')_{/F} \to A \times A' \xrightarrow{\mathrm{pr}_2} A'$$

is in $\mathcal{W}$, and hence F is aspheric iff A' is aspheric.

Proof of lemma. The composition is fibering, as both factors are. This reminds us of the "fibration condition" L 5 on $\mathcal{W}$ (page 164), as yesterday (page 219), where we stated that this condition follows from the conditions Loc 1) to Loc 3) reviewed yesterday. This condition asserts that a fibration with aspheric fibers is in $\mathcal{W}$ – hence the lemma.

Remarks. The results stated in prop. 4 and its corollary give a lot of elbow freedom for getting new aspheric functors in terms of old ones, with target category B – provided B satisfies the two assumptions: stability under binary products (a property frequently met with, although the standard test categories $\blacktriangle$, $\square$ and $\bigcirc$ lack it...), and the asphericity of the presheaves $\mathbf{Hom}(b,y)$, for any two objects b, y in B. A slightly stronger condition (indeed an equivalent one, for fixed b, when b admits already a section over $e_{B\hat{}}$ and if $B\hat{}$ is totally aspheric, hence an aspheric object is even aspheric over $e_{B\hat{}}$...) is *contractibility* of the objects b of B, for the homotopy interval structure on $B\hat{}$ defined by $\mathcal{W}_B$ (i.e., in terms of homotopy intervals aspheric over $e_{B\hat{}}$). (Compare with propositions on pages 121 and 143.) Whereas this latter assumption admittedly is quite a stringent one, it is however of a type which has become familiar to us in relation with test categories, where it seems a rather common lot. [p. 230]

If B satisfies these conditions (as in prop. 4 and if A is totally aspheric, we see from prop. 4 that the category $\mathbf{Asph}(A,B)$ of aspheric functors from A to B is stable under binary products. Now, a non-empty object C of (Cat) stable under binary products gives rise to a category $C\hat{}$ which is clearly totally aspheric for any basic localizer $\mathcal{W}$, and in particular for the usual one $\mathcal{W}_0$ corresponding to usual weak equivalence. A fortiori, such a category C is 1-connected (which is easily checked too by down-to-earth direct arguments). Thus, the reflections of page 225 apply, and imply that if $\mathbf{Asph}(A,B)$ is non-empty, i.e., if there is at least *one* aspheric functor $i : A \to B$, then there is a canonical transitive system of isomorphisms between all functors

[later, we'll write $\mathcal{W}_\infty$ for this instead...]

(*) $$\mathrm{Hot}_B \to \mathrm{Hot}_A$$

of the type $\overline{i^*}$, and hence there is a canonical functor (*), defined up to unique isomorphism, as announced in the remark on page 225.

13.6.

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Asphericity structures: a bunch of useful tautologies.

The generalities on aspheric maps of the last three sections should be more than what is needed to develop now the notion of *asphericity structure* – which, together with the closely related notion of *contractibility structure*, tentatively dealt with before, and the various "test-notions" (e.g., *test categories* and *test functors*) seems to me the main pay-off so far of our effort to come to a grasp of a general formalism of "homotopy models".

In the case of asphericity structures, just as for the kindred notion of a contractibility structure, in all instances I could think of a present, the underlying category M of an asphericity structure is *not* an object in (Cat) nor even a "small category", but is "large" – namely the cardinality of $\mathrm{Ob}\, M$ and $\mathrm{Fl}(M)$ are not in the "universe" we are working in, still less is M an object of $\mathcal{U}$ – all we need instead, as usual, is that M be a $\mathcal{U}$-category, namely that for any two objects of M, $\mathrm{Hom}(x, y)$ be an element of $\mathcal{U}$. Till now, the universe $\mathcal{U}$ has been present in our reflections in a very much implicit way, in keeping with the informal nature of the reflections, which however by and by have become more formal (as [p. 231] I finally let myself become involved in a minimum of technical work, needed for keeping out of the uneasiness of "thin air conjecturing"). An attentive reader will have felt occasionally this implicit presence of $\mathcal{U}$, for instance in the definition of the basic modelizer (Cat) (which is, as all modelizers, a "large" category), and in our occasional reference to various categories as being "small" or "large". He [or she] will have noticed that whereas modelizers are by necessity large categories (just as (Hot) itself, whose set of isomorphism classes of objects is large), test categories are supposed to be small (and often even to be in (Cat)) – with the effect that $A^\wedge$, the category of presheaves on A, is automatically a $\mathcal{U}$-category (which would not be the case if A was merely assumed to be a $\mathcal{U}$-category).

An *asphericity structure* (with respect to the basic localizer $\mathcal{W}$) on the $\mathcal{U}$-category M consists of a subset

$$M_{\mathrm{as}} \subset \mathrm{Ob}\, M, \tag{1}$$

whose elements will be called the *aspheric objects* of M (more specifically, the $\mathcal{W}$-aspheric objects, if confusion may arise), this subset being submitted to the following condition:

(Asstr) There exists an object A in (Cat), and a functor $i : A \to M$, satisfying the following two conditions:

(i) For any a in A, $i(a) \in M_{\mathrm{as}}$, i.e., i factors through the full subcategory (also denoted by M_{as}) of M defined by M_{as}.

(ii) Let

$$i^* : M \to A^\wedge, \quad x \mapsto i^*(x) = (a \mapsto \mathrm{Hom}(i(a), x)),$$

then we have

$$M_{\mathrm{as}} = (i^*)^{-1}(A^\wedge_{\mathrm{as}}), \tag{2}$$

where $A^\wedge_{\mathrm{as}}$ is the subset of $\mathrm{Ob}\,A^\wedge$ of all ($\mathcal{W}$-)aspheric objects of $A^\wedge$, i.e., the presheaves F on A such that the object $A_{/F}$ of (Cat) is $\mathcal{W}$-aspheric.

In other words, for x in M, we have the equivalence

(2 bis) $\quad x \in M_{\mathrm{as}} \Leftrightarrow i^*(x) \in A^\wedge_{\mathrm{as}}, \quad$ i.e., $\quad A_{/x} (\stackrel{\mathrm{def}}{=} A_{/i^*(x)}) \in (\mathrm{Cat})_{\mathrm{as}},$

where $(\mathrm{Cat})_{\mathrm{as}}$ is the subset of $\mathrm{Ob}\,(\mathrm{Cat})$ made up with all $\mathcal{W}$-aspheric objects of (Cat) – i.e., objects X such that $X \to \Delta_0$ (= final object) be in $\mathcal{W}$.

Thus, an asphericity structure on M can always be defined by a functor

$$i : A \to M,$$

with A in (Cat), by the formula (2); and conversely, any such functor defined an asphericity structure M_{as} on M, admitting i as a "*testing functor*" (i.e., a functor satisfying (i) and (ii) above), provided only we assume that [p. 232]

(3) For any a in A, $i^*(i(a))$ is an aspheric object of $A^\wedge$.

This latter condition is automatically satisfied if i is fully faithful, for instance, if it is the inclusion functor of a full subcategory A of M. Thus, *any small full subcategory A of M defines an asphericity structure on M*, and we'll see in a minute that any asphericity structure on M can be defined this way.

Proposition 1. *Let M_{as} be an asphericity structure on M, and A, B objects in* (Cat)*, and*

$$A \xrightarrow{f} B \xrightarrow{j} M$$

be functors, with j factoring through M_{as}.

a) *If f is aspheric, then j is a testing functor iff $i = jf$ is.*

b) *If j is fully faithful, and if $i = jf$ is a testing functor, then f is aspheric, and j is a testing functor too.*

Proof. a) follows trivially from

$$B^\wedge_{\mathrm{as}} = (f^*)^{-1}(A^\wedge_{\mathrm{as}}),$$

which is one of the ways of expressing that f is aspheric (criterion (ii') of prop. 1, 214). And the first assertion in b) is a trivial consequence of the definition of testing functors, and of asphericity of f (by criterion (iii) on p. 214) – and the second assertion of b) now follows from a).

Corollary. *Let $i : A \to M$ be a testing functor for (M, M_{as}), and let B any* small *full subcategory of M_{as} containing $i(A)$. Then the induced functor $A \to B$ is aspheric, and the inclusion functor $B \hookrightarrow M$ is a testing functor.*

Remarks. This shows, as announced above, that any asphericity structure on M can be defined by a small full subcategory of M (for instance, the smallest full subcategory of M containing $i(A)$). We have been slightly floppy though, while we defined testing functors by insisting that the source should be in Cat (which at times will be convenient), whereas the B we got here is merely small, namely *isomorphic* to an object of (Cat), but not necessarily in (Cat) itself. This visibly is an "inessential floppiness", which could be straightened out trivially, either by enlarging accordingly the notion of testing functor, or by submitted
[p. 233] M to the (somewhat artificial, admittedly) restriction that all its small subcategories should be in (Cat). This presumably will be satisfied by most large categories we are going to consider, and it shouldn't be hard moreover to show that any $\mathfrak{U}$-category is isomorphic to a category M' satisfying the above extra condition.

A little more serious maybe is the use we are making here of the name of a "testing functor", which seems to be conflicting with an earlier use (def. 5, p. 175 and def. 6, p. 176), where we insisted for instance that the source A should be a test category. That's why I have been using here the name "testing functor" rather than "test functor", to be on the safe side formally speaking – but this is still playing on words, namely cheating a little. Maybe the name "*aspherical functor*" instead of "testing functor" would be less misleading, thinking of the case when M is small itself, and endowing M with the canonical asphericity structure, for which

$$M_{\mathrm{as}} = M,$$

(admitting the identity functor as a testing functor) – in which case the "testing functors" $A \to M$ are indeed just the aspherical functors. The drawback is that when M is small, it may well be endowed with an asphericity structure M_{as} different from the previous one, in which case the proposed extension of the name "aspheric functor" again leads to an ambiguity, unless specified by "M_{as}-*aspheric*" (where, after all, M_{as} could be *any* full subcategory of M) – but then the notion reduces to the one of an aspheric functor $A \to M_{\mathrm{as}}$. But the same after all holds even for large M – the notion of a "testing functor" $A \to M$ (with respect to an asphericity structure M_{as} on M) does not really depend on the *pair* (M, M_{as}), but rather on the (possibly large) category M_{as} itself – namely it is no more no less than a functor

$$A \to M_{\mathrm{as}}$$

which satisfies the usual asphericity condition (iii) (of prop. 1, p. 214), with the only difference that M_{as} may not be small, and therefore $M_{\mathrm{as}}^{\wedge}$ may not be a $\mathfrak{U}$-category (and we are therefore reluctant to work with this latter category at all, unless we first pass to the next larger universe $\mathfrak{U}' \ldots$).

This short reflection rather convinces us that the designation of "testing functors" as introduced on the page before, by the alternative name of M_{as}-*aspheric functors*, or just *aspheric functors* when no confusion is likely to arise as to existence and choice of M_{as}, is satisfactory indeed. I'll use it tentatively, as a synonym to "testing functor", and it will appear soon enough if this terminology is a good one or not – namely if it is suggestive, and not too conducive to misunderstandings. [p. 234]

Proposition 2. *Let (M, M_{as}) be an asphericity structure, and let*

$$i : A \to M$$

be an M_{as}-aspheric functor, hence $i^* : M \to A^\wedge$. Let

$$W = (i^*)^{-1}(\mathcal{W}_A) \subset \mathrm{Fl}(M). \tag{4}$$

Then W is a mildly saturated subset of $\mathrm{Fl}(M)$*, independent of the choice of (A, i).*

Mild saturation of W follows from mild saturation of $\mathcal{W}_A$, the set of $\mathcal{W}$-weak equivalences in $A^\wedge$ (which follows from mild saturation of $\mathcal{W}$ and the definition

$$\mathcal{W}_A = i_A^{-1}(\mathcal{W})\ ).$$

To prove that W_i for (A, i) is the same as $W_{i'}$ for (A', i'), choose a small full subcategory B of M_{as} such that i and i' factor through B, let $j : B \to M$ be the inclusion, we only have to check $W_i = W_j$ (and similarly for $W_{i'}$), which follows from $i^* = f^* j^*$ (where $f : A \to B$ is the induced functor) and the relation

$$\mathcal{W}_A = (f^*)^{-1}(\mathcal{W}_A),$$

which follows from f being aspheric by prop. 1 b), and the known property (iv) (prop. 1, p. 214) of asphericity, qed.

Remark. Once we prove that $\mathcal{W}$ is even strongly saturated, it will follow of course that the sets $\mathcal{W}_A$, and hence W above, are strongly saturated too.

We'll call W the set of *weak equivalences* in M, for the given asphericity structure M_{as} in M. We are interested now in giving a condition on M_{as} ensuring that conversely, M_{as} is known when the corresponding set W of weak equivalences is. We'll assume for this that M has a final object e_M, and we'll call the asphericity structure (M, M_{as}) "*aspheric*" if e_M is aspheric, i.e., $e_M \in M_{\mathrm{as}}$ (which does not depend of course on the choice of e_M). Now we get the tautology:

Proposition 3. *Let (M, M_{as}) be an asphericity structure, and $u : x \to y$ a map in M. If y is aspheric, then x is aspheric iff u is aspheric.*

Corollary. *Assume M admits a final object e_M, and that e_M is aspheric (i.e., (M, M_{as}) is aspheric). Then an object x in M is aspheric iff the map $x \to e_M$ is aspheric.*

Next question then is to state the conditions on a pair (M, W), with $W \subset \mathrm{Fl}(M)$, for the existence of an aspheric asphericity structure on M, admitting W as its set of weak equivalences. We assume beforehand [p. 235]

M admits a final object e_M. A n.s. condition is the existence of a small category A and a functor

$$i : A \to M$$

satisfying the following two conditions:

(i) for x in M of the form $i(a)$ ($a \in \mathrm{Ob} A$) *or* e_M, $i^*(x) \in A^\wedge_{\mathrm{as}}$,

(ii) $W = (i^*)^{-1}(\mathcal{W}_A)$.

Another n.s. condition, which looks more pleasant I guess, is that there exist a small full subcategory B of M, containing e_M, with inclusion functor $j : B \to M$, such that

$$W = (j^*)^{-1}(\mathcal{W}_B). \tag{5}$$

In any case, if we want a n.s. condition on W for it to be the set of weak equivalences for some asphericity structure on M (not necessarily an aspheric one, and therefore maybe not unique), we get: there should exist a small full subcategory B of M, with inclusion functor j, such that (5) above holds. (Here we do not assume e_M to exist, still less B to contain it.) Similarly, the corollary of prop. 1 implies that a subset M_{as} of $\mathrm{Ob}\, M$ is an asphericity structure on M, iff there exists a small full subcategory B in M, such that

$$M_{\mathrm{as}} = (j^*)^{-1}(B^\wedge_{\mathrm{as}}); \tag{6}$$

moreover, if M admits a final object e_M, the asphericity structure M_{as} is aspheric iff B can be chosen to contain e_M.

Remarks 2. The relationship between aspheric asphericity structures M_{as} on M, and sets $W \subset \mathrm{Fl}(M)$ of "weak equivalences" in M satisfying the condition above (and which we may call "weak equivalence structures" on M), in case M admits a final object e_M, is reminiscent of the relationship between "contractibility structures" $M_{\mathrm{c}} \subset \mathrm{Ob}\, M$ on M, and those "homotopism structures" $h_M \subset \mathrm{Fl}(M)$ on M which can be described in terms of such a contractibility structure (cf. sections 51 and 52). Both pairs can be viewed as giving two equivalent ways of expressing one and the same kind of structure – the structure concerned by the first pair being centered on asphericity notions, whereas the second is concerned with typical homotopy notions rather. We'll see later that any contractibility structure defines in an evident way an asphericity structure, and in the most interesting cases (e.g., canonical modelizers), it is uniquely determined by the latter.

[p. 236] To sum up some of the main relationships between the three asphericity notions just introduced (M_{as}, W, M_{as}-aspheric functors), let's state one more tautological proposition, which is very much a paraphrase of the display given earlier (prop. on p. 214) of the manifold aspects of the notion of an aspheric map between small categories:

Proposition 4. *Let* (M, M_{as}) *be an asphericity structure, A a small category, $i : A \to M$ a functor, factoring through M_{as}. Consider the following conditions on i:*

(i) *i is M_{as}-aspheric (or "testing functor" for M_{as}), i.e.,*

$$M_{\text{as}} = (i^*)^{-1}(A^{\wedge}_{\text{as}}).$$

(i') *For x in M_{as}, $i^*(x)$ is aspheric, i.e.,*

$$M_{\text{as}} \subset (i^*)^{-1}(A^{\wedge}_{\text{as}}).$$

(i") *(Here, B is a given small full subcategory of M_{as} containing $i(A)$ and which* "generates" *the asphericity structure M_{as}, namely such that the inclusion functor $j : B \hookrightarrow M$ is M_{as}-aspheric, i.e., $M_{\text{as}} = (j^*)^{-1}(B^{\wedge}_{\text{as}})$.) For any x in B, $i^*(x)$ is aspheric, i.e.,*

$$B \subset (i^*)^{-1}(A^{\wedge}_{\text{as}}).$$

(ii) *For any map u in M, u is a weak equivalence iff $i^*(u)$ is, i.e.,*

$$W = (i^*)^{-1}(\mathcal{W}_A).$$

(ii') *If the map u in M is a weak equivalence, so is $i^*(u)$, i.e.,*

$$W \subset (i^*)^{-1}(\mathcal{W}_A).$$

(ii") *(Here, B is given as in (i") above) For any map u in M, $i^*(u)$ is a weak equivalence, i.e.,*

$$\mathrm{Fl}(B) \subset (i^*)^{-1}(\mathcal{W}_A).$$

The conditions (i)(i')(i") are equivalent and imply all others, and we have the tautological implications (ii) $\Rightarrow$ (ii') $\Rightarrow$ (ii"). If M admits a final object e_M and if A and the asphericity structure M_{as} are aspheric, then all six conditions (except the last) are equivalent; and all six are equivalent if moreover $e_M \in \mathrm{Ob}\, B$.

Proof. The implications (i) $\Rightarrow$ (i') $\Rightarrow$ (i") are tautological, on the other hand (i") just means that the induced functor $f : A \to B$ is aspheric, which by prop. 1 a) implies that i is aspheric. On the other hand (i) $\Rightarrow$ (ii) by the definition of W (cf. prop. 2). If e_M exists and A and the asphericity structure M_{as} are aspheric, and if B contains e_M, then (ii") implies that the maps $x \to e_M$ for x in B are transformed by i^* into a weak equivalence, and as $e_{A^{\wedge}}$ is aspheric, this implies $i^*(x)$ is aspheric, i.e., (i"), which proves the last statement of the proposition – all six conditions are equivalent in this case. If no B is given, but still assuming A and (M, M_{as}) aspheric, we can choose a generating subcategory for (M, M_{as}) large enough in order to contain $i(A)$ and e_M, and we get that conditions (i) to (ii') are equivalent, qed. [p. 237]

Examples. Totally aspheric asphericity structures.

76 It's time to give some examples.

1) Take $M = (\mathrm{Cat})$, $M_{\mathrm{as}} =$ set of $\mathcal{W}$-aspheric objects in (Cat). We then got an asphericity structure, as we see by taking any weak test category A in (Cat) (def. 2 on page 172), and the functor

$$i_A : a \mapsto A_{/a} : A \to (\mathrm{Cat}),$$

which satisfies indeed (i) and (ii) of p. 231 above.

We may call

$$((\mathrm{Cat}), (\mathrm{Cat})_{\mathcal{W}-\mathrm{as}} \overset{\mathrm{def}}{=} (\mathrm{Cat})_{\mathrm{as}}) \tag{7}$$

the "basic asphericity structure", giving rise (by taking the corresponding set W of "weak equivalences") to the "basic modelizer" $((\mathrm{Cat}), \mathcal{W})$. Turning attention towards the former corresponds to a shift in emphasis; whereas previously, our main emphasis has been dwelling consistently with the notion of "weak equivalence", namely with giving on a category M a bunch of *arrows* W, here we are working rather with notion of "aspheric *objects*" as the basic notion. One hint in this direction comes from prop. 1 on p. 214, when we saw that for a functor $f : A \to B$ between small categories, giving rise to $f^* : B^\wedge \to A^\wedge$ (a map between asphericity structures, as a matter of fact), asphericity of f can always be expressed in manifold ways as a property of f relative to the notion of aspheric *objects* in $A^\wedge$ and $B^\wedge$, but not as a property relative to the notion of weak equivalences, unless A and B are assumed to be aspheric.

Coming back to the case of the "basic asphericity structures" (7), we get more general types of "aspheric" functors $A \to (\mathrm{Cat})$ than functors i_A, by taking any weak test category A and any weak test functor $i : A \to (\mathrm{Cat})$ (cf. def. 5, p. 175), provided however the objects $i(a)$ are aspheric. It would seem though that, for a given A, even assuming A to be a *strict* test category say (and even a "contractor" moreover), and restricting to functors $i : A \to (\mathrm{Cat})$ which factor through $(\mathrm{Cat})_{\mathrm{as}}$ (to make them eligible for being "aspheric functors"), the condition for i to be a weak test functor, namely for i^* to be "model preserving", is substantially stronger than mere "asphericity": indeed, the latter just means that i^*
[p. 238] transforms weak equivalences into weak equivalences, i.e., gives rise to a functor

$$\mathrm{Hot} = \mathcal{W}^{-1}(\mathrm{Cat}) \to \mathrm{Hot}_A = \mathcal{W}_A{}^{-1} A^\wedge,$$

whereas the latter insists that this functor moreover should be an equivalence of categories. Theorem 1 (p. 176) gives a hint though that the two conditions may well be equivalent – this being so at any rate provided the objects $i(a)$ in (Cat) are, not only aspheric, but even *contractible*. This reminds us at once of the "silly question" of section 46 (p. 95), which was the starting point for the subsequent reflections leading up to the theorem 1 recalled above; and, beyond this still somewhat technical result, the ultimate motivation for the present reflections on asphericity structures. The main purpose for these, I feel, is to lead up to a comprehensive answer to the "silly question". We'll have to come back to this very soon!

2) Take $M = (\text{Spaces})$, the category of topological spaces, and M_{as} the spaces which are weakly equivalent to a point. We get an asphericity structure, indeed an *aspheric* asphericity structure (I forgot to make this evident specification in the case 1) above), as we see by taking $A = \triangle$ for instance, and

$$i : \triangle \to (\text{Spaces})$$

the "geometric realization functor" for simplices, which satisfies conditions (i) and (ii) of page 231, as is well known (cf. the book of Gabriel-Zisman); as a matter of fact, i is even a test functor for the modelizer (M, W) (which is one way of stating the main content of GZ's book). [Gabriel and Zisman 1967]

3) The two examples above are *aspheric* asphericity structures, and such moreover that (M, W) is ($\mathcal{W}$-)modelizing. These extra features however are not always present in the next example

(8) $$M = A^{\wedge}, \quad M_{\text{as}} = A^{\wedge}_{\text{as}}, \quad A \text{ a small category,}$$

which is indeed tautologically an asphericity structure, by taking the canonical inclusion functor (which is fully faithful)

$$A \to A^{\wedge},$$

satisfying the conditions (i)(ii) above (p. 231). This shows moreover that the corresponding notion of weak equivalence is the usual one, (I forgot to state the similar fact in example 2), sorry):

$$W = \mathcal{W}_A.$$

The asphericity structure is aspheric iff A is aspheric. A functor

$$A' \to A$$

where A' is another small category, is M_{as}-aspheric as a functor from A' to $A^{\wedge}$, iff it is aspheric. [p. 239]

This last example suggests to call an asphericity structure (M, M_{as}) "*totally aspheric*" if M is stable under finite products, and if the final object of M, as well as the product of any two aspheric objects of M, is again aspheric; in other words, if any finite product in M whose factors are aspheric is aspheric. We have, in this respect:

Proposition 5. *Let (M, M_{as}) be an asphericity structure, where M is stable under finite products. The following conditions are equivalent:*

(i) *M is totally aspheric, i.e., e_M and the product of any two aspheric objects of M are aspheric.*

(ii) *There exists a small subcategory B of M, stable under finite products (i.e., containing a final object e_M of M and, with any two objects x and y, a product $x \times y$ in M), and which generates the asphericity structure (i.e., $j : B \hookrightarrow M$ is M_{as}-aspheric).*

(iii) *There exists a small category A such that $A^{\wedge}$ is totally aspheric (def. 1, p. 170), and a M_{as}-aspheric functor $A \to M$.*

The proof is immediate. Of course, the asphericity structure in example 3) is totally aspheric iff $A^\wedge$ is totally aspheric in the usual sense referred to above.

The canonical functor $(\mathrm{Hot}_M) \to (\mathrm{Hot})$.

77 Let

$$\mathcal{M} = (M, M_{\mathrm{as}}) \tag{9}$$

be an asphericity structure, which will be referred to also as merely M, when no ambiguity concerning M_{as} is feared. We'll write

$$\mathrm{Hot}_{\mathcal{M}} \ \text{(or simply } \mathrm{Hot}_M) \ = W^{-1}M, \tag{10}$$

where W of course is the set of weak equivalences in M. We are now going to define a canonical functor

$$\mathrm{Hot}_{\mathcal{M}} \to (\mathrm{Hot})_{\mathcal{W}} \quad \text{(or simply } (\mathrm{Hot}) \overset{\mathrm{def}}{=} \mathcal{W}^{-1}(\mathrm{Cat})), \tag{11}$$

defined at any rate up to canonical isomorphism. For this, take any aspheric (namely, M_{as}-aspheric) functor

$$i : A \to M,$$

and consider the composition

$$M \xrightarrow{i^*} A^\wedge \xrightarrow{i_A} (\mathrm{Cat}). \tag{12}$$

[p. 240] The three categories in (12) are endowed respectively with sets of arrows W, $\mathcal{W}_A$, $\mathcal{W}$, and the two functors satisfy the conditions

$$W = (i^*)^{-1}(\mathcal{W}_A) \quad \text{and} \quad \mathcal{W}_A = (i_A)^{-1}(\mathcal{W}), \tag{13}$$

hence $W = (i_{M,i})^{-1}(\mathcal{W})$, where

$$i_{M,i} : M \to (\mathrm{Cat})$$

is the composition $i_A i^*$. Therefore, we get a functor

$$\overline{i_{M,i}} : \mathrm{Hot}_M \to (\mathrm{Hot}), \tag{14}$$

which a priori depends upon the choice of (A, i). If we admit strong saturation of $\mathcal{W}$, it follows that this functor is "conservative", namely an arrow in Hot_M is an isomorphism, provided its image in (Hot) is.

To define (11) in terms of (14), we have to describe merely a transitive system of isomorphisms between the functors (14), for varying pair (A, i). Therefore, consider two such (A, i) and (A', i'), choose a small full subcategory B of M_{as} containing both $i(A)$ and $i'(A')$ (therefore, B is a generating subcategory for the asphericity structure M_{as}), and consider the inclusion functor $j : B \hookrightarrow M$. From prop. 1 b) it follows that the functors

$$f : A \to B, \quad f' : A' \to B$$

induced by i, i' are aspheric. Using this, and the criterion (i) of asphericity (prop. 1 on p. 214) we immediately get two isomorphisms

$$\begin{tikzcd}[column sep=small] & \overline{i_{M,j}} \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \overline{i_{M,i}} & & \overline{i_{M,i'}} \end{tikzcd},$$

and hence an isomorphism

$$\overline{i_{M,i}} \xrightarrow{\sim} \overline{i_{M,i'}},$$

depending only on the choice of B. As a matter of fact, it doesn't depend on this choice. To see this, we are reduced to comparing the two isomorphisms arising from a B and a B' such that $B \subset B'$, in which case the inclusion functor $g : B \to B'$ is aspheric (prop. 1). We leave the details of the verification (consisting mainly of some diagram chasing and some compatibilities between the $\lambda_i(F)$-isomorphisms of prop. 1 on page 214) to the skeptical reader. As for transitivity for a triple (A, i), (A', i'), (A'', i''), it now follows at once, by using a B suitable simultaneously for all three. [p. 241]

Having thus well in hand the basic functor (11) (which in case of example 3) above with $M = A^\wedge$, reduces to the all-important functor $\mathrm{Hot}_A \to (\mathrm{Hot})$), we cannot but define an asphericity structure to be *modelizing*, as meaning that this canonical functor is an equivalence of categories. In the case of $M = A^\wedge$ above, this means that A is a pseudo-test category – a relatively weak test notion still. It appears as just as small bit stronger than merely assuming the pair (M, W) to be modelizing, i.e., to be a "*modelizer*", namely assuming Hot_M to be equivalent (in some way or other...) to (Hot). Maybe we should be a little more cautious with the use of the word "modelizing" though, and devise a terminology which should reflect very closely the hierarchy of progressively stronger test notions

$$\begin{aligned}&(\text{pseudo-test cat.}) \supset (\text{weak test cat.}) \supset \\ &\qquad\qquad (\text{test cat.}) \supset (\text{strict test cat.}) \supset (\text{contractors})\end{aligned}$$

which gradually has peeled out of our earlier reflections, by pinpointing corresponding qualifications for an asphericity structure, as being "pseudo-modelizing", "weakly modelizing", "modelizing", "strictly modelizing", and ???. It now appears that a little extra reflection is needed here – for today it's getting a little late though!

18.6 and 19.6.

Test functors and modelizing asphericity structures: the outcome (at last!) of an easy "observation".

78 It is about time now to get a comprehensive treatment, in the context of asphericity structures, of the relationships suggested time ago in the "observation" and the "silly question" of section 46 (pages 94 and 95). The former is concerned with test-functors from test categories to modelizers, the second more generally with model-preserving maps between modelizers, having properties similar to the map $M \to A^\wedge$ stemming from a test-functor. For the time being, as I found but little time for the "extra reflection" which seems needed, only the first situation is by now reasonably clear in my mind.

As no evident series of "asphericity structure"-notions has appeared, paralleling the series of test-notions recalled by the end of last Monday's reflections (see above), I'm going to keep (provisionally only, maybe) the name of a *modelizing asphericity structure* (M, M_{as}) as one for which the canonical functor

$$\mathrm{Hot}_M = W^{-1}M \to (\mathrm{Hot}) \tag{1}$$

[p. 242] (cf. (14), p. 240) is an equivalence. This notion at any rate is satisfactory for formulating the following statement, which comes out here rather tautologically, and which however appears to me as exactly what I had been looking for in the "observation" recalled above:

Theorem 1. *Let (M, M_{as}) be a* modelizing *asphericity structure, A a pseudo-test category, and*

$$i : A \to M$$

a functor, factoring through M_{as} (i.e., $i(a)$ is aspheric for any a in A). We assume moreover M has a final object e_M. Then the six conditions (i) to (ii") of prop. 4 (p. 236) (the first of which expresses that i is M_{as}-aspheric) are equivalent, and they are equivalent to the following condition:

(iii) *The functor $i^* : M \to A^\wedge$ gives rise to a functor*

$$\mathrm{Hot}_M = W^{-1}M \to \mathrm{Hot}_A = \mathcal{W}_A{}^{-1}A^\wedge$$

(i.e., (ii') of prop. 4 holds, namely $i^*(W) \subset \mathcal{W}_A$), and *this functor moreover is an equivalence of categories.*

Proof. For the first statement (equivalence of conditions (i) to (ii")), by prop. 4 we need only show that the asphericity structure and A as aspheric. But this follows from the assumptions and from the

Lemma. *Let (M, M_{as}) be any modelizing asphericity structure, then a final object of M is aspheric (i.e., the structure is "aspheric"). In particular, if A is a pseudo-test category (i.e., $(A^\wedge, A^\wedge_{\mathrm{as}})$ is a modelizing asphericity structure), then A is aspheric.*

We only have to prove the first statement. It is immediate that e_M is a final object of $W^{-1}M = \mathrm{Hot}_M$ (this is valid whenever we got a localization $W^{-1}M$ of a category M with final object e_M), hence its image in $(\mathrm{Hot}) = \mathcal{W}^{-1}(\mathrm{Cat})$ is a final object. Thus, we are reduced to proving the following

Corollary. *Let* (M, M_{as}) *be any asphericity structure, consider the composition*

$$\varphi_M : M \to \mathrm{Hot}_M = W^{-1} \xrightarrow{\text{can.}} (\mathrm{Hot}),$$

then we get

$$M_{\mathrm{as}} = \left\{ x \in \mathrm{Ob}\, M \;\middle|\; \varphi_M(x) \textit{ is a final object of } (\mathrm{Hot}) \right\}.$$

Indeed, using the construction of $\mathrm{Hot}_M \to (\mathrm{Hot})$ in terms of a given M_{as}-aspheric functor $B \to M$, we are reduced to the same statement, with (M, M_{as}) replaced by $(B^\wedge, B^\wedge_{\mathrm{as}})$. The statement then reduced to: the object B in (Cat) is aspheric iff its image in (Hot) is a final object (this can be viewed also as the particular case of the corollary, when $M = (\mathrm{Cat})$, $M_{\mathrm{as}} = (\mathrm{Cat})_{\mathrm{as}}$). Now, this follows at once from strong saturation of $\mathcal{W}$, which (as we announced earlier) followed from Loc 1) to Loc 3) (and will be proved in part V of the notes). The reader who fears a vicious circle may till then restrict use of the theorem to the case when we assume beforehand that e_M and $e_{A^\wedge}$ are aspheric. [p. 243]

It is now clear that the six conditions of prop. 4 are equivalent, and they are of course implied by (iii). Conversely, they imply (iii), as follows from the fact that in the canonical diagram (commutative up to canonical isomorphism)

$$\begin{tikzcd} \mathrm{Hot}_M \arrow[rr] \arrow[dr] & & \mathrm{Hot}_A \arrow[dl] \\ & (\mathrm{Hot}) & \end{tikzcd},$$

the two downwards arrows are equivalences, qed.

The theorem above seems to me to be exactly *the* "something very simple-minded surely" which I was feeling to get burningly close, by the end of March, nearly three months ago (p. 89); at least, to be "just it" as far as the case of *test-functors* is concerned. We may equally view this theorem as giving the precise relationship between the notion of a weak test functor or a *test functor* (the latest version of which (in the context of $\mathcal{W}$-notions) appears in section 65 (def. 5 and 6, pp. 175 and 176)), and the notion of *aspheric functors*, more precisely of M_{as}-aspheric functors, introduced lately (p. 233). This now is the moment surely to check if the terminology of test functors and weak ones introduced earlier, before the relevant notion of asphericity structures was at hand, is really satisfactory indeed, and if needed, adjust it slightly.

So let again

$$i : A \to M$$

be a functor, with A small, and (M, M_{as}) an asphericity structure. We don't assume beforehand, neither that A is a test-category or the like, nor that (M, M_{as}) be modelizing. We now paraphrase def. 5 (p. 175) of weak test functors as follows:

[it looks like there are only two conditions, but see 1)–3) below. . .]

Definition 1. The functor i above is called a *pseudo-test functor*, if it satisfies the following three conditions:

[p. 244] a) The corresponding functor $i^* : M \to A^\wedge$ is "model-preserving", by which is meant here that

$$W = (i^*)^{-1}(\mathcal{W}_A),$$

and the induced functor

$$\mathrm{Hot}_M = W^{-1}M \to \mathrm{Hot}_A = {\mathcal{W}_A}^{-1}A^\wedge$$

is an equivalence.

b) The functor $i_A : A^\wedge \to (\mathrm{Cat})$ is model preserving (for $\mathcal{W}_A, \mathcal{W}$), which reduces to the canonical functor

$$\mathrm{Hot}_A = {\mathcal{W}_A}^{-1}A^\wedge \to (\mathrm{Hot}) = \mathcal{W}^{-1}(\mathrm{Cat})$$

being an equivalence (as we know already that $\mathcal{W}_A = (i_A)^{-1}(\mathcal{W})$, by definition of $\mathcal{W}_A$).

Condition b) just means that A is a pseudo-test category, i.e., that the asphericity structure $(A^\wedge, A^\wedge_{\mathrm{as}})$ it defines is modelizing. By the corollary of lemma above, it implies that A is aspheric (which corresponds to condition c) of def. 5 in loc. cit.). Condition a) comes in two parts, the first just meaning that i is M_{as}-aspheric – this translation being valid, at any rate, in case we assume already A aspheric, and the given asphericity structure is aspheric, i.e., M admits a final object e_M, and e_M is aspheric. We certainly do want a pseudo-test functor to be (at the very least) M_{as}-aspheric, so we should either strengthen condition a) to this effect (which however doesn't look as nice), or assume beforehand (M, M_{as}) aspheric. At any rate, if we use the first variant of the definition, condition a) in full then is equivalent (granting b)) to:

a') The functor i is M_{as}-aspheric, *and* (M, M_{as}) is modelizing,

which in turn implies that (M, M_{as}) is aspheric. So we may as well assume asphericity of (M, M_{as}) beforehand! At any rate, we see that the notion we are after can be decomposed into three conditions, namely:

1) A is a pseudo-test category, i.e., $(A^\wedge, A^\wedge_{\mathrm{as}})$ is modelizing.
2) (M, M_{as}) is modelizing.
3) The functor i is M_{as}-aspheric.

The two first conditions are just conditions on A and on (M, M_{as}) respectively, the third is just the familiar asphericity condition on i.

To get the notion of a *weak test-functor* (def. 5, p. 175) we have to be [p. 245] just one step more specific in 1), by demanding that A be even a weak test category, namely that the functor

$$i_A^* = j_A : (\mathrm{Cat}) \to A^\wedge$$

be model-preserving (which implies that i_A is too).

Following def. 6 (p. 176), we'll say that i is a *test functor* if i *and* the induced functors $i_{/a} : A_{/a} \to A \to M$ are weak test functors. Using the definition of a test category (def. 3, p. 173), we see that this just means that the following conditions hold:

1') A is a test category.

2') (M, M_{as}) is modelizing (i.e., same as 2) above).

3') The functor i and the induced functors $i_{/a} : A_{/a} \to M$ are M_{as}-aspheric.

This last condition merits a name by itself, independently of other assumptions:

Definition 2. Let (M, M_{as}) be any asphericity structure, A a small category, and $i : A \to M$ a functor. We'll say that i is *totally M_{as}-aspheric* (or simply *totally aspheric*, if no confusion is feared), if i and the induced functors $i_{/a} : A_{/a} \to M$ are M_{as}-aspheric (for any a in A). We'll say that i is *locally M_{as}-aspheric* (or simply *locally aspheric*) if for any a in A, the induced functor $i_{/a} : A_{/a} \to M$ is M_{as}-aspheric.

Thus, i is totally aspheric iff it is aspheric and locally aspheric. On the other hand, i is a test functor iff A is a test category, (M, M_{as}) is modelizing, and i is totally aspheric.

In order for i to be locally aspheric, it is n.s. that i factor through M_{as} and for any x in M_{as}, $i^*(x)$ be *aspheric over $e_{A^\wedge}$*; if B is any subcategory of M containing $i(A)$ and generating the asphericity structure, it is enough in this latter condition to take x in B.

Remarks. Thus, we see that the three gradations for the test-functor notion, as suggested by the definitions 5 and 6 of section 65 and now by the present context of asphericity structures, just amount to gradations for the test conditions on the category A itself (namely, to be a pseudo-test, a weak test or just a plain test -category), and a two-step gradation on the asphericity condition for i (namely, that i be either just aspheric in the two first cases, or totally aspheric in the third), while these two asphericity conditions on i are of significance, independently of any specific assumption which we may make on either A or (M, M_{as}). This [p. 246]
seems to diminish somewhat the emphasis I had put formerly upon the notion of a test functor and its weak variant, and enhance accordingly the notion of an aspheric functor (with respect to a given asphericity structure) and the two related asphericity notions for i which spring from it (namely, the notions of locally and of totally aspheric functors), which now seem to come out as the more relevant and the more general ones.

To be wholly happy, we still need the relevant reformulation, in terms of asphericity structures, of the main result of section 65, namely theorem 1 (p. 177) characterizing test functors with values in (Cat), under the assumption that the objects $i(a)$ be contractible. This theorem, I recall, has been the main outcome of the "grinding" reflections taking their start with the "observation" on p. 94 about ten days earlier. The appreach then followed, as well as the contractibility assumption for the objects $i(a)$ made in the theorem, retrospectively look awkward – it is clear that the relevant notions of asphericity structures, and of aspheric functors into these, had been lacking. The theorem stated last

(p. 242) looks indeed a lot more satisfactory than the former, except however in one respect – namely that the asphericity condition on i, in theorem 1 of p. 177, can be expressed by asphericity over $e_{A^\wedge}$ *of just* $i^*(\triangle_1)$ *alone*, rather than having to take $i^*(x)$ for all x in some fixed generating subcategory of (M, M_{as}) containing $i(A)$. To recover such kind of minimal criterion in a more general case than (Cat), we'll have to relate the notion of an asphericity structure with the earlier one of a *contractibility structure*; the latter in our present reflections has faded somewhat into the background, while at an earlier stage the homotopy and contractibility notions had invaded the picture to an extent as to overshadow and nearly bring to oblivion the magic of the "asphericity game".

[The rest of this page is unreadable in my scan of the typescript.]

[p. 247] 20.6.

Asphericity structure generated by a contractibility structure: the final shape of the "awkward main result" on test functors.

79 I would like now to write out the relationship between asphericity structures, and contractibility structures as defined in section 51 D) (pages 117–119). First we'll need to rid ourselves of the smallness assumption for a generating category of an asphericity structure:

Proposition 1. *Let M be a $\mathcal{U}$-category ($\mathcal{U}$ being our basic universe), and N any full subcategory. The following conditions are equivalent:*

(i) *There exists an asphericity structure M_{as} in M such that* (a) $M_{\mathrm{as}} \supset N$ *and* (b) *N admits a* small *full subcategory N_0 generating the asphericity structure, i.e., such that the inclusion functor i_0 from N_0 to M be M_{as}-aspheric.*

(ii) *There exists a small full subcategory N_0 of N, such that for any x in N, $i_0^*(x)$ in $N_0^\wedge$ be aspheric, where $i_0 : N_0 \to M$ is the inclusion functor.*

(iii) *The couple (N, N) is an asphericity structure.*

Moreover, when these conditions are satisfied, the asphericity structure M_{as} in (i) is unique.

Of course, we'll say it is the asphericity structure on M *generated* by the full subcategory N of M, and the latter will be called a *generating subcategory* for the given asphericity structure.

The proposition is a tautology, in view of the definitions and of prop. 1 (p. 232) and its corollary. The form (ii) or (iii) of the condition shows that it depends only upon the category N, not upon the way in which N is embedded in a larger category M. We may call a category N satisfying condition (iii) above an *aspherator*, by which we would like to express that this category represents a standard way of generating asphericity structures, through any full embedding of N into a category M. This condition is automatically satisfied if N is small, more generally it holds if N is equivalent to a small category. It should be kept in mind that the condition depends both on the choice of the basic universe $\mathcal{U}$ and on the choice of the basic localizer $\mathcal{W}$ in the corresponding large category (Cat). It looks pretty sure that the condition is not always satisfied (I doubt it is for $N = (\mathrm{Sets})$ say), but I confess I didn't sit down to make an explicit example.

Let now (M, M_{c}) be a contractibility structure, such that there exists a *small* full subcategory C of M which generates the contractibility structure, i.e., such that (a) the objects of C are "contractible", i.e., are in M_{c} and (b) any object of M_{c} is C-contractible (i.e., contractible for the homotopy interval structure admitting the intervals made up with objects of C as a generating family). Independently of any smallness assumption upon C, we gave in section 51 D) (p. 118), under the name of "basic assumption" (Bas 4), the n.s. condition on a full subcategory C of a given category M, in order that C can be viewed as a generating set of contractible objects, for a suitable contractibility structure [p. 248]

$$M_{\mathrm{c}} \subset \mathrm{Ob}\, M$$

in M (which is uniquely defined by C). It turns out that in case C contains a final object of M, and is stable under binary products in M, the condition (Bas 4) depends only upon the category structure of C, and not upon the particular way this category is embedded in another one M.

In loc. sit. we did not impose, when defining a contractibility structure, a condition that there should exist a small set of generators for the structure. From now on, we'll assume that the (possibly large) categories M we are working with are $\mathcal{U}$-categories, and that "contractibility structure" means "$\mathcal{U}$-contractibility structure", namely one such that M admit a small full subcategory C, generating the structure.

We recall too that in the definition of a contractibility structure (M, M_{c}), it has always been understood that M is stable under finite products.

Proposition 2. *With the conventions above, let (M, M_{c}) be any contractibility structure. Then:*

a) *The full subcategory M_{c} of M generates an asphericity structure M_{as} in M (cf. prop. 1).*

b) *Any small full subcategory C of M which generates the contractibility structure and such that $C\hat{}$ be totally aspheric,* generates *the asphericity structure M_{as}, i.e., the inclusion functor $i : C \to M$ is M_{as}-aspheric.*

c) *The asphericity structure M_{as} is totally aspheric (cf. prop. 5, p. 239).*

The first statement a) can be rephrased, by saying that the category M_{c} of contractible objects of M is an *aspherator*. To prove this, we use the fact that M_{c} is stable under finite products, which implies that we can find a small full subcategory C of M_{c}, stable under such products, and which generates the contractibility structure. The stability condition upon C implies that $C\hat{}$ is totally aspheric. Therefore, a) and b) will be proved, if we prove that any small full subcategory C of M_{c}, such that $C\hat{}$ is totally aspheric, satisfies the conditions of prof. 1 (with $N = M_{\mathrm{c}}$), namely that for any x in M_{c}, $i^*(x)$ is an aspheric object of $C\hat{}$, where [p. 249]

$i : C \to M$ (or $C \to M_c$, equivalently) is the inclusion functor. But from the fact that i^* commutes with finite products, it follows that $i^*(x)$ is contractible in $C^\wedge$, for the homotopy interval structure admitting as a generating family of homotopy intervals the intervals in $C^\wedge$ made up with objects of C. The assumption of total asphericity upon $C^\wedge$ implies that these homotopy intervals are aspheric over $e_{C^\wedge}$, and from this follows (as already used a number of times earlier) that $i^*(x)$ too is aspheric over $e_{C^\wedge}$, and hence aspheric as $e_{C^\wedge}$ is aspheric (because of the assumption of total asphericity). This proves a) and b), and c) follows via prop. 5 (p. 239).

We'll call of course the asphericity structure described in prop. 2 the *asphericity structure generated by the given contractibility structure*.

Remark 1. Clearly, not every asphericity structure can be generated by a contractibility structure, as a necessary condition (presumably not a sufficient one) is total asphericity. We don't expect either, in case it can, that the generating contractibility structure is uniquely defined; however, we do expect in this case that there should exist a canonical (largest) choice – we'll have to come back upon this in due course. For the time being, let's only remark that all modelizing asphericity structures met with so far, it seems, do come from contractibility structures.

It occurs to me that the last statement was a little hasty – after all we have met with test categories which are not strict ones, hence $(A^\wedge, A^\wedge_{\mathrm{as}})$ is a modelizing asphericity structure (it would be enough even that A be a pseudo-test category), which isn't totally aspheric, and a fortiori
[p. 250] does not come from a contractibility structure. Thus, I better correct the statement, to the effect that, it seems, all modelizing *totally aspheric* asphericity structures met with so far are generated by suitable contractibility structures.

Let M be a category endowed with a contractibility structure M_c, hence also with an asphericity structure M_{as}, and let

$$i : A \to M$$

be a functor from a small category A to M. We want to give n.s. conditions for i to be a *test functor*, in terms of homotopy notions in M and in $A^\wedge$. To this end, it seems necessary to refresh memory somewhat and recall some relevant notions which were developed in part III of our notes (sections 54 and 55).

It will be convenient to call an object F of $A^\wedge$ *locally aspheric* (resp. *totally aspheric*), if its product in $A^\wedge$ by any object of A, and hence also its product by any aspheric object of $A^\wedge$, is aspheric (resp. and if moreover F itself is aspheric). With this terminology, $A^\wedge$ is totally aspheric iff every aspheric object of $A^\wedge$ is totally aspheric, and if moreover A is aspheric, i.e., $e_{A^\wedge}$ is aspheric. Note that F is locally aspheric iff the map $F \to e_{A^\wedge}$ is aspheric, or what amounts to the same, if this map is universally in $\mathcal{W}_A$, or equivalently, if the corresponding functor $A_{/F} \to A$ is aspheric. If A itself is aspheric, and in this case only, this condition implies already that F is aspheric, i.e., that F is totally aspheric. We'll denote by

$$A^{\wedge}_{\text{tot.as}} \quad (\text{resp.} \quad A^{\wedge}_{\text{loc.as}})$$

the full subcategory of $A^{\wedge}$ made up with the totally aspheric (resp. locally aspheric) objects of $A^{\wedge}$. Thus we get

$$\begin{tikzcd}[column sep=small] & A^{\wedge}_{\text{tot.as}} = A^{\wedge}_{\text{loc.as}} \cap A^{\wedge}_{\text{as}} \arrow[dl, hook] \arrow[dr, hook] & \\ A^{\wedge}_{\text{loc.as}} & & A^{\wedge}_{\text{as}} \end{tikzcd} .$$

Now recall that (section 54) in terms of the set $\mathcal{W}_A$ of weak equivalences in $A^{\wedge}$, we constructed a homotopy structure on $A^{\wedge}$, more specifically a homotopy interval structure, admitting as a generating family of homotopy intervals

$$\mathbb{I} = (I, \delta_0, \delta_1)$$

the set of intervals such that I be a locally aspheric object of $A^{\wedge}$, i.e.,

$$I \in \operatorname{Ob} A^{\wedge}_{\text{loc.as}}.$$

Let [p. 251]

$$h = h_{\mathcal{W}_A}$$

be this homotopy structure, hence a corresponding notion of *h-equivalence* or *h-homotopy* $\underset{h}{\sim}$ for arrows in $A^{\wedge}$, a corresponding notion of h-homotopisms, i.e., a set of arrows

$$W^h \subset \operatorname{Fl} A^{\wedge}, \quad \text{such that} \quad W^h \subset \mathcal{W}_A,$$

a notion of h-homotopy interval (namely an interval such that δ_0 and δ_1 be h-homotopic, for which it is sufficient, but not necessary, that I be locally aspheric...), and last not least, a notion of *contractible objects*, making up a full subcategory

$$A^{\wedge}_{\text{c}} \subset A^{\wedge}, \quad \text{such that} \quad A^{\wedge}_{\text{c}} \subset A^{\wedge}_{\text{loc.as}}.$$

The latter inclusion, in case A is aspheric, can be equally written

$$A^{\wedge}_{\text{c}} \subset A^{\wedge}_{\text{tot.as}} \quad (\text{if } A \text{ aspheric}).$$

Coming back now to the contractibility structure (M, M_{c}), and a functor $i : A \to M$, we are interested in the corresponding functor

$$u = i^* : M \to M' = A^{\wedge},$$

where both members will be viewed as being endowed with their respective homotopy structures – the one of M being of the most restrictive type envisioned in section 51, namely it is defined in terms of a contractibility structure, whereas the one of M' is a priori defined in terms of a homotopy interval structure, but not necessarily in terms of a contractibility structure. Now this situation has been described in section 53, as far as compatibility conditions with homotopy structures are concerned, independently by the way of any special assumption on M' (such as being a category of presheaves on some small category A), or on the functor u, except for commuting to finite products. Compatibility of u with the homotopy structures on M, M' can be expressed by either one of the following six conditions, which are all equivalent:

H 1) u transforms homotopic arrows of M into homotopic arrows in M'.

H 2) u transforms homotopisms in M into homotopisms in M'.

H 3) u transforms any homotopy interval $\mathbb{I} = (I, \delta_0, \delta_1)$ in M into a homotopy interval in M' (i.e., if two sections of an object I over e_M are homotopic, so are $u(\delta_0)$ and $u(\delta_1)$).

H 3') Same as H 3), but $\mathbb{I}$ being restricted to be in a given family of homotopy intervals, generating for the homotopy structure in M.

[p. 252] H 4) u transforms any contractible object x of M into a contractible object of M'.

H 4') Same as H 4), with x being restricted to be in a given subcategory C of M, generating for the contractibility structure of M.

Remark 2. It should be noted that the condition upon C stated in H 4'), namely that the (given) C should be generating for the contractibility structure M_c of M, means exactly two things: (a) $C \subset M_c$, and (b) the family of all intervals $\mathbb{I} = (I, \delta_0, \delta_1)$ made up with objects of C (these intervals are necessarily homotopy intervals for the homotopy structure of M) *generates* the homotopy structure of M, i.e., two arrows in M are homotopic iff they can be joined by a chain of arrows, two consecutive among which being related by an elementary homotopy involving an interval of that family. This reminder being made, it follows that C is equally eligible for applying criterion H 3'), which means that we get still another equivalent formulation of H 4'), by demanding merely that for any two sections δ_0, δ_1 of x over e_M, the corresponding sections of $u(x)$ should be homotopic.

After these preliminaries, we can state at last the following generalization of the main result of section 65 (th, 1, p. 176), concerning test functors with values in (Cat):

Theorem 1. *Let (M, M_c) be a contractibility structure (cf. p. 248), C a small full subcategory of M generating the contractibility structure (cf. remark 2 above), A a small category, and $i : A \to M$ a functor, factoring through M_c. We consider M as endowed equally with the asphericity structure M_{as} generated by M_c (cf. prop. 2). Then the following conditions on i are equivalent:*

(i) *The functor $i^* : M \to A^\wedge$ deduced from i is compatible with the homotopy structures on M and on $A^\wedge$ (cf. pages 250–251 for the latter), i.e., i^* satisfies either one of the six equivalent conditions H 1) to H 4') above.*

(ii) *The functor i^* is locally M_{as}-aspheric (def. 2, p. 245), i.e., for any x in M_{as}, $i^*(x)$ is a locally aspheric object of $A^\wedge$ (cf. p. 250), i.e., $i^*(x)$ is aspheric over $e_{A^\wedge}$.*

(iii) *Same as (ii), but with x restricted to be in C.*

Corollary 1. *In order for i to be totally M_{as}-aspheric (def. 2, p. 245) it is n.s. that A be aspheric, and that the equivalent conditions of th. 1 be satisfied.* [p. 253]

Indeed, it follows at once from the definitions and from the fact that the asphericity structure (M, M_{as}) is aspheric, i.e., that e_M is aspheric, that i is totally M_{as}-aspheric iff it is locally M_{as}-aspheric (i.e., condition (ii)), and if moreover A is aspheric, hence the corollary.

Corollary 2. *In order for i to be a test functor, it is n.s. that A be a test category, that (M, M_{as}) be modelizing, and that the equivalent conditions of theorem 1 be satisfied.*

This follows from cor. 1 and the reformulation of the notion of a test-functor, given p. 245.

This corollary contains the main result of section 65 as a particular case, when taking $M = (\mathrm{Cat})$ with the usual contractibility structure, and $C = \{\Delta_1\}$, except that in loc. sit. we did not have to assume beforehand that A be a test category, but only that A is aspheric: this condition, plus condition (iii) above (namely, $i^*(\Delta_1)$ locally aspheric) implies already that A is a test category. In order to get also this extra result, we state still another corollary:

Corollary 3. *Assume*

$$\mathbb{I} = (I, \delta_0, \delta_1, \mu), \quad \textit{with} \quad I \in \operatorname{Ob} C,$$

is a multiplicative interval *in M, i.e., an interval endowed with a multiplication μ, admitting δ_0 as a left unit and δ_1 as a left zero element (cf. section 49, p. 108 and section 51, p. 120 – where such an interval was provisionally called a "contractor"). Assume moreover that for any x in M_{c}, the two compositions $x \to e_M \overset{\delta_0,\delta_1}{\rightrightarrows} I$ are distinct (which is the case for instance if* $\operatorname{Ker}(\delta_0, \delta_1)$ *exists in M and is a strict initial object $\varnothing_M$ of M (i.e., an initial object such that any map $x \to \varnothing_M$ in M is an isomorphism), and moreover $\varnothing_M \notin M_{\mathrm{c}}$). Then the conditions of th. 1 imply that A is a local test category, and hence a test category provided A is aspheric.*

Indeed, $i^*(\mathbb{I})$ is a multiplicative interval in $A^\wedge$ which is locally aspheric, and (as follows immediately from the assumptions on $\mathbb{I}$) separating – hence $A^\wedge$ is a local test category.

This array of immediate corollaries of th. 1 do convince me that this statement is indeed "the" natural generalization of the "awkward" main result of section 51. All we have to do is to prove theorem 1 then. [p. 254]

The three conditions of theorem 1 can be rewritten simply as

(i) $i^*(C) \subset A^\wedge_{\mathrm{c}}$ (using H 4'),

(ii) $i^*(M_{\mathrm{as}}) \subset A^\wedge_{\mathrm{loc.as}}$,

(iii) $i^*(C) \subset A^\wedge_{\mathrm{loc.as}}$,

and because of

$$C \subset M_{\mathrm{as}}, \quad A^\wedge_{\mathrm{c}} \subset A^\wedge_{\mathrm{loc.as}},$$

it follows tautologically that (i) and (ii) both imply (iii). On the other hand, (iii) ⇒ (i) by criterion H 3'), and the definition of the homotopy interval structure of $A^\wedge$ in terms of $A^\wedge_{\mathrm{loc.as}}$. Thus, we are left with proving (i) ⇒ (ii). But (i) can be rewritten as

$$i^*(M_{\mathrm{c}}) \subset A^\wedge_{\mathrm{c}},$$

which implies

$$(*) \qquad i^*(M_{\mathrm{c}}) \subset A^\wedge_{\mathrm{loc.as}}.$$

That the latter condition implies (ii) now follows from the corollary to the following tautology (with $N = M_{\mathrm{c}}$), which should have been stated as a corollary to prop. 1 above (p. 247):

Lemma. *Let (M, M_{as}) be an asphericity structure, generated by the full subcategory N of M, let A be a small category, and $i : A \to M$ a functor* factoring through N. *Then i is M_{as}-aspheric iff for any x in N, $i^*(x)$ is an aspheric object of $A^\wedge$.*

Corollary. *The functor i is locally M_{as}-aspheric (resp. totally M_{as}-aspheric) iff $i^*(N) \subset A^\wedge_{\mathrm{loc.as}}$ (resp. $i^*(N) \subset A^\wedge_{\mathrm{tot.as}}$).*

[p. 255] **Remark 3.** Assume in theorem 1 that A is totally aspheric. Then condition (ii) just means that i is M_{as}-aspheric, and condition (iii) that $i^*(x)$ is aspheric for any x in C (as we got $A^\wedge_{\mathrm{loc.as}} = A^\wedge_{\mathrm{as}} = A^\wedge_{\mathrm{tot.as}}$). If moreover C satisfies the condition of cor. 2 above, then these conditions imply that A is a strict test category, and if (M, M_{as}) is modelizing, that i is a test functor as stated in corollary 2. These observations sum up the substance of the restatement of the main result of section 51, given in theorem 2 of p. 178, in the present general context.

80

Reminders and questions around canonical modelizers.

In the preceding section, we associated to any contractibility structure M_{c} on a category M, an asphericity structure M_{as} "generated" by the former in a natural sense. It is this possibility of associating (in a topologically meaningful way) an asphericity structure to a given contractibility structure, which singles out the latter structure type, among the three essential distinct "homotopy flavored" kind of structures developed at length in sections 51 and 52, in preference to the two weaker notions of a homotopy interval structure, and of a homotopism structure (or, equivalently, or a "homotopy relation"). The association

$$(*) \qquad M_{\mathrm{c}} \mapsto M_{\mathrm{as}} \quad (\longleftrightarrow \text{corresponding notion of weak equivalence } W_{\mathrm{a}} \text{ in } M)$$

finally carried through in the last section, had been foreshadowed earlier (cf. p. 110 and p. 142), but was pushed off for quite a while, in order to "give precedence" to the other approach in view by then towards more general test functors than before, leading up finally to the "awkward main result" of section 65. In (*), as the asphericity structure M_{as} is totally aspheric and a fortiori aspheric, M_{as} and the corresponding notion of weak equivalence $W_{\mathrm{a}} = W_{M_{\mathrm{c}}}$ (which of course should not be confused with the notion of homotopism associated to M_{c}, giving a considerably smaller set of arrows W_{c}) determine each other mutually. Till the writing up of section 51, it was the aspect "weak equivalence" W_{a} which was in the fore, whereas the conceptually more relevant aspect of "aspheric objects" did not appear in full light before I was through with grinding out the "awkward approach" (cf. p. 188).

It is time now to remember the opposite association

$$(**)\qquad W_{\mathrm{a}} \mapsto \text{homotopy structure } h_{W_{\mathrm{a}}},$$

associating to a notion of weak equivalence in M, i.e., to any saturated subset [p. 256]

$$W_{\mathrm{a}} \subset \mathrm{Fl}(M),$$

a corresponding homotopy structure $h_{W_{\mathrm{a}}}$, a homotopy interval structure as a matter of fact (section 54, p. 131). Here we are primarily interested of course in the case when W_{a} is associated to a given asphericity structure M_{as} in M, which we may as well assume to be aspheric, so to be sure that W_{a} and M_{as} determine each other. I recall that the weak interval structure $h_{W_{\mathrm{a}}}$ can be described by the generating family of homotopy intervals, consisting of all intervals

$$\mathbb{I} = (I, \delta_0, \delta_1)$$

such that $I \to e_M$ be universally in W_{a}, i.e., such that for any object x in M, the projection

$$x \times I \to x$$

be in W_{a}. For pinning down further the exact relationship between contractibility structures (which may be viewed as just special types of homotopy interval structures) and asphericity structures, we are thinking of course more specifically of *totally aspheric* asphericity structures, in view of prop. 2 c) of the preceding section (p. 248). It is immediate in this case that for an object I of M, $I \to e_M$ is in $\mathrm{U}W_{\mathrm{a}}$ (i.e., is universally in W_{a}) if (and only if, of course) I is aspheric, i.e., $I \to e_M$ is in W_{a}. The most relevant questions which come up here, now seem to me the following:

1) If M_{as} is generated by a contractibility structure M_{c}, is the homotopy structure $h_{W_{\mathrm{a}}}$ associated to M_{as} (indeed, a homotopy interval structure as recalled above) also the one defined by M_{c}, using intervals in M_{c} as a generating family of homotopy intervals?

2) Conversely, what extra conditions on a given asphericity structure M_{as} on M are needed (besides total asphericity) to ensure that the corresponding homotopy structure $h_{W_{\mathrm{a}}}$ on M comes from a contractibility structure M_{c} (i.e., admits a generating family of homotopy intervals which are *contractible*), and that moreover M_{c} *generates* M_{as}? [p. 257]

Before looking up a little these questions, I would like however to carry through at once the "idyllic picture" of canonical modelizers, foreshadowed in section 50 (p. 110), as I feel that this should be possible at present at no costs. Taking into account the reflections of the later sections 57 and 59, we get the following set-up.

Let M be a $\mathcal{U}$-category, stable under finite products, endowed with a functor

$$\pi_0^M \text{ or } \pi_0 : M \to (\text{Sets}), \tag{1}$$

on which we make no assumptions for the time being. We are thinking of the example when M is a totally 0-connected category (cf. prop. on page 142 for this notion) and π_0 is the "connected components" functor, or when $M = (\text{Spaces})$ and π_0 corresponds to taking sets of arc-wise connected components. According to section 54 (p. 131), we introduce a corresponding homotopy interval structure h on M, admitting the generating family of homotopy intervals made up with those intervals $\mathbb{I} = (I, \delta_0, \delta_1)$ for which $I \to e_M$ is "universally in W_{π_0}", namly

$$\pi_0(x \times I) \to \pi_0(x) \quad \text{bijective for any } x \text{ in } M.$$

(Under suitable conditions on π_0, this homotopy interval structure h on M is the widest one "compatible with π_0" in the sense of page 130, namely such that π_0 transforms homotopisms into isomorphisms – cf. proposition p. 133.) We are interested in the case when this homotopy structure on M can be described by a contractibility structure M_{c} on M, which is then unique of course, hence well-defined in terms of π_0. Therefore, the asphericity structure generated by M_{c} is equally well defined in terms of π_0, and likewise the corresponding notion W_{a} of "weak equivalence". We then get a canonical functor

$$\mathrm{Hot}_M = W_{\mathrm{a}}^{-1} M \to (\mathrm{Hot}) \tag{2}$$

(section 77, p. 239). We'll have to find still a suitable extra condition on the functor π_0, implying that this functor is canonically isomorphic
[p. 258] deduced from (2) by composing with the canonical functor

$$(\mathrm{Hot}) \xrightarrow{\pi_0} (\text{Sets}), \tag{3}$$

which can be defined using a very mild extra condition on the basic localizer $\mathcal{W}$ (namely, $f \in \mathcal{W}$ implies $\pi_0(f)$ bijective, cf. condition L a) on page 165). Thus, there seems to be a little work ahead after all – in order to deduce something like a one to one correspondence, say, between pairs (M, π_0) satisfying suitable conditions, and certain types of asphericity structures (M, M_{as}) (which will have to be assumed totally aspheric, and presumably a little more still).

The case of special interest to us is the one when the asphericity structure we get on M in terms of the functor π_0^M is *modelizing*, hence even strictly modelizing (i.e., (M, W_{a}) is a strict modelizer), as (M, M_{as}) is totally aspheric. If we assume moreover that the category M is totally

0-connected and that π_0 is just the "connected components" functor, then the modelizing asphericity structure we got on M is canonically determined by the mere category structure of M, and deserves therefore to be called *the canonical* (*modelizing*) *asphericity structure on* M. A *canonical modelizer* (M, W) is a modelizer which can be obtained from a canonical asphericity structure (M, M_{as}) by $W = W_{\mathrm{a}}$ = corresponding set of weak equivalences (for M_{as}).

Remark. The slight sketchy definition we just gave for the canonical modelizing asphericity structures, and accordingly for the canonical modelizers, is essentially complete. The point however which requires clarification is the relationship between the "connected components" functor on the corresponding (totally 0-connected) category M, and the composition of the canonical functors (2) and (3).

25.6. [p. 259]

81 I didn't find much time since Monday for mathematical pondering – the little I got nonetheless has been enough for convincing myself that things came out more nicely still than I expected by then. One main point being that, provided the basic localizer satisfies the mild extra assumption Loc 4) below, any contractibility structure M_{c} on a category M with finite products can be recovered, in the simplest imaginable way, in terms of the associated asphericity structure M_{a}, or equivalently, in terms of the corresponding set W_{a} of "weak equivalences"; namely, M_{c} is the set of contractible objects in M, for the homotopy interval structure defined in terms of all intervals made up with objects of M_{a}. This implies that the canonical map

Contractibility as the common expression of homotopy, asphericity and 0-connectedness notions. (An overall review of the notions met with so far.)

(1) $$\mathrm{Homtp}_4(M) = \mathrm{Cont}(M) \hookrightarrow \mathcal{W}\text{-Asph}(M),$$

from the set of contractibility structures on M to the set of asphericity structures on M relative to the basic localizer $\mathcal{W}$ (or "$\mathcal{W}$-*asphericity structures*"), is *injective*. In other words, we may view a contractibility structure (on a category M stable under finite products), which is an *absolute* notion (namely independent of the choice of a basic localizer $\mathcal{W}$), as a "*particular case*" of a $\mathcal{W}$-asphericity structure (depending on the choice of $\mathcal{W}$), namely as "equivalent" to a $\mathcal{W}$-asphericity structure, satisfying some extra conditions which we'll have to write down below.

This pleasant fact associates immediately with two related ones. The first is just a reminder of our reflections of sections 51 and 52, namely that the set of contractibility structures on M can be viewed as one among four similar sets of "homotopy structures" on M

(2) $$\mathrm{Homtp}_4(M) \hookrightarrow \mathrm{Homtp}_3(M) \hookrightarrow \mathrm{Homtp}_2(M) \xrightarrow{\sim} \mathrm{Homtp}_1(M),$$

corresponding to the four basic "homotopy notions" met with so far, namely (besides contractibility structure Homtp_4) the *homotopy interval structures* (Homtp_3), the *homotopism structures* (Homtp_2), and the *homotopy relations* between maps (Homtp_1). In the sequel, if

(3) $$M_{\mathrm{c}} \subset \mathrm{Ob}\, M$$

is a given contractibility structure on M, we'll denote by

$$(4) \qquad J_{\mathrm{c}} \subset \mathrm{Int}(M), \quad W_{\mathrm{c}} \subset \mathrm{Fl}(M), \quad R_{\mathrm{c}} \subset \mathrm{Fl}(M) \times \mathrm{Fl}(M)$$

the corresponding other three homotopy structures on M, where $\mathrm{Int}(M)$ denotes the set of all "intervals" $\mathbb{I} = (I, \delta_0, \delta_1)$ in M, i.e., objects of M endowed with two sections δ_0, δ_1 over the final object e_M of M.

[p. 260] The second fact alluded to above is concerned with behavior of the $\mathcal{W}$-asphericity notions, for varying $\mathcal{W}$, more specifically for a pair

$$(5) \qquad \mathcal{W} \subset \mathcal{W}' \subset \mathrm{Fl}((\mathrm{Cat}))$$

of two basic localizers, $\mathcal{W}$ and $\mathcal{W}$', such that $\mathcal{W}$ "refines" $\mathcal{W}$'. It then follows that for any small category A, we have

$$(6) \qquad \mathcal{W}_A \subset \mathcal{W}'_A,$$

and accordingly, that *for any $\mathcal{W}$-asphericity structure*

$$(7) \qquad M_{\mathcal{W}} \subset \mathrm{Ob}\, M$$

on M, there exists a unique $\mathcal{W}'$-asphericity structure $M_{\mathcal{W}'}$ on M,

$$(8) \qquad M_{\mathcal{W}} \subset M_{\mathcal{W}'},$$

such that for any small category A, a functor $A \to M$ which is $M_{\mathcal{W}}$-aspheric (*with respect to* $\mathcal{W}$) *is also $M_{\mathcal{W}'}$-aspheric* (*with respect to* $\mathcal{W}'$). This is merely a tautology, which we didn't state earlier, because there was no compelling reason before to look at what happens when $\mathcal{W}$ is allowed to vary. Thus, we get a canonical map

$$(9) \qquad \mathcal{W}\text{-Asph}(M) \to \mathcal{W}'\text{-Asph}(M),$$

with the evident transitivity property for a triple

$$\mathcal{W} \subset \mathcal{W}' \subset \mathcal{W}'',$$

in other words we get a functor

$$\mathcal{W} \mapsto \mathcal{W}\text{-Asph}(M)$$

from the category of all basic localizers (the arrows between localizers being inclusions (5)) to the category of sets. The relation of the canonical inclusion (1) with this functorial dependence of $\mathcal{W}$-Asph(M) on $\mathcal{W}$ is expressed in the commutativity of

$$\begin{tikzcd}[column sep=small] & \mathrm{Homtp}_4(M) = \mathrm{Cont}(M) \arrow[dl, hook] \arrow[dr, hook] & \\ \mathcal{W}\text{-Asph}(M) \arrow[rr] & & \mathcal{W}'\text{-Asph}(M) \end{tikzcd}.$$

It is time to write down the "mild extra assumption" on $\mathcal{W}$ needed to ensure injectivity of (1), namely the familiar enough condition:

Loc 4) The set $\mathcal{W} \subset \mathrm{Fl}((\mathrm{Cat}))$ of weak equivalences in (Cat) is "compatible" with the functor $\pi_0 : (\mathrm{Cat}) \to (\mathrm{Sets})$, i.e.,

$$f \in \mathcal{W} \Rightarrow \pi_0(f) \text{ bijective.} \tag{10}$$

(Cf. pages 213–214 for the conditions Loc 1) to Loc 3).) [p. 261]

Among all basic localizers satisfying this extra condition, there is one coarsest of all, which we'll call $\mathcal{W}_0$, defined by the condition

$$f \in \mathcal{W}_0 \Leftrightarrow \pi_0(f) \text{ bijective.} \tag{11}$$

for any map f in (Cat). It is clear too that among all basic localizers there is a finest, which we'll call $\mathcal{W}_\infty$, and which can be described as

$$\mathcal{W}_\infty = \bigcap \mathcal{W}, \quad \text{intersection of the set of all basic localizers in (Cat).} \tag{12}$$

We'll see in part V of the notes that $\mathcal{W}_\infty$ is none else than just the usual notion of weak equivalence we started with, at the very beginning of our reflections (cf. section 17). Thus, functoriality of (1) with respect to $\mathcal{W}$ implies that (1) for arbitrary $\mathcal{W}$ can be described in terms of the particular case $\mathcal{W}_\infty$, as the composition [see also Cisinski 2004]

$$\mathrm{Cont}(M) \hookrightarrow \mathcal{W}_\infty\text{-}\mathrm{Asph}(M) \to \mathcal{W}\text{-}\mathrm{Asph}(M). \tag{13}$$

On the other hand, the strongest version of injectivity of (1), for different $\mathcal{W}$'s, is obtained for $\mathcal{W}_0$, i.e., taking the map

$$\mathrm{Cont}(M) \hookrightarrow \mathcal{W}_0\text{-}\mathrm{Asph}(M). \tag{14}$$

This last map seems to me of special significance, because the two sets it relates correspond to "*absolute*" notions (not depending on the choice of some $\mathcal{W}$), and which moreover are both "*elementary*", in the sense that they do not depend on anything like consideration of non-trivial homotopy or (co)homology invariants of objects of (Cat). As a matter of fact, the notion of a contractibility structure corresponds to the algebraic translation of one of the most elementary and intuitive topological notions, namely contractibility; whereas the notion of a $\mathcal{W}_0$-asphericity structure can be expressed just as "elementarily" in terms of the functor

$$\pi_0 : (\mathrm{Cat}) \to (\mathrm{Sets}), \tag{15}$$

which we may call the "basic" connected components-functor, which is nothing but the algebraic counterpart of the basic intuitive notion of connected components of a space. We'll denote by

$$M_0 = M_{\mathcal{W}_0} \subset \mathrm{Ob}\, M, \quad W_0 \subset \mathrm{Fl}(M) \tag{16}$$

the set of $\mathcal{W}_0$-aspheric objects and the set of $\mathcal{W}_0$-weak equivalences, associated to a given contractibility structure M_c on M. The objects of M_0 merit the name of 0-*connected objects* of M (with respect to M_c), and [p. 262]

26.6. This name for *maps* is improper though, as it rather suggests the property of being "universally in W_0".

the arrows of M in W_0 merit the name of 0-*connected maps* (with respect to M_c). These two notions of 0-connectedness determine each other in an evident way (valid for any $\mathcal{W}$-aspheric $\mathcal{W}$-asphericity structure, for any basic localizer $\mathcal{W}\dots$). Explicitly, this can be expressed by

$$(17)\qquad \begin{cases} x \in M_0 \Leftrightarrow \big((x \to e_M) \in W_0\big) \\ (f : x \to y) \in W_0 \Leftrightarrow \big(\pi_0(M_{0/x}) \to \pi_0(M_{0/y}) \text{ bijective}\big). \end{cases}$$

The injectivity of (1) can be restated by saying that *any contractibility structure M_c on a category M* (with finite products) *can be recovered in terms of the corresponding notion of* 0-*connected objected objects of M*, or, equivalently, in terms of the corresponding notion of 0-connected maps in M. Still another way of phrasing this result, is in terms of the canonical functor

$$M \to W_0^{-1}M = \mathrm{Hot}_{(M,M_0),\mathcal{W}_0} \to \mathrm{Hot}_{\mathcal{W}_0} = \mathcal{W}_0{}^{-1}(\mathrm{Cat}) \xrightarrow{\approx} (\mathrm{Sets}),$$

which is a functor

$$(18)\qquad \pi_0 : M \to (\mathrm{Sets})$$

canonically associated to the contractibility structure. We may say that *the contractibility structure M_c can be recovered in terms of the corresponding functor π_0*, more accurately still, in terms of the isomorphism class of the latter. Indeed, in terms of this functor π_0, we recover M_0 and W_0 by the relations:

$$(19)\qquad \begin{aligned} M_0 &= \big\{x \in \mathrm{Ob}\, M \;\big|\; \pi_0(x) \text{ is a one-point set}\big\}, \\ W_0 &= \big\{f \in \mathrm{Fl}(M) \;\big|\; \pi_0(f) \text{ is bijective}\big\}. \end{aligned}$$

This shows that the "nice" main fact mentioned at the beginning of today's notes, namely (essentially) injectivity of (1) (and, moreover an explicit description of a way how to recover an M_c in terms of the corresponding $\mathcal{W}$-asphericity structure) is not really dependent on relatively sophisticated notions such as "basic localizers" and corresponding "asphericity structures", but can be viewed as an "elementary" result (namely independent of any consideration of "higher" homotopy or homology invariants, apart from π_0) about the relationship between contractibility structures, and corresponding 0-connectedness notions; the latter may at will be expressed in terms of either one of the three structural data

$$(20)\qquad M_0, \quad W_0, \quad \text{or} \quad \pi_0.$$

[p. 263] This relationship has been "in the air" since section 50 (p. 109–110), and I kind of turned around it consistently up to section 60, without really getting to the core. One reason for this "turning around" has been, I guess, that I let myself be distracted, not to say hypnotized, by the "canonical" π_0 functor on a category M (which makes really good sense only when M is "totally 0-connected" as a category, a condition which should mean, more or less I suppose, that the 0-connected objects of M, defined in terms of the mere category structure of M, define a $\mathcal{W}_0$-asphericity structure on M). Even after realizing (in section 59) that one should generalize the description of a contractibility structure in terms of a "connected components functor" π_0, to the case of a functor $\pi_0 : M \to (\text{Sets})$ given beforehand, and satisfying suitable restrictions (which I did not try to elucidate), I still was under the impression that the contractibility structures one could get this way must be of an extremely special nature. In order to become aware of the fact that this is by no means so, namely that any contractibility structure could be obtained from a suitable functor π_0, it would have been necessary to notice that such a structure M_c defines in a natural way a functor π_0. There was indeed the realization that M_c should allow to define a notion of weak equivalence (cf. page 136), but it wasn't clearly realized by then that at the same time as a notion of weak equivalence W in M, we should also get a canonical functor

$$M \to W^{-1}M \to (\text{Hot}),$$

namely something a lot more precise still than a functor with values in (Sets) merely! But rather than push ahead in this direction, I then decided (p. 138) that it would be "unreasonable" to go on still longer pushing of investigation of test functors with values in (Cat), following the approach which had been on my mind for quite a while by then, and finally sketched (with the promise of a corresponding generalization of the former "key result" on test functors) in section 47. Retrospectively, the whole "grinding" part III of these notes now looks as a rather heavy and long-winded digression, prompted by this approach to still a particular case of test functors (namely with values in (Cat), and more stringently still, factoring through $(\text{Cat})_\text{as}$).

This particular case has been of no use in the present part IV of the notes, developing the really relevant notions in terms of asphericity structures. *Technically speaking, it now appears that most of the reflections of part III are superseded by part IV* – the main exception being the development of the various homotopy notions in sections 51 and 52. [p. 264] On the other hand, it is clear that the main ideas which are coming to fruition in part IV all originated during the awkward grinding process in part III!

The "modelizing story" so far has turned out as the interplay of three main sets of notions. One is made up with the "*test-notions*", centering around the notion of a *test-category*, as one giving rise to the most elementary type of "*modelizers*", namely the so-called "*elementary modelizers*" $(A\hat{\ }, \mathcal{W}_A)$. This was developed in part II (while part I was concerned with the initial motivation of the reflections, namely stacks, forgotten for the time being!). The second set of notions concerns the so-called "*homotopy notions*", developed at some length in sections 51 to 55, summarized in the diagram (2) above. They constitute the main technical content of part III of the notes, with however one major shortcoming: the relationship between these notions, and 0-connectedness notions

(20), was only partially understood in part III, namely as a one-way relationship merely, associating to suitable 0-connectedness notions in a category M, a corresponding homotopy structure in M. The third set of notions may be called "*asphericity notions*", they center around the notions of *aspheric objects* and *aspheric maps* (in a category endowed with a so-called *asphericity structure*), and more specifically around the notion of an aspheric map in (Cat), whose formal properties turn out to be the key for the development of a theory of asphericity structures. The first and the third set of notions (namely test notions and asphericity notions) depend on the choice of a "*basic localizer*" $\mathcal{W}$ in (Cat), whereas the second set, namely homotopy notions, is "absolute", i.e., does not depend on any such choice, nor on any knowledge of homotopy or (co)homology invariants.

Whereas the test notions are essentially concerned with modelizers, namely getting descriptions of the category of homotopy types (Hot) in terms of elementary modelizers $A\hat{}$ (as being ${\mathcal{W}_A}^{-1}A\hat{}$), it appears that the homotopy notions, as well as the asphericity notions, are independent of any modelizing notions and assumptions. In a deductive presentation of the theory, the test notions would come last, whereas they came first in these notes – as an illustration of the general fact that the deductive approach will present things roughly in opposite order in which they have been discovered! The test notions, as an outcome of the attempt to get a picture of modelizers, have kept acting as a constant guideline in the whole reflection, even though technically speaking they
[p. 265] are "irrelevant" for the development of the main properties of homotopy and asphericity notions and their interplay.

By the end of part IV, there has been some floating in my mind as to whether which among the two structures, namely contractibility structures or asphericity structures, should be considered as "the" key structure for an understanding of the modelizing story. There was a (justified) feeling, expressed first at the beginning of section 67, that in some sense, asphericity structures were "more general" than contractibility structures, which caused me for a while to view them as the more "basic" ones. I would be more tempted at present to hold the opposite view. *The notion of a contractibility structure now appears as a kind of hinge between the two main sets of notions besides the test notions, namely between homotopy notions and asphericity notions.* On the one hand, as displayed in diagram (2), the notion of a contractibility structure appears as the most stringent one among the four main types of homotopy structures. On the other hand, by (1) it can be equally viewed as being a special case of an asphericity structure, and as such it gives rise to (and can be expressed by) either one of the following four asphericity-flavored data on a category M (for any given basic localizer $\mathcal{W}$ satisfying Loc 4)):

[also listed, but scratched out in the typescript, was $\pi_0 : M \to (\mathrm{Sets})$]

$$(21)\qquad \begin{cases} M_{\mathcal{W}} \subset \operatorname{Ob} M, & \mathcal{W}_{M_c} \text{ or simply } \mathcal{W}_M \subset \operatorname{Fl}(M) \\ \varphi_M^{\mathcal{W}} \text{ or } \varphi_M : M \to \mathrm{Hot}_{\mathcal{W}}. \end{cases}$$

Restricting to the case when $\mathcal{W}$ is either $\mathcal{W}_0$ or $\mathcal{W}_\infty$, we get the corresponding structures on M, namely the three structures M_0, W_0, π_0 of (20) plus the three extra structures:

$$M_\infty = M_{\mathcal{W}_\infty} \subset \operatorname{Ob} M, \quad W_\infty = (\mathcal{W}_\infty)_M \subset \operatorname{Fl}(M),$$
$$\varphi_M : M \to (\mathrm{Hot}) = \mathcal{W}_\infty{}^{-1}(\mathrm{Cat}), \tag{22}$$

which can be referred to as ∞-*connected objects* of M, ∞-*connected arrows* of M, and the "*canonical functor*" from M to (Hot). Putting together (4), (20), (22), we see that a contractibility structure M_c gives rise to ten different structures (including M_c itself) and is determined by each one of these, [p. 266]

(23)
$$\begin{cases} M_\mathrm{c} \subset \operatorname{Ob} M, & J_\mathrm{c} \subset \operatorname{Int}(M), \quad W_\mathrm{c} \subset \operatorname{Fl}(M), \quad R_\mathrm{c} \subset \operatorname{Fl}(M) \times \operatorname{Fl}(M) \\ M_0 \subset \operatorname{Ob} M, & \qquad\qquad\qquad\quad W_0 \subset \operatorname{Fl}(M), \quad \pi_0 : M \to (\mathrm{Sets}) \\ M_\infty \subset \operatorname{Ob} M, & \qquad\qquad\qquad\quad W_\infty \subset \operatorname{Fl}(M), \quad \varphi_M : M \to (\mathrm{Hot}), \end{cases}$$

namely: *contractible objects*, *homotopy intervals*, *homotopisms*, *homotopy relation for maps*, 0-*connected objects*, 0-*connected maps*,[†] the *connected components functor*, ∞-*connected* (or "aspheric", more accurately $\mathcal{W}_\infty$-aspheric) *objects*, $\mathcal{W}_\infty$-*equivalences* (or simply "weak equivalences"), and the (would-be "modelizing") *canonical functor from M to homotopy types*. Moreover, it should be remembered that, just as the structures J_c, W_c, R_c are by no means unrestricted (the fact that they stem from a contractibility structure being a substantial restriction), the two asphericity structures (namely, the 0-asphericity structure M_0 and the ∞-asphericity structure M_∞) are subject to extra conditions which will be written down below, implying among others that they are totally aspheric (hence π_0 and φ_M are compatible with finite products).

[†] This name is inadequate, cf. note p. 262.

We may complement the ten structure data (23) above, by the following two

$$R_0, R_\infty \subset \operatorname{Fl}(M) \times \operatorname{Fl}(M), \tag{24}$$

defined by

$$(f,g) \in R_0 \Leftrightarrow \big(\pi_0(f) = \pi_0(g)\big), \quad (f,g) \in R_\infty \Leftrightarrow \big(\varphi_M(f) = \varphi_M(g)\big).$$

More generally, for any basic localizer $\mathcal{W}$, we may define

$$R_\mathcal{W} \subset \operatorname{Fl}(M) \times \operatorname{Fl}(M), \quad (f,g) \in R_\mathcal{W} \Leftrightarrow \big(\varphi_M^\mathcal{W}(f) = \varphi_M^\mathcal{W}(g)\big). \tag{25}$$

It is not clear however that M_c, or equivalently $M_\mathcal{W}$ or $\mathcal{W}_M$, can be recovered in terms of the equivalence relation $R_\mathcal{W}$ among maps of M.

Among the three series of structures appearing in (23), we have the tautological relations

$$\begin{cases} M_\mathrm{c} \subset M_\infty \subset M_0 \\ W_\mathrm{c} \subset W_\infty \subset W_0 \\ R_\mathrm{c} \subset R_\infty \subset R_0 \\ \begin{tikzcd} M \arrow[r, "\varphi_M"] \arrow[rr, bend right, "\pi_0"'] & (\mathrm{Hot}) \arrow[r, "\overline{\pi_0}"] & (\mathrm{Sets}) \end{tikzcd} \quad \text{(commutative).} \end{cases} \tag{26}$$

Finally, we may also display some of the main functors defined in terms of a given contractibility structure M_{c}:

$$
(27)\qquad
\begin{array}{ccccccccc}
M & \longrightarrow & \overline{M} & \longrightarrow & \mathrm{Hot}_M & \longrightarrow & (\mathrm{Hot}) & \to & \mathrm{Hot}_{\mathcal{W}} & \to & (\mathrm{Sets}) \ . \\
\| & & \| & & \| & & & & \wr\wr \\
W_{\mathrm{c}}^{-1}M & & W_{\infty}^{-1}M & & \mathrm{Hot}_{\mathcal{W}_\infty} & & & & \mathrm{Hot}_{\mathcal{W}_0}
\end{array}
$$

[p. 267] 26.6.

82 Yesterday I have been busy mainly with the readjustment of the overall perspective on the main notions developed so far, which has sprung from the new fact stated at the beginning of yesterday's notes: namely that an arbitrary contractibility structure M_{c} (on a category M stable under finite products) can be recovered in terms of the associated $\mathcal{W}$-asphericity structure, where $\mathcal{W}$ is any basic localizer satisfying (besides the condition Loc 1) to Loc 3) of p. 213–214) the extra assumption Loc 4) of p. 260. It seems about time now to enter into a little more technical specifications along the same lines – and to start with, give a proof of the "new fact"! Let's state it again in full:

Proof of injectivity of $\alpha : \mathrm{Contr}(M) \hookrightarrow \mathcal{W}\text{-}\mathrm{Asph}(M)$. Application to **Hom** *objects and to products of aspheric functors $A \to M$.*

Theorem 1. *Let M be a $\mathcal{U}$-category stable under finite products, $M_{\mathrm{c}} \subset \mathrm{Ob}\,M$ a contractibility structure on M, admitting a small full subcategory C which generates the structure. Let moreover $\mathcal{W}$ be any basic localizer satisfying* Loc 4) *(compatibility with the π_0-functor* $(\mathrm{Cat}) \to (\mathrm{Sets})$*), and let $M_{\mathcal{W}} \subset \mathrm{Ob}\,M$ the $\mathcal{W}$-asphericity structure generated by M_{c}, $W = \mathcal{W}_{M_{\mathrm{c}}}$ the corresponding set of "$\mathcal{W}$-equivalences" or "weak equivalences" in* $\mathrm{Fl}(M)$*. Consider the homotopy structure h_W associated to W, i.e.* (*cf. section 54*)*, the homotopy structure associated to the homotopy interval structure J generated by the set J_0 of all intervals*

$$\mathbb{I} = (I, \delta_0, \delta_1)$$

in M such that

$$I \in M_{\mathcal{W}}.$$

Then h_W is the homotopy structure on M associated to the contractibility structure M_{c}, and hence M_{c} can be described in terms of $M_{\mathcal{W}}$ (or of $\mathcal{W}_{M_{\mathrm{c}}} = W$) as the set of objects which are contractible for h_W, i.e., such that the map $x \to e_M$ is an h_W-homotopism.

Proof. Let M_{c}' be the set of h_W-contractible objects of M, clearly we have

$$M_{\mathrm{c}} \subset M_{\mathrm{c}}' \subset M_{\mathcal{W}} \overset{\mathrm{def}}{=} M_{\mathrm{a}}.$$

The theorem amounts to saying that M_{c} generates the homotopy interval structure J (by which we mean that the set of intervals of M made up with objects of M_{c} generates the structure J). Indeed, because of $M_{\mathrm{c}} \subset M_{\mathrm{c}}'$, this will imply that J is associated to a contractibility structure, namely to M_{c}'. But for an object x of M to be in M_{c}', i.e., to be contractible for the structure J, amounts to be contractible for M_{c}, and hence by saturation of M_c, to be in M_c, hence $M'_c = M_c$ – which yields what we want. By the description of J in terms of J_0, we are now reduced to proving the following [p. 268]

Lemma 1. *Let I be an object of $M_a = M_{\mathcal{W}}$. Then any two sections of I (over e_M) are M_c-homotopic.*

Let, as in the previous section, $\mathcal{W}_0 \supset \mathcal{W}$ be the largest of all basic localizers satisfying Loc 4), i.e.,

$$\mathcal{W}_0 = \left\{ f \in \mathrm{Fl}((\mathrm{Cat})) \;\middle|\; \pi_0(f) \text{ a bijection} \right\},$$

therefore, we have

$$M_{\mathcal{W}} \subset M_{\mathcal{W}_0} \stackrel{\text{def}}{=} M_0,$$

and we are reduced to proving the lemma for $\mathcal{W}_0$ instead of $\mathcal{W}$, i.e., for M_0 instead of $M_{\mathcal{W}}$. We'll use the small full subcategory C of M_c generating the contractibility structure M_c, we may assume that C is stable under finite products. Hence $C^\wedge$ is totally $\mathcal{W}_0$-connected, i.e., totally 0-connected. Moreover, as $C \subset M_c$, and e_M is in C, it follows that every object of C has a section (over $e_M = e_C$) – which implies that every non-empty object of $C^\wedge$ has a section, i.e., $C^\wedge$ is "strictly totally 0-connected" (cf. p. 144 and 149). Note that (by prop. 2 b) of p. 248) we have

$$M_0 = \left\{ x \in \mathrm{Ob}\, M \;\middle|\; i^*(x) \text{ is 0-connected in } C^\wedge \right\},$$

where i is the inclusion functor:

$$i : C \to M.$$

We have to prove that any two sections δ_0, δ_1 of an object I of M_0 are M_c-homotopic, or what amounts to the same, C-homotopic. This translates readily into the statement that $i^*(\delta_0)$ and $i^*(\delta_1)$ are C-homotopic in $C^\wedge$. Thus, we are reduced to the following lemma (in the case of the topos $C^\wedge$):

Lemma 2. *Let $\mathcal{C}$ be a totally 0-connected topos such that any non-empty object of C has a section, and let C be a small full generating subcategory, whose elements are 0-connected. Then for any 0-connected object I of $\mathcal{C}$, and any two sections δ_0, δ_1 of I, these are C-homotopic, i.e., they can be joined by a finite chain of sections, any two consecutive among which can be obtained as the images of two sections s_i, t_i of an objects x_i of C, by means of a map $h_i : x_i \to I$.*

This lemma is essentially a restatement (cleaned from extraneous assumptions due to an awkward conceptual background) of the proposition of page 149 (section 60), and the proof will be left to the reader.

[p. 269] **Application to relation between contractibility and objects $\mathbf{Hom}(X,Y)$.** I would like to review here a few things along these lines, which were somewhat scattered in the notes before (section 51 E) p. 121 and section 57 p. 143 notably), and at times came out awkwardly because of inadequate conceptual background. We assume M endowed with a contractibility structure M_c, and use the notations of the previous section, especially concerning the subsets of $\operatorname{Ob} M$ and of $\operatorname{Fl}(M)$

(1) $$M_c \subset M_\infty \subset M_{\mathcal{W}} \subset M_0, \quad W_c \subset W_\infty \subset \mathcal{W}_M \subset W_0.$$

Let X be an object of M, such that the object

(2) $$I = \mathbf{Hom}(X,X)$$

exists in M (NB a priori it is an object in $M^\wedge$, we assume it to be representable). We assume that X is endowed with a section c, hence a section δ_1 of I, corresponding to the constant endomorphism of X with value c. We'll denote by δ_0 the section of I corresponding to the identity of X. Thus,

(3) $$\mathbb{I} = (I, \delta_0, \delta_1)$$

becomes an interval of M, and the composition law of $I = \mathbf{Hom}(X,X)$ turns it into a *multiplicative interval*, admitting respectively δ_0 and δ_1 as left unit and left zero element. Then the following conditions are equivalent:

[in the typescript we have π_0 and π_1 as the elements; I assume this is a typo?]

(i) X is contractible.

(ii) $\mathbf{Hom}(X,X)$ is contractible, i.e., $I \in M_c$.

(ii') $\mathbf{Hom}(X,X)$ is 0-connected, i.e., $I \in M_0$.

(ii") The two sections δ_0, δ_1 of $I = \mathbf{Hom}(X,X)$ are homotopic.

(iii) For any object Y in M such that $\mathbf{Hom}(Y,X)$ exists in M, $\mathbf{Hom}(Y,X)$ is contractible.

(iii') For any Y as in (iii), $\mathbf{Hom}(Y,X)$ is 0-connected.

(iv) For any Y in M such that $\mathbf{Hom}(X,Y)$ exists in M, the canonical map

(4) $$Y \to \mathbf{Hom}(X,Y)$$

is a homotopism (more accurately still, Y as a subobject of $\mathbf{Hom}(X,Y)$ is a deformation retract). Moreover, X is 0-connected.

(iv') For any Y in M as in (iv), the map (4) induces a bijection

$$\pi_0(Y) \to \pi_0(\mathbf{Hom}(X,Y)),$$

moreover X is 0-connected.

[p. 270] NB Of course, the 0-connectedness notion and the functor π_0 used here are those associated to the given structure M_c. The case dealt with in section 57 (p. 143) is essentially (it seems) the one when these notions are the ones canonically defined in terms of the category structure of M alone. In view of the inclusions (1), we could throw in a handful more obviously equivalent conditions, using M_∞, $M_\mathcal{W}$ or W_∞, $\mathcal{W}_M$, instead of M_c, etc. but it seems this would confuse the picture rather than complete it.

Next thing is to look at the canonical map

(5) $$\overline{\Gamma}(X) \to \pi_0(X)$$

defined for any object X of M, where $\Gamma(X)$ denotes the set of sections of X, and $\overline{\Gamma}(X)$ denotes the set of corresponding homotopy classes of sections. (This map was considered in a slightly different case on page. 144) The map (5) can be viewed as the particular case of

$$\overline{\mathrm{Hom}}(Y,X) \to \mathrm{Hom}_{\mathrm{Sets}}(\pi_0(Y), \pi_0(X)),$$

obtained when $Y = e_M$, hence $\pi_0(Y) =$ one-point set. By the standard description

(6) $$\pi_0(X) \simeq \boldsymbol{\pi}_0(C_{/X}) \quad (= \pi_0(i^*(X)))$$

we immediately get that (5) is always surjective, as any element in $\pi_0(X)$ is induced by an element $x \to X$ of $C_{/X}$, hence by a section

$$e_M \to x \to X,$$

where $e_M \to x$ is a section of the (contractible) object x in C. But (5) is in fact bijective. To see this, let's note that the map (5) is isomorphic to the corresponding map in (Sets), with M replaced by $C\hat{\ }$ and X by $i^*(X)$, C remaining the same, as follows from (6) and the formula

$$\Gamma(X) \simeq \overline{\Gamma}(i^*(X)).$$

The latter is a particular case of the

Lemma 3. *Let Y,X be two objects of M, with Y in C. Then the natural map*

(7) $$\overline{\mathrm{Hom}}(Y,X) \to \overline{\mathrm{Hom}}(i^*(Y), i^*(X))$$

between sets of homotopy classes of maps (in M and in $C\hat{\ }$ respectively, the latter being endowed with the contractibility structure generated by C) is bijective.

The verification is tautological, due to the fact that for any object I of C (which is going to play the role of a homotopy interval for deformations), we got

$$\mathrm{Hom}_M(I \times Y, X) \xrightarrow{\sim} \mathrm{Hom}_{C\hat{\ }}(I \times Y, i^*(X)).$$

Now, the map (5) in case of a strictly totally 0-connected category (here $C\hat{\ }$) has been dealt with in section 58 (p. 114), it follows easily that the map is bijective – hence the [p. 271]

Proposition 1. *Let (M, M_c) be any contractibility structure,*

$$\pi_0 : M \to (\text{Sets}), \quad \pi_0(X) \simeq \boldsymbol{\pi}_0(M_{c/X})$$

the corresponding "connected components functor", X any object of M. Then the canonical map (5) above is bijective.

Corollary. *Let X, Y be two objects of M, such that* **Hom**(X, Y) *exists in M. Then the natural map*

$$\overline{\mathrm{Hom}}(X, Y) \to \pi_0(\mathbf{Hom}(X, Y)) \tag{8}$$

is bijective.

To finish these generalities on **Hom**'s, I would like to generalize to the context of contractibility structures the results of prop. 4 p. 228 (section 74) on products of aspheric functors. We start within a context of asphericity structures:

Proposition 2. *Let (M, M_a) be a $\mathcal{W}$-asphericity structure (for a basic localizer $\mathcal{W}$), C a full subcategory of M generating this structure, i.e., such that $C \to M$ is M_a-$\mathcal{W}$-aspheric, A a small category,*

$$i : A \to M$$

a M_a-$\mathcal{W}$-aspheric functor. We assume M stable under binary products.

a) *Let b_0 in M be such that for any y in C,* **Hom**(b_0, y) *exists in M, and for any x in M_a, $x \times b_0$ is in M_a. Let $i_{b_0} : A \to M$ be the constant functor with value b_0, and consider the product functor*

$$i \times i_{b_0} : a \mapsto i(a) \times b_0 : A \to M.$$

This functor is M_a-$\mathcal{W}$-aspheric iff for any y in C, **Hom**(b_0, y) *is in M_a (a condition which does not depend on i nor even on A).*

b) *Let B be a full subcategory of M, such that for any b_0 in B, x in M_a, and y in C, we have $x \times b_0 \in M_a$ and* **Hom**(b_0, y) *exists in M, and is in M_a. Assume A totally $\mathcal{W}$-aspheric, let $i' : A \to M$ be any functor factoring through B, then the product functor*

$$i \times i' : a \mapsto i(a) \times i'(a) : A \to M$$

is M_a-$\mathcal{W}$-aspheric.

The proof is word by word the same as for the analogous statements p. 228–229, and therefore left to the reader. (The analogous statement to cor. 1 p. 229 is equally valid.)

[p. 272] The case I've in mind is when M_a is generated by a contractibility structure M_c, and $C = B = M_c$. The condition in b) boils down to existence of **Hom**(b_0, y) in M, when b_0 and y are both contractible objects of M – as a matter of fact, in all cases I'm having in mind, the **Hom** exists even without any contractibility assumption. Thus we get:

Corollary. *Let (M, M_c) be a contractibility structure, such that for any two contractible objects x, y,* **Hom**(x, y) *exists in M (a condition which presumably is even superfluous...). Consider the $\mathcal{W}$-asphericity structure $M_{\mathcal{W}}$ on M generated by M_c. Then the product of a $M_{\mathcal{W}}$-$\mathcal{W}$-aspheric functor $i : A \to M$ with a functor $i' : A \to M$ factoring through M_c is again $M_{\mathcal{W}}$-$\mathcal{W}$-aspheric. In particular, the category of all aspheric functors from A to M which factor through M_c is stable under binary products.*

From this one can deduce as on p. 230 that if the latter category is non-empty, i.e, if there does exist an aspheric functor $A \to M$ through M_c, then one can define a canonical functor

(9) $$\mathrm{Hot}_{M,\mathcal{W}} \to \mathrm{Hot}_{A,\mathcal{W}},$$

defined up to unique isomorphism, via a transitive system of isomorphisms between the functors deduced from aspheric functors $A \to M$ factoring through M_c.

Tautologies on $\operatorname{Im}\alpha$, and related questions.

83 After stating and proving theorem 1 of the previous section, I forgot to give an answer to the most natural question arising from it – namely how to characterize those $\mathcal{W}$-asphericity structures on a category M stable under finite products, which can be generated by a contractibility structure. The reason for this is surely that I have nothing better to offer than a tautology: let

$$M_a \subset \operatorname{Ob} M$$

be the given $\mathcal{W}$-asphericity structure, we assume beforehand that this structure is totally aspheric (which is a necessary condition for M_a to come from a contractibility structure M_c). If W_a is the corresponding set of $\mathcal{W}$-equivalences, it follows that the homotopy structure h_{W_a} is also the one defined by the homotopy interval structure J generated by the set J_0 of intervals of M made up with objects of M_a. Let [p. 273]

$$M_c \subset \operatorname{Ob} M$$

be the corresponding set of contractible objects (which may not be a contractibility structure on M). The conditions now are the following:

a) M_c generates the homotopy interval structure J, or equivalently, any two sections of an object of M_a are M_c-homotopic.

This condition is clearly equivalent to saying that J is indeed generated by a contractibility structure, and the latter is necessarily M_c.

b) M_c generates the $\mathcal{W}$-asphericity structure M_a, i.e., there exists a small full subcategory B of M_c such that the inclusion functor $i : B \to M$ be M_a-$\mathcal{W}$-aspheric, i.e., for any x in M_a, $i^*(x)$ in $B\hat{}$ is $\mathcal{W}$-aspheric (i.e., $B_{/x}$ is $\mathcal{W}$-aspheric in (Cat)).

One would like too a n.s. condition on a functor

$$\varphi : M \to \mathrm{Hot}_{\mathcal{W}},$$

for this functor to be isomorphic to the one associated to a contractibility structure on M – with special interest in the case when $\mathcal{W} = \mathcal{W}_0$, i.e., when the functor reduces to a functor

$$\pi_0 : M \to (\text{Sets}).$$

Here again, I have nothing to offer except a tautological statement, which isn't worth the trouble writing down. Nor do I have at present a compelling feeling that there should exist such a characterization, under suitable exactness assumptions on M say, and possibly assuming too that M is stable under the operation **Hom**. Here also arises the question *whether a functor $\varphi : M \to \mathrm{Hot}_{\mathcal{W}}$ stemming from a contractibility structure M_{c} on M can have non-trivial automorphisms* – a question closely connected to the "inspiring assumption" of section 28.

[p. 274]

29.6.

84 *A silly (provisional) answer to the "silly question" – and the new perplexity $f_!(M_{\mathrm{as}}) \subset M'_{\mathrm{as}}$?*

[Grothendieck (June 27, 1983)]

Two days have passed without writing any notes. Much of the time I spent on writing mathematical letters – one pretty long one to Gerd Faltings, who (on my request) had sent me preprints of his recent work, notably on the Tate conjectures for abelian varieties and on the Mordell conjecture, and had expressed interest hearing about some ideas and conjectures on "anabelian algebraic geometry". I had been impressed, from a glance upon the last of his manuscripts, to see three key conjectures proved in about forty pages, while they were being considered as quite out of reach by the people supposed to know. Some "anabelian" conjectures of mine are closely related to the Tate and Mordell conjectures just proved by Faltings – Deligne had pointed out to me about two years ago that a certain fixed-point conjecture (which I like to view as the basic conjecture at present in the anabelian program) implied Mordell's, so why loose one's time on it! I have the feeling Faltings is the kind of chap who may become interested in things which are supposed to be too far off to be worth looking at, that's why I had written him a few words, under the moment's inspiration. – Another letter, not quite as long, was an answer to a very long and patient letter of Tim Porter, telling me about a number of things which have been done by homotopy people, and which I was of course wholly ignorant of! His letter has been the first echo I got from someone who read part of the notes on "Pursuing stacks", and I was glad he could make some sense of what he read so far, and conversely – that not all he was telling me was going wholly "above my head"!

Apart from this and the (not unpleasant!) daily routine, I spent a fair bunch of hours on scratchwork, centering around trying to figure out the right notion of morphism for asphericity structures. It was a surprise that the notion should be so reticent for revealing itself – as a matter of fact, I am not quite sure yet if I got the right notion, in sufficient generality I mean. Time will tell – for the time being, the notion of morphism I have to offer, while maybe too restrictive, looks really seducing, because it parallels so perfectly the formalism of aspheric functors $i : A \to M$

and the corresponding (would-be modelizing) functor $i^* : M \to A^\wedge$. Again, it has been a surprise, right now, that after uncounted hours of unconvincing efforts today and yesterday, and almost wholly unrelated to these, this pretty set-up would come out within ten minutes reflection!

Proposition. *Let (M, M_a), (M', M'_a) be two $\mathcal{W}$-asphericity structures ($\mathcal{W}$ a basic localizer), $B \subset M_a$ a full subcategory of M generating the asphericity structure M_a, and* [p. 275]

$$f^* : M \to M'$$

a functor, admitting a left adjoint $f_!$ (hence f^* commutes to inverse limits). We consider the following six conditions, paralleling those of prop. 4 of section 75 (p. 236):

(i) $M_a = (f^*)^{-1}(M'_a)$,

(i') $M_a \subset (f^*)^{-1}(M'_a)$, *i.e.*, $f^*(M_a) \subset M'_a$,

(i") $B_a \subset (f^*)^{-1}(M'_a)$, *i.e.*, $f^*(B) \subset M'_a$,

(ii) $W_a = (f^*)^{-1}(W'_a)$,

(ii') $W_a \subset (f^*)^{-1}(W'_a)$, *i.e.*, $f^*(W_a) \subset W'_a$,

(ii") $\mathrm{Fl}(B)_a \subset (f^*)^{-1}(W'_a)$, *i.e.*, $f^*(\mathrm{Fl}(B)) \subset W'_a$,

where W_a, W'_a are the sets of $\mathcal{W}$-equivalences in M and M' respectively. We'll make the extra assumption (I nearly forgot, sorry!):

(Awk) *There exists a M'_a-$\mathcal{W}$-aspheric functor*

$$i' : A \to M'$$

(A a small category), such that $i = f_! i' : A \to M$ factors through B.

Under these conditions and with these notations, the following holds: The conditions (i), (i'), (i") are equivalent and imply the three others, which satisfy the tautological implications (ii) $\Rightarrow$ (ii') $\Rightarrow$ (ii"). If M and M' admit final objects and their asphericity structures are aspheric, then the fiver first conditions (i) to (ii') are equivalent, and if moreover $e_M \in \mathrm{Ob}\, B$, all six are equivalent.

Proof: reduction to loc. sit., using the commutative diagram

(1)
$$\begin{tikzcd} M \arrow[rr, "f^*"] \arrow[dr, "i^*"'] & & M' \arrow[dl, "i'^*"] \\ & A^\wedge & \end{tikzcd}$$

(up to isomorphism), and the relations

$$M'_a = (i'^*)^{-1}(A^\wedge_a), \quad W'_a = (i'^*)^{-1}(\mathcal{W}_A).$$

This proof shows moreover that the conditions (i), (i'), (i") are equivalent each to i being M_a-$\mathcal{W}$-aspheric. Thus, if these conditions are satisfied (let's say then that f^* is a *morphism* for the asphericity structures), we [I can't read this footnote]
may use i^* and i'^* for describing the canonical functors from the local- [p. 276]

izations of M, M' to $\mathrm{Hot}_{\mathcal{W}}$, and therefore from (1) get a *commutative* diagram (up to canonical isomorphism)

(2)
$$\begin{tikzcd} \mathrm{Hot}_{M,\mathcal{W}} \arrow[rr, "\overline{f^*}"] \arrow[dr] & & \mathrm{Hot}_{M',\mathcal{W}} \arrow[dl] \\ & \mathrm{Hot}_{\mathcal{W}} & \end{tikzcd} .$$

From this follows:

Corollary. *Let f^* be a morphism for the given asphericity structure, and assume these to be* modelizing. *Then the induced functor for the localizations*
$$\overline{f^*} : W_{\mathrm{a}}^{-1}M = \mathrm{Hot}_{M,\mathcal{W}} \to W_{\mathrm{a}}'^{-1}M' = \mathrm{Hot}_{M',\mathcal{W}}$$
is an equivalence of categories.

This is the answer, at last, of the "silly question" of section 45 (p. 95)!

We have still to comment though on the restrictive conditions we had to make, for getting the equivalences stated in the proposition, and the result stated in the corollary. These conditions are twofold: a) existence of a left adjoint $f_!$, which presumably, in most circumstances we are going to meet, will be equivalent with f^* commuting to inverse limits. It's a pretty restrictive condition, but of a rather natural kind, often met with in the modelizing situations; b) this is the "awkward" condition (Awk), which can be equally stated as follows: there exists a small full subcategory B' of M' (NB we may take the full subcategory defined by $i'(\mathrm{Ob}A)$), *generating* the asphericity structure M'_{a}, and such moreover that
$$f_!(B') \subset M_{\mathrm{a}},$$
(so that we can choose C, a full subcategory of M generating the asphericity structure, such that C contains $f_!(B')$. Another way of phrasing this condition on f^* (preliminary to the choice of C) is that the full subcategory of M'

(3) $\quad (f_!)^{-1}(M_{\mathrm{a}}) \cap M'_{\mathrm{a}}$ generates the asphericity structure of M'.

If this condition, plus the condition (i) say, which we express jointly by saying that f^* is a "morphism of asphericity structures", did imply [p. 277] the condition (stronger than (3))

(4)
$$f_!(M'_{\mathrm{a}}) \subset M_{\mathrm{a}},$$

we would replace (Awk) by this condition (4), which doesn't look awkward any longer, and does not refer to any A or i' whatsoever (and the six conditions (i) to (ii"), with the exception of (i") and (ii"), do not make any reference to any given aspheric functor). In any case, one might think of defining a *morphism* f^* of asphericity structures as a functor admitting a left adjoint satisfying (4), and such moreover that (i) above, or equivalently (i'), namely the nice symmetric relation to (4)

(5)
$$f^*(M_{\mathrm{a}}) \subset M'_{\mathrm{a}},$$

is satisfied. This notion for a morphism, stricter than the one we adopted provisionally, looks a lot nicer indeed – the trouble is that I could not make up my mind if the restriction (4) is not an unreasonable one. It would be reasonable indeed, if for those cases which are the most interesting for us, and primarily for any functor

$$f^* = i^* : M \to A^\wedge$$

associated to an M_{a}-$\mathcal{W}$-aspheric functor

$$i : A \to M$$

for any given "nice" asphericity structure (M, M_{a}), this condition is satisfied. Of course, in this case, i^* does admit a left adjoint

$$i_! : A^\wedge \to M,$$

provided only M is stable under (small) direct limits, which we'll assume without any reluctance! Thus, we are led to the following

Question 1. Under which conditions is it true, for an M_{a}-$\mathcal{W}$-aspheric functor $i : A \to M$ (where M is endowed with an asphericity structure M_{a}) that

(6) $$i_!(A^\wedge_{\mathrm{a}}) \subset M_{\mathrm{a}},$$

i.e., $i_!$ (the canonical extension of i to $A^\wedge$, commuting with direct limits) takes aspheric objects into aspheric objects?

Yesterday and today I pondered mainly about the typical case when $M = B^\wedge$, endowed with its canonical asphericity structure, where B is a small category, and moreover i is a functor $i : A \to B \subset B^\wedge$. This then brings us to the related [p. 278]

Question 1'. Let $i : A \to B$ be an aspherical map in (Cat), under which restrictive conditions on A, B, i (if any) does

$$i_! : A^\wedge \to B^\wedge$$

take aspheric objects into aspheric ones, i.e., do we have

(7) $$i_!(A^\wedge_{\mathrm{a}}) \subset B^\wedge_{\mathrm{a}}?$$

[here in the typescript, AG reverts to putting an extra "s" in the subscript for asphericity structures, but I'm sticking to just "a" from now on]

The question makes sense even without assuming i to be aspheric. If we drop this asphericity assumption on i, we are reduced, for a given B, (by replacing A by $A_{/F}$, where F is a given aspheric object of $A^\wedge$) to the case when A is aspheric and $F = e_{A^\wedge}$, i.e., to looking at whether the element

(8) $$i_!(e_{A^\wedge}) = \varinjlim_{A}{}_{(B^\wedge)} i(a)$$

in $B^\wedge$ is aspheric. This element appears as the direct limit in $B^\wedge$ of aspheric elements $i(a)$, the indexing category A being itself aspheric.

Thus, it is rather tempting to hope that the limit might well be aspheric too. Let's assume $\mathcal{W} = \mathcal{W}_\infty$ = usual weak equivalence, it turns out that when A is 1-connected, so is (8), which would seem to give some support to the hope that (8) is aspheric if A is. However, this is definitely not so, unless at the very least we assume B to be equally aspheric (which is however a rather natural assumption, as it follows from A aspheric if we assume moreover i to be aspheric). To see this, we take i to be cofibering with 0-connected fibers – the cofibration assumption implies that the direct limit over A can be computed first fiberwise and then take a limit over B (the most general version of associativity for direct limits!), whereas the 0-connectedness assumption on the fibers implies that the limit taken over the fiber A_b is just b itself, hence (8) is isomorphic to

$$\varinjlim_{b}{}_{(B^\wedge)} b = e_{B^\wedge},$$

which isn't aspheric except precisely when B is; now for topological reasons it is easy to find an aspheric A cofibered with 0-connected fibers over a non-aspheric B.

Thus, in the question of asphericity of (8) we'll better assume both A *and* B aspheric. If either A has a final object or B has an initial object, asphericity of (8) is more or less trivial in any case. For certain categories B even without initial object, (8) is always aspheric when A is, this I checked at any rate for the ordered category

$$B = \begin{tikzcd}[row sep=small] \alpha \arrow[dr] & \\ & \gamma \\ \beta \arrow[ur] & \end{tikzcd},$$

giving rise to a rather interesting computation. This suggested, for arbitrary B again, to look at the dual of the category above as A:

(9)
$$A = \begin{tikzcd}[row sep=small] \alpha & \\ & \gamma \arrow[ul] \arrow[dl] \\ \beta & \end{tikzcd}.$$

Here the question then amounts to *whether for a diagram*

$$\begin{tikzcd}[row sep=small] & a \\ c \arrow[ur] \arrow[dr] & \\ & b \end{tikzcd}$$

in B, the amalgamated sum

(10)
$$a \amalg_c b$$

in $B^\wedge$ is aspheric. This I know to be true if either $c \to a$ or $c \to b$ are monomorphisms (as stated in section 70). Now, I don't really expect this to be true in general, even if B is such an excellent category as Δ say, however, I didn't push through and make a counterexample. If we now want to remember the asphericity condition on i which we dropped, this condition, when A has an initial object γ, just means that $c = i(\gamma)$ is an initial object of B. Even with this extra condition, I do not really expect (10) to be necessarily aspheric. Therefore, I do not expect the inclusion (7) to hold for any aspheric functor i, even when A (and hence B) are assumed to be aspheric, without some extra condition, on A or B say.

The condition that $A^\wedge$ be *totally aspheric* seems in this context a rather natural one – for instance, when reducing the question whether (7) holds to the question of asphericity of an object (8) (where A stands for $A_{/F}$), and if we do not want to loose the asphericity assumption for i when taking the composition $A_{/F} \to A \xrightarrow{i} B$, we would like that asphericity of F imply asphericity of $A_{/F} \to A$ – which means precisely that A is totally aspheric. The trouble is that in the would-be counterexample above with A given by (9), A is stable under Inf, i.e., under binary products, hence $A^\wedge$ *is* totally aspheric indeed – thus it is doubtful that this extra condition in the "question 1'" above is quite enough. If it does fail indeed, the next best would be to try the stronger condition "A is a strict test category" (NB the category (9) isn't a test category!), which leads to the question whether (8) is aspheric when A is a test category (no longer a strict one though!) and $i : A \to B$ an aspheric functor. But I confess I have no idea at present how to handle this question, and I am dubious there will come out any positive result along these lines, even when assuming A and B to be both stable under binary products say, and to be contractors and what-not!

Another typical case for "Question 1" (p. 277) is the case when [p. 280]

$$M = (\mathrm{Cat}),$$

endowed with the usual asphericity structure, giving rise to the notion of "test functor"

$$i : A \to (\mathrm{Cat})$$

we have been working with almost from the very start. The most important case of all is of course the canonical functor

$$i_A : a \mapsto A_{/a} : A \to (\mathrm{Cat}),$$

and we well know that this functor is aspheric (for the natural asphericity structure of (Cat)) if A is a weak test category, and in this case it is true indeed that not only i_A^*, but equally $i_{A!}$ is modelizing, and transforms aspheric objects into aspheric objects. But we know too that for more general test functors, for instance the standard inclusion

$$i : \mathbb{\Delta} \hookrightarrow (\mathrm{Cat}), \tag{11}$$

it is no longer true in general that $i_!$ be modelizing – so is it at all reasonable to expect

$$i_!(A^\wedge_{\mathrm{a}}) \subset (\mathrm{Cat})_{\mathrm{a}}? \tag{12}$$

Definitely, I'll have to find out the answer in the typical case (11), whether I like it or not!

30.6.

85 *Digression on left exactness properties of $f_!$ functors, application to the inclusion $i : \triangle \hookrightarrow (\mathrm{Cat})$.*

[Thomason (1980)]

Just got an impressive heap of reprints and preprints by Tim Porter (announced in his letter two days ago). Many titles have to do with "shape theory", "coherence" and "homotopy limits" (corresponding to proobjects in various modelizers, as I understand it from his letter). The one which most attracts my attention though is "Cat as a closed model category" by R.W. Thomason – the title comes quite as a surprise, as my ponderings on the homotopy structure of (Cat) had left me with the definite feeling that there wasn't a closed model structure on (Cat), with the usual notion of weak equivalence. I'll have to have a closer look at this paper definitely, before starting on part V of these notes!

It was getting prohibitively late yesterday, so I had to stop. It then occurred to me that the (admittedly provisional) notion of *morphism* of asphericity structures I had proposed yesterday (p. 275) is definitely stupid, because there is no reason whatever that it should be stable under composition! [p. 281] This surely was the reason for the feeling of uneasiness caused by the condition (Awk), which surely deserves its name! This flaw of course disappears, if we strengthen the condition, as suggested on p. 277, by condition 4 – that $f_!$ should take aspheric objects into aspheric ones. This makes all the more imperative the task of finding out whether this condition is reasonable, and otherwise, what to put in as a substitute. The test case now seems to be really the one when f^* is the usual nerve functor

$$f^* = i^* : (\mathrm{Cat}) \to \triangle^\wedge,$$

associated to the standard inclusion

$$i : \triangle \to (\mathrm{Cat}).$$

I never before had a closer look upon the corresponding functor

$$i_! : \triangle^\wedge \to (\mathrm{Cat}),$$

left adjoint to the nerve functor, except just rectifying a big blunder (p. 22), and convincing myself that $i_!$ did *not* take weak equivalences into weak equivalences. But why not aspheric objects into aspheric ones?

Pondering a bit over this matter today, and trying to see whether $i_!$ takes *contractible* objects into contractible ones, this brought up the question whether $i_!$ commutes with finite products – which will allow then to make use of the generalities on morphisms of contractibility structures, and will imply that $i_!$ is indeed such a morphism (as it transforms $\triangle$, which generates the contractibility structure of $\triangle^\wedge$, into contractible elements of (Cat)). As a matter of fact, a number of times in my scratchwork during the last four months I've met with the question of when a functor of the type $f_!$ commutes to various types of finite inverse limits, and this now is the occasion for writing down some useful general facts in this respect, which had remained somewhat in the air so far. It seems that the relevant facts can be summed up in two steps.

Proposition 1. *Let M be a category stable under (small) direct limits, A a small category, $f : A \to M$ a functor, and*

$$f_! : A^\wedge \to M, \quad f^* : M \to A^\wedge$$

the corresponding pair of adjoint functors.

a) *Assume M stable under binary products, and that for any x in M, the product functor $y \mapsto y \times x$ commutes to arbitrary direct limits (with small categories of indices of course). Then, in order for $f_!$ to commute with binary products, is it n.s. that for any two objects a, b of A, the map* [p. 282]

$$(*) \qquad f_!(a \times b) \to f(a) \times f(b)$$

in M be an isomorphism.

b) *Assume M stable under fibered products, and that base change in M commutes to arbitrary direct limits (with small indexing categories of course). Then, in order for $f_!$ to commute with fibered products, it is n.s. that it does so for any diagram*

$$\begin{array}{ccc} b & & c \\ & \searrow \quad \swarrow & \\ & F & \end{array} ,$$

with b, c objects in A, F in $A^\wedge$.

Corollary. *Assume M stable under finite inverse limits, and that base change in M commutes with arbitrary direct limits. The $f_!$ is left exact iff it satisfies the conditions a) and b) above, and moreover transforms $e_{A^\wedge}$ into a final object of M (which also means, if A has a final object e_A, that $f(e_A)$ is a final object of M).*

Proof of proposition. a) As we have, by definition of $f_!$, for any F and G in $A^\wedge$

$$f_!(F) = \varinjlim_{A_{/F}} f(a), \quad f_!(G) = \varinjlim_{A_{/G}} f(b),$$

deduced from the corresponding relations in $A^\wedge$

$$F = \varinjlim_{A_{/F}} a, \quad G = \varinjlim_{A_{/G}} b,$$

the looked for bijectivity of

$$f_!(F \times G) \to f_!(F) \times f_!(G)$$

will follow from the assumption (*), and from the following lemma (applied in both categories $A^\wedge$ and M), the proof of which is immediate:

[p. 283] **Lemma.** *Let M be a category stable under direct limits and under binary products, and such that the product functors $y \mapsto y \times x$ commute to direct limits. Then binary products are distributive with respect to direct limits, i.e., if*

$$u : I \to M, \quad v : J \to M$$

are two functor of small categories with values in M, we have

$$\varinjlim_{I\times J} u(i) \times v(j) \xrightarrow{\sim} \big(\varinjlim_{I} u(i)\big) \times \big(\varinjlim_{J} v(j)\big).$$

Proof of b). Follows from a), applying it to the induced functor

$$f_{/F} : A_{/F} \to M_{/f_!(F)},$$

in order to prove commutation of $f_!$ with a fibered product corresponding to a diagram

$$(**) \qquad \begin{array}{ccc} G & & H \\ & \searrow \;\; \swarrow & \\ & F & \end{array}$$

in $A^\wedge$.

Remark 1. Part b) does not look as nice as part a), as we would like to be able to take F equally in A – what I first expected to get out. If we make only this weaker assumption, it will follow at once that $f_!$ commutes to fibered products corresponding to diagrams (**) *with F in A*. It doesn't seem finally that from this follows commutation of $f_!$ to all fibered products, unless making some extra assumptions, such as $f_!$ transforming monomorphisms into monomorphisms (which is a necessary condition anyhow). We'll have to come back upon this – for the time being we need only to deal with products, for which a) is adequate.

The "second step" now is a tautology:

Proposition 2. *Under the preliminary assumptions of prop. 1 above, assume moreover that f is fully faithful, and that $f(A)$ "generates M by strict epimorphisms", i.e., that for any object P in M, we have*

$$(***) \qquad P \xleftarrow{\sim} \varinjlim_{A_{/P}} f(x).$$

Then $f_!$ commutes to finite products in $A^\wedge$ of objects of A, and to fibered products in $A^\wedge$ of objects of A.

[p. 284] *Proof.* That $f_!$ transforms final object into final object follows from the formula (***) above, by taking P to be the final object of M. To get commutation to binary products, we apply the formula for $P = a \times_M b$, and notice that

$$A_{/P} \simeq A_{/P'}, \quad \text{with } P' = a \times_{A^\wedge} b,$$

because the inclusion $f : A \hookrightarrow M$ is full (I forgot to say we may assume f to be the inclusion functor of a full subcategory). But we have too

$$f_!(P') = \varinjlim_{A_{/P'}} f(x),$$

hence $f_!(P') \simeq P$. The proof for fibered products is similar, or can be reduced to the case of products by considering the induced functor

$$f_{/a} : A_{/a} \to M_{/a}.$$

Remark 2. The various variants of the notion of a generating subcategory A of a category M have been dealth with in some detail in SGA 4 vol. 1 exp. I par. 7 (p. 45–60) (Springer LN 269), cf. more specifically prop. 7.2 on page 47, summarizing the main relationships. The strongest of all notions considered there is generation by strict epimorphisms, which can be expressed by the formula (***) above, and is also equivalent to the *functor f^* being fully faithful*.

[Artin, Grothendieck, and Verdier (SGA 4.1)]

We come back now to the case $M = (\mathrm{Cat})$ – it is immediate that the preliminary exactness conditions on M of prop. 1 are satisfied. Thus, putting together prop. 1 and 2, we get:

Proposition 3. *Let A be a full subcategory of* (Cat)*, generating* (Cat) *by strict epimorphisms, $i : A \hookrightarrow (\mathrm{Cat})$ the inclusion functor. Then the corresponding functor*

$$i_! : A^\wedge \to (\mathrm{Cat})$$

commutes to finite products, and also to fibered products in $A^\wedge$ over an object of $A^\wedge$ coming from A.

The most familiar case when this applies is indeed the case of the canonical inclusion of Δ into (Cat), as it is well-known that the corresponding i^*, namely the nerve functor, is fully faithful. From this follows that *a fortiori*, for any full subcategory A of (Cat) containing Δ, A generates (Cat) by strict epimorphisms, and hence proposition 3 applies.

hasty!

Remark 3. I doubt however, even for $A = \Delta$, the most typical case, that $i_!$ is even left exact, because it looks unlikely to me, in view of the explicit description of $i_!(K_\bullet)$ for a given ss complex $K_\bullet$, in terms of "generators and relations" (cf. Gabriel-Zisman's book), that $i_!$ should transform monomorphisms into monomorphisms. For what follows, this is irrelevant anyhow.

[p. 285]

[Gabriel and Zisman (1967)]

[there is an unreadable footnote here]

Bimorphisms of contractibility structure as the (final?) answer. Does the notion of a map of asphericity structures exist?

86 After these somewhat painful preliminaries, it seems to me that firm ground is in sight at last! The main feeling that finally comes out of these reflections, is that we have a very good hold on checking whether or not a functor $f_! : A^\wedge \to M$ commutes to finite products, and moreover, that this condition is satisfied in many "good" cases, in more cases at any rate than I suspected. In case of the inclusion functor of $\triangle$ into (Cat), more generally of any subcategory A of (Cat) consisting of contractible objects, containing $\triangle_1$, and large enough in order to generate (Cat) by strict epimorphisms (or, as we'll say, to *generate strictly* the category (Cat)), it follows that we have the inclusion

$$(1) \qquad i_!(A^\wedge_{\mathrm{c}}) \subset (\mathrm{Cat})_{\mathrm{c}},$$

i.e., $i_!$ transforms *contractible* objects into contractible objects, or, equivalently, is a *morphism for the homotopy structures on $A^\wedge$ and* (Cat). (Note that the two first assumptions on the subcategory A of (Cat) imply that the homotopy structure on $A^\wedge$ generated by the "intervals" made up with elements of A, is a contractibility structure, admitting A as a generating set of contractible objects. The situation is really nice only when assuming that $A^\wedge$ is totally aspheric though, which will imply that A is a test category (even a strict one) and that the $\mathcal{W}$-asphericity structure on $A^\wedge$ associated to the contractibility structure just described is the usual one (cf. prop. 2 b) p. 248, where we take C to be A).)

The inclusion (1) of course strongly resembles the problematic inclusion (4) (p. 277) of yesterday's notes, namely, here to $i_!$ taking aspheric objects into aspheric ones. I doubt though this latter is true even for the standard case $A = \triangle$? But before trying at all costs to see whether it holds or not, the point I wish to make is that for the purpose we have in view, namely defining suitable conditions, stable under composition, on a pair $(f_!, f^*)$ of adjoint functors between two asphericity structures M and M', – for this purpose, a relation of the type (1), namely [p. 286]

$$(2) \qquad f_!(M'_{\mathrm{c}}) \subset M_{\mathrm{c}},$$

is just as good as the relation (4) on p. 277, provided we make the evident assumption needed for (2) to make sense, namely *the given asphericity structures on M, M' to be associated to contractibility structures*. This condition (2) is (considerably!) weaker than

$$(2') \qquad f_!(M'_{\mathrm{a}}) \subset M_{\mathrm{a}},$$

due to the inclusion $M'_{\mathrm{c}} \subset M'_{\mathrm{a}}$ and to the fact that (with the notations and using the results of section 81) the inclusion (2) is implied by the apparently weaker inclusion (under the assumption Loc 4)):

$$f_!(M'_{\mathrm{c}}) \subset M_0,$$

where $M_{\mathrm{c}} \subset M_{\mathrm{a}} \subset M_0$. On the other hand, (2) *is evidently stable under composition*.

To be specific about "as good as", let's come back to the statement of the "pretty" proposition on p. 275, somewhat marred by the "awkward" condition (Awk) we had to throw in, which looks so ugly because of its lack of stability under compositions. Let's replace this condition by the slightly stronger one (2), which is stable under composition. Condition (2) does imply (Awk) indeed, as we see by taking for A a small full subcategory of M_{c}, generating the asphericity structure of M, and for B any full subcategory of M contained in M_{a} and big enough in order a) to contain $f_!(A)$ and b) to generate the asphericity structure of M. We may take for instance for A any subcategory of M'_{c} closed under finite products in M' and generating the contractibility structure, and accordingly for B, so that the pair (A, B) will match for any choice of the basic localizer $\mathcal{W}$. Thus, the six conditions (i) to (ii”) of the proposition are equivalent, provided in the last (ii”) we assume moreover that B contain a final object e_M of M. Moreover, it is clear now that the conditions can be viewed also as a property of the functor

$$f_{!\mathrm{c}} : M'_{\mathrm{c}} \to M_{\mathrm{c}} \tag{3}$$

induced by $f_!$, which we may adequately express by saying that this functor (3) is $\mathcal{W}$-*aspheric* (where the two sides of (3) are categories which need not be small, but which are at any rate $\mathcal{W}$-*aspherators* in the sense of section 79, p. 247).

Stated this way, the condition just obtained on the pair of adjoint functors $(f_!, f^*)$, between the two categories M, M' endowed with contractibility structures, namely (2) and [p. 287]

$$f^*(M_{\mathrm{a}}) \subset M'_{\mathrm{a}}, \tag{4}$$

are not quite symmetric, as the condition on $f_!$ is expressed in terms of contractible objects, whereas the condition on f^* is expressed in terms of aspheric ones. However, assuming that $\mathcal{W}$ satisfies Loc 4) and using theorem 1 p. 252, we see that (4) is equivalent to the condition

$$f^*(M_{\mathrm{c}}) \subset M'_{\mathrm{c}}, \tag{5}$$

which does not depend any longer on the choice of $\mathcal{W}$!

Finally, we are led to a notion of pure homotopy theory, in terms of contractibility structures alone, without the intrusion of the choice of a basic localizer $\mathcal{W}$ and corresponding asphericity notions. We may call the pair a “*bimorphism*” of contractibility structures, and introduce it via the following summing-up statement:

“Scholie”. *Let (M, M_{c}) and (M', M'_{c}) be two contractibility structures, each admitting a small generating subcategory for the contractibility structure, say C and C' respectively. Let $(f_!, f^*)$ be a pair of adjoint functors*

[a “scholium” is a critical or explanatory comment extracted from preexisting propositions]

$$f_! : M' \to M, \quad f^* : M \to M'.$$

Let's consider the following inclusion conditions

(!) $f_!(M'_{\mathrm{c}}) \subset M_{\mathrm{c}}$,

(!') $f_!(C') \subset M_0$,

(∗) $f^*(M_c) \subset M'_c$,

(∗') $f^*(C) \subset M'_0$,

where M_0 and M'_0 are (as in section 81) the sets of "0-connected" objects in M and M' respectively, for the given contractibility structures.

a) *Clearly, in view of*

$$C \subset M_c \subset M_0, \quad C' \subset M'_c \subset M'_0,$$

condition (!) implies (!') and condition (∗) implies (∗'). Moreover, it is even true that conditions (∗) and (∗') are equivalent, and the same holds for (!) and (!') provided $f_!$ commutes with finite products. We'll say that $(f_!, f^*)$ is a bimorphism *for the given contractibility structures, if the inclusions (!) and (∗) do hold.*

[p. 288] b) *Assume we got a bimorphism $(f_!, f^*)$, i.e., (!) and (∗) above hold. Let $\mathcal{W}$ be a basic localizer, hence the sets $M_{\mathcal{W}}$ and $M'_{\mathcal{W}}$ of $\mathcal{W}$-aspheric objects in M and M' respectively, and the sets $\mathcal{W}_M$ and $\mathcal{W}_{M'}$ of $\mathcal{W}$-equivalences in M and M'. Then the following relations hold*

($\mathcal{W}$) $M_{\mathcal{W}} = (f^*)^{-1}(M'_{\mathcal{W}})$,

($\mathcal{W}$') $\mathcal{W}_M = (f^*)^{-1}(\mathcal{W}_{M'})$.

c) *Under the assumption of b), hence ($\mathcal{W}$') holds and f^* gives rise to a functor $\overline{f^*}$ between the localizations*

$$\mathrm{Hot}_{M,\mathcal{W}} = \mathcal{W}_M^{-1}M \quad \text{and} \quad \mathrm{Hot}_{M',\mathcal{W}} = \mathcal{W}_{M'}^{-1}M',$$

the following diagram is commutative up to canonical isomorphism:

$$\begin{tikzcd} \mathrm{Hot}_{M,\mathcal{W}} \arrow[rr, "\overline{f^*}"] \arrow[dr] & & \mathrm{Hot}_{M',\mathcal{W}} \arrow[dl] \\ & \mathrm{Hot}_{\mathcal{W}} & \end{tikzcd},$$

where the vertical functors are the canonical functors of section 77. In particular, if the latter are equivalences (i.e., M and M' are $\mathcal{W}$-modelizing), then so is $\overline{f^*}$.

d) *Assume merely that the inclusion (!) holds, and let $\mathcal{W}$ as in b)c) be a basic localizer, satisfying moreover Loc 4). Then the following conditions on the pair $(f_!, f^*)$ are equivalent:*

(i) *The pair is a bimorphism, i.e., (∗) (or equivalently, (∗')) holds.*

(ii) *The inclusion $\subset$ in ($\mathcal{W}$) above holds.*

(iii) *The inclusion $\subset$ in ($\mathcal{W}$') above holds.*

(iv) *The functor induced by $f_!$,*

$$f_{!c} : M'_c \to M_c$$

is a $\mathcal{W}$-aspheric functor between the aspherators M'_c, M_c.

(v) (*If A is a given small category, and*

$$i' : A \to M'$$

a given functor, factoring through M'_{c}, *and* $M'_{\mathcal{W}}$-$\mathcal{W}$-*aspheric.*) *The composition*

$$i = f_! i' : A \to M$$

is $M_{\mathcal{W}}$-$\mathcal{W}$-*aspheric.*

Comments. In terms of what is known to us since sections 81 and 82, and notably that any two sections of a 0-connected object of M or of M' are homotopic, the Scholie is just a long tautology. I have taken great case however in stating it, so as to get as clear a view as possible of exactly what the relevant relationships are. In a) the conditions (∗) and (∗') are just different ways of stating that f^* is a morphism of contractibility structures, which can still be expressed in various other ways, compare p. 251–252. Similarly, if we assume that $f_!$ commutes with finite products, then (!) or (!') can be viewed as expressing that $f_!$ is a morphism of homotopy structures (in opposite direction), which again could be expressed in various other ways, for instance in terms of the homotopy relation for maps, or in terms of homotopisms. This implies that in any case f^* induces a functor between the strict localizations

$$\overline{f^*}^{\mathrm{c}} : W_{\mathrm{c}}^{-1} M = \overline{M} \to W_{\mathrm{c}}'^{-1} M' = \overline{M'},$$

and similarly for $f_!$, when $f_!$ commutes to finite products. I doubt however that the latter localized map is of geometric relevance, except maybe in the cases when $f_!$ gives rise to relations similar to ($\mathcal{W}$) and ($\mathcal{W}$') in b) above, and both $f_!$ and f^* are model-preserving with respect to $\mathcal{W}$-equivalences, and define quasi-inverse equivalences between the localizations $\mathrm{Hot}_{M,\mathcal{W}}$ and $\mathrm{Hot}_{M',\mathcal{W}}$ – a rather exceptional case indeed.

After stating the Scholie, there is scarcely a doubt left in my mind about the notion of bimorphism, which finally peeled out of reflections, being a relevant one. This Scholie, rather than the "seducing" proposition of section 84 (p. 275), now seems to me the adequate "answer" of the not-so-silly-after-all question of section 46 (cf. page 95 – nearly 200 pages ago!). The only minor uncertainty remaining in my mind is whether or not in the notion of a bimorphism of contractibility structures, we should insist that $f_!$ should commute to finite products. It seems that it will be hard to check condition (!), except via an apparently weaker form such as (!'), and using commutation of $f_!$ to finite products. But on the other hand, the Scholie makes good sense without assuming such commutation property. If the notion of a bimorphism is going to be useful, time only will tell which terminology use is the best.

More puzzling however is this facit, that we still have not been able to give a satisfactory definition of a *morphism* of *asphericity* structures, independently of any restrictive assumption on these, such as being generated by a contractibility structure. Even when making such an assumption, it is not wholly clear yet that there isn't a good notion of

["facit" comes from the Latin verb "*facere*", to do, so it means the result. Or it could be a typo for "fact", which essentially means the same. . .]

[p. 290]

a morphism f^* of contractibility structures, without having to assume there exists a left adjoint $f_!$. Maybe a little more pondering on the situation would be useful. If it should become clear that (except for the obvious notion of *equivalence* of asphericity structures) there was *not* any reasonable notion of a morphism of asphericity structures, this would probably mean that the notion of an asphericity structure should not be viewed as a main structure type in homotopy theory in its own right, but rather, mainly, as an important by-product of a contractibility structure. At any rate, since section 81, I feel that the main emphasis has definitely been shifting towards contractibility structures, which seems now *the* main type of structure dominating in the modelizing story, as this "story" is gradually emerging into light.

Part V

Homology and cohomology (abelianization of homotopy types)

1.7. [p. 291]

Comments on Thomason's paper on closed model structure on (Cat).

87 I couldn't resist last night and had to look through Thomason's preprint on the closed model structure of (Cat). The paper is really pleasant reading – and it gives exactly what had been lacking me in my reflections lately on the homotopy theory of (Cat) – namely, a class of neat monomorphisms $Y \to X$ which all have the property that cobase change by these preserves weak equivalence, the so-called *Dwyer maps*. I had hoped for a while that "open immersions" and their duals, the "closed immersions" (namely sieve and cosieve maps, in Thomason's wording), have this property, and when it turned out they hadn't, I had been at a loss of what stronger property to put instead, wide enough however to allow for the standard factorization statements for a map to go through. The definition of a Dwyer map is an extremely pretty one – it is an open immersion $Y \to X$ such that the induced map from Y into its closure $\overline{Y}$ should have a right adjoint. Now this implies that $Y \to \overline{Y}$ is aspheric, and I suspect that this extra condition on an open immersion $Y \to X$ should be sufficient to imply that it has the "cofibration property" above with respect to weak equivalence. It would mean in a sense that the given open immersion is very close to being a closed immersion too, without however being a direct summand necessarily. The dual notion is that of a closed immersion such that the corresponding interior $\mathring{Y}$ of Y in X gives a map $\mathring{Y} \to Y$ which has a left adjoint, or only which is "coaspheric". By Quillen's duality principle, if one notion works well for pushouts, so does the other. – With this notion in hands, it shouldn't be difficult now to get a closed model structure on (Cat) a lot simpler than Thomason's. Visibly, he was hampered by the standing reflex: homotopy = semi-simplicial algebra, which caused him to pass by the detour of the category $\triangle^\wedge$ of semi-simplicial complexes, rather than just working in (Cat) itself. I'll have to come back upon this in part V of the notes, where

[Thomason (1980)]

[These Dwyer maps are not closed under retracts, so Cisinski (1999) introduced the *pseudo-Dwyer maps* which are, and *they* are now called Dwyer maps.]

I intend to investigate the homotopy properties of (Cat) and elementary modelizers $A^\wedge$, including the existence of closed model structures.

Some comments of Thomason's at the end of his preprint, about application to algebraic K-theory, seem to indicate that the notion of "integration" and "cointegration" of homotopy types I have been interested in, has been studied (under the name holim and hocolim) in the context of closed model categories by Anderson (his paper appeared in 1978). As I am going to develop some ideas along these lines in part VI on derivators, I should have a look at what Anderson does, notably what his assumptions on the indexing categories are. Thomason seems to believe that the closed model structure of (Cat) is essential for being able to take homotopy limits – whereas it is clear a priori to me that the notion depends only on the notion of weak equivalence. Indeed, he seems to consider the possibility of taking homotopy limits in (Cat) as the main application of his theorem, and in order to apply Anderson's results, believes it is necessary to be able to give concrete characterizations for "fibrations" and "cofibrations" of his closed model structure. Now, it turns out that the case he is interested in (for proving "Lichtenbaum's conjecture") is a typical case of *direct* homotopy limits, namely "integration" – which can be described directly in (Cat) in such an amazingly simple way (as sketched in section 69, p. 198–199). Thus, I feel for this application, the closed model structure is wholly irrelevant. As for cointegration, I do not expect that there is a comparably simple construction of this operation within the modelizer (Cat), but presumably there is when taking ∞-Gr-stacks as models (as suggested by the "geometric" approach to cohomology invariants, via stacks, where the operation of "direct image", namely cohomology precisely, is the obvious one, whereas inverse images are more delicate to define, by an adjunction property with respect to direct images…). When working in (Cat), cointegration of homotopy types should be no more nor less involved than in any closed model category say, and involve intensive recourse to "fibrations" (in the sense of the closed model structure, or more intrinsically, in the sense that base change by these should preserve weak equivalence). Now the latter have become quite familiar to me during my long scratchwork on cohomology properties of maps in (Cat), and I'll have to try to put it down nicely in part V of the notes.

[Anderson (1978)]

[p. 292]

[aka the Quillen–Lichtenbaum conjecture]

88 *Review of pending questions and topics (questions 1) to 5), including characterizing canonical modelizers).*

The present "part IV" on asphericity structures (and their relations to contractibility structures) turns out a lot longer than I anticipated, and the end is not yet quite in sight! Therefore, before pursuing, I would like to make a review of the questions along these lines which seem to require elucidation, and then decide which I'm going to deal with, before going over to part V.

1) Whereas the relevant notions of "morphisms" and "bimorphisms" for contractibility structures seem to me well understood, there remains a certain feeling of uneasiness with respect to asphericity structures, which haven't got yet a reasonable notion of morphism. Thus, I have left unanswered the questions raised in section 84 around the inclusion

[p. 293]

condition

$$f_!(M_{\rm a}) \subset M'_{\rm a},$$

so it is not impossible that, while relying on the mere feeling that the inclusion is just not reasonable and that the answer to the specific questions are presumably negative, I am about to miss some unexpected important fact! It seems I developed kind of a block against checking – maybe the answers are well-known and Tim Porter will tell me. . . Maybe I better leave the question for a later moment, as it will ripen by itself if I leave it alone. . .

2) I should at last introduce contractors, and morphisms of such. When I set out on part IV of the notes, I expected that the notion of a contractor would be one main notion, alongside with the notion of a canonical modelizer – it turns out that so far I didn't have much use yet for one or the other. Contractors can be viewed as categories generating contractibility structures, just as aspherators are there for generating asphericity structures. However, whereas any small category is an aspherator, the same is definitely not so for contractors, as we demand that every object in C should be contractible, for the homotopy structure in $C^\wedge$ generated by C itself. If we except the case of a contractor equivalent to the final category (a so-called *trivial* contractor), a contractor is a strict test category – thus the notion appears somewhat as a hinge notion between the test notions, and the "pure" homotopy notions and more specifically, contractibility structures. Writing up some scratchnotes I got should be a pure routine matter.

3) A lot more interesting seems to me to try and resolve a persistent feeling of uneasiness which has been floating, throughout the long-winded reflections on homotopy and asphericity structures in parts III and IV of the notes. This is tied up with this fact, that my treatment of the main notions, namely contractibility and asphericity, has been consistently *non-autodual*. More specifically, when a category M is endowed with either a contractibility or an asphericity structure, it does *not* follow that the opposite category $M^{\rm op}$ is too in a natural way. When defining homotopy relations and homotopism structures (section 51, A) and B)), these were autodual notions, but the notion of a homotopy interval structure, which we used in order to pass from a contractibility [p. 294]
structure to the corresponding notions of homotopy equivalence between maps and of homotopisms, is highly non-autodual too. It breaks down altogether when M is a "pointed" category, namely contains an object which is both initial and final – in this case, for any homotopy interval structure on M, any two maps in M are homotopic, and any map is a homotopism, hence any object is contractible!

Our initial motivation, namely understanding "modelizers" for ordinary homotopy types, made it very natural to get involved in non-autodual situations, as the homotopy category (Hot) itself, and the usual model categories for it, displays strongly non-autodual features. (Thus, whereas the usual test categories all have a final object, it is easy to see that a test category cannot possibly have an initial object.) However, the homotopy and asphericity notions we then came to develop make sense and are familiar indeed not only in the "modelizing story", but in any situation whatever which turns up, giving rise to anything like a "homology" or a "homotopy" theory. To give one specific example, starting with an abelian category $\mathcal{A}$, the corresponding homology theory is concerned with the category of $\mathcal{A}$-valued complexes, say $\mathrm{K}^\bullet(\mathcal{A})$. The most basic notions here are the three homotopy notions and two asphericity notions, namely: homotopy equivalence between maps, homotopism, contractible objects, and quasi-isomorphisms (= "weak equivalences"), and acyclic (= "aspheric") objects. The two first homotopy notions determine each other in the usual way, and define the third, namely an object is contractible (or null-homotopic) iff the map $0 \to X$, or equivalently $X \to 0$, is a homotopism. On the other hand, if we use the mapping-cylinder construction for a map, the set of contractible objects determines the set of homotopisms, as the maps whose mapping cylinder is contractible. Likewise, weak equivalences determine aspheric objects, and conversely if the mapping cylinder construction is given. The question that now comes to mind immediately is whether the two sets of notions, the three pure homotopy notions (determining each other), and the two "asphericity notions" (determining each other too), mutually determine each other, as in the non-commutative set-up we have worked in so far. We can also remark that the functor

$$\mathrm{H}^0 : \mathrm{K}^\bullet(\mathcal{A}) \to \mathcal{A}$$

visibly plays the part of the functor π_0 in the non-commutative set-up, it gives rise moreover to the H^i functors (any integer i) by composing
[p. 295] with the iterated shift functor (where the shift of X is just the mapping cylinder of $0 \to X$), thus the set of aspheric objects formally from the functor H^0 (as the objects X such that $\mathrm{H}^i(X) = 0$ for all i), much as in the non-commutative set-up the functor π_0 for a contractibility structure determines the latter, and hence also the corresponding asphericity structure.

Thus, the question arises of formulating the basic structures, namely contractibility and asphericity structures, in an autodual way, applying both to the autodual situation just described, and to the non-autodual one we have been working out in the notes – and if possible even, in all situations met with so far where a homology or homotopy theory of some kind of other has turned up. Of course, it is by no means sure a priori that we can do so, by keeping first nicely apart the two sets of notions (contractibility and asphericity), namely defining them separately, and then showing that a contractibility structures determines an asphericity structures, and is determined by the latter. Maybe we'll have to define from the outset a richer kind of structure, where both "pure" homotopy notions and asphericity notions are involved. Also, the familiar generalization of mapping cylinders, namely homotopy fibers and cofibers, and the corresponding long exact sequences, will evidently play an important role in the structure to be described. Now, this again ties in with the corresponding structure of a derivator, as contemplated in section 69, namely "integration" and "cointegration" of diagrams in a given category.

Definitely, this reflection is going to lead well beyond the scope of the present part IV – it relates rather to part C) of the working program envisioned by the end of May (section 71, p. 207–210), and rather belong to part VI of these notes, which presumably will center around the notion of a derivator. There is however a more technical question, and of more limited scope, which deserves some thought and goes somewhat in the same direction, namely: how to define (for a given contractibility or asphericity structure on M) an *induced* structure on a category $M_{/a}$, where a is in M? There is a little perplexity in my mind, even when M is of the type $C^\wedge$ say, with C a contractor and a in C, taking the canonical structure on $C^\wedge$ – because with the most evident choice of an "induced" asphericity structure on $(C^\wedge)_{/a} \simeq (C_{/a})^\wedge$, namely the usual notion of aspheric objects, this structure will practically never be totally aspheric (unless we take a to be a final object of C), hence will not be associated to a contractibility structure – whereas we expect that the contractibility structure of $C^\wedge$ should induce one on $C^\wedge{}_{/a}$. Presumably, the "correct" notions of induced structure, in the case of asphericity structure and the corresponding notion of weak equivalence, should be considerably stronger than the one I just envisioned, and correspond to the intuition of "*fiberwise* homotopy types" over the object a (visualized as a space-like object). A careful description of such induced structures seems to me to be needed, and the natural place to be the present part IV of the notes.

[I guess it is because this question is already growing in prominence that AG later decided to include the present section in part V!]

[p. 296]

4) A little reflection on semi-simplicial homotopy notions (and their analogons when $\triangle$ is replaced by a general test category Δ) seems needed, in order to situate the following fact: ss homotopy notions, namely for ss objects in any category A, behave well with respect to *any* functor

$$A \to B, \tag{1}$$

without having to assume that this functor commutes with finite products, whereas in the context of homotopy structures, when have a functor between categories endowed say with homotopy interval structures (for instance, with contractibility structures), such a functor

$$M \to N \tag{2}$$

behaves well with respect to homotopy notions only, it would seem, if we assume beforehand it commutes with finite products (plus, of course, that it transforms a given generating family of homotopy intervals of M into homotopy intervals of N). In case

$$M = \mathbf{Hom}(\Delta^{\mathrm{op}}, A), \quad N = \mathbf{Hom}(\Delta^{\mathrm{op}}, B),$$

and (2) comes from a functor (1) which does not commutes to finite products, neither does (2) – and still (2) is well-behaved with respect to ss homotopy notions! It should be noted of course that the semi-simplicial homotopy notions in M can be defined, even without assuming that in A finite products exist, namely in situations when M does not admit finite products – and hence, strictly speaking, the set-up of section 51 does not apply. All this causes a slightly awkward feeling, which I would like to clarify and see what's going on. I suspect it should be simple enough to do it here and now.

First, assume that A is stable under finite products, and under direct sums (with small indexing set say – for what we want to do with $\triangle\!\!\!\triangle$, finite direct sums even would be enough). We'll use sums only with [p. 297] summands equal to a chosen final object e of A, in order to get a functor

$$I \mapsto I_A : (\text{Sets}) \to A,$$

where I_A is the "constant" object of A with value I, namely a sum of I copies of e (sometimes also written $I \times e$). Using this functor, we get a functor

$$\Delta^\wedge = \mathbf{Hom}(\Delta^{\text{op}}, (\text{Sets})) \to M = \mathbf{Hom}(\Delta^{\text{op}}, A),$$

which I denote by

$$K \mapsto K_A,$$

associating to any ss set the corresponding "constant" (relative to A) ss object of A. On the other hand, because A admits finite products, so does M, which enables us to make use of the homotopy notions developed in sections 51 etc. Thus, if

[I think it's a bit unclear here whether the test category Δ is actually assumed to be $\triangle\!\!\!\triangle$ after all…]

$$\mathbb{I} = (I, \delta_0, \delta_1)$$

is any interval in $\Delta^\wedge$, considering the corresponding "A-constant" interval $\mathbb{I}_A$ in M, we get homotopy notions in M, which we may refer to as $\mathbb{I}$-*homotopy* (dropping the subscript A). They can all be deduced from the elementary $\mathbb{I}$-homotopy between maps in M, which is expressed in the known way, in terms of a map in M

$$h : \mathbb{I}_A \times X \to Y,$$

where X and Y are the source and target in M of the two considered maps, between which we want to find an elementary $\mathbb{I}$-homotopy h. This map h decomposes componentwise into

$$(*) \qquad h_n : (I_n)_A \times X_n \to Y_n,$$

and each h_n can be interpreted, in view of the definition of I_n, as a map

$$(*') \qquad h'_n : I_n \to \text{Hom}_A(X_n, Y_n),$$

provided we assume that taking products in A is distributive with respect to the sums we are taking, whence

$$(I_n)_A \times X_n \simeq (I_n)_{X_n} = \text{direct sum of } I_n \text{ copies of } X_n.$$

Now to give h, or equivalently a sequence of maps h_n in (*) "functorial in n for variable n", amounts to giving a sequence of maps (*'), satisfying a corresponding compatibility relation for variable n. The point of course is that (for any I in $\Delta^\wedge$ and X, Y in M) *the set of data* (*') *plus the compatibility condition make sense, formally, independently of any exactness assumptions on* A. Thus, it can be taken as the formal ingredient of a definition of "elementary $\mathbb{I}$-homotopy" between two maps in M, without any assumptions whatever on the category A we start with. The standard case is the one when $I = \mathbb{\Delta}_1$, the "unit interval", but never mind. The definition works just as well, when $\mathbb{\Delta}$ is replaced by any (let's say small) category Δ whatever. On the other hand, it is immediate that for a functor $M \to N$ as above, induced by a functor $A \to B$, for two maps in M, any elementary homotopy between them gives rise to an elementary homotopy of their images in N – which is just the well-known fact (in case $\Delta = \mathbb{\Delta}$, $I = \mathbb{\Delta}_1$) that $M \to N$ is compatible with simplicial homotopy notions. [p. 298]

In order to fit this into the general framework of section 51, let's remark that if A is a full subcategory of a category A', then for a pair of maps in M, the elementary $\mathbb{I}$-homotopies between these are the same as when considering the given maps as maps in $M' = \mathbf{Hom}(\Delta^{\mathrm{op}}, A')$, in which M is embedded as a full subcategory. Now, any (small, say) category A can be embedded canonically into $A' = A^\wedge$ as a full subcategory, and any functor

$$f : A \to B$$

embeds in the corresponding functor

$$f_! : A^\wedge \to B^\wedge,$$

hence the functor

$$\varphi : M \to N$$

embeds in the corresponding functor

$$\varphi' : M' \to N', \quad M' = \mathbf{Hom}(\Delta^{\mathrm{op}}, A^\wedge), N' = \mathbf{Hom}(\Delta^{\mathrm{op}}, B^\wedge).$$

As $A^\wedge$, $B^\wedge$ satisfy the required exactness properties, it follows that the $\mathbb{I}$-homotopy notions in M', N' can be interpreted in terms of the notions of section 51, with respect to $\mathbb{I}_A$ and $\mathbb{I}_B$, defined now as (componentwise) constant objects of $A^\wedge$ and $B^\wedge$ respectively. Still, $f_!$ commutes to finite products only if f does, so we are still left with explaining why $M' \to N'$ is well-behaved with respect to $\mathbb{I}$-homotopy notions. Equivalently, we need only see this in the case of (2) $M \to N$, when A and B are supposed to have the required exactness properties to allow for the interpretation given above of the $\mathbb{I}$-homotopy notions in terms of the formalism of section 51, and when moreover f (as $f_!$ above) commutes with sums. [p. 299]
This now is readily expressed by the relations

$$\begin{aligned}
&\varphi'(\mathbb{I}_A) \simeq \mathbb{I}_B,\\
&\varphi'(\mathbb{I}_A \times X) \xrightarrow{\sim} \varphi'(\mathbb{I}_A) \times \varphi'(X),
\end{aligned}$$

i.e., while φ' does *not* commute to finite products in general, however it *does* commute to the products which enter in the description of elementary homotopies (as these products can be expressed in terms of direct sums in A, B, and f commutes to these).

These reflections suggest that the notions of homotopy interval structures and contractibility structures may be generalized, in a way that the underlying category need no longer be stable under finite products nor even admit a final object; and likewise, the notion of a morphism of such structures may be generalized, without assuming that the underlying functor should commute with finite products. The thought that this kind of generalization may be needed had already occurred before in these notes, in connection with the corresponding situation about twenty five years ago, when the notion of a site was developed. But at present, the extension doesn't seem urgent yet, and I better stop here this long digression!

The remaining questions possibly to deal with in part IV are all concerned with modelizers. I'll try to be brief!

5) Consider an "algebraic structure type", and the category M of its set-theoretic realizations. I am looking for a comprehensive set of sufficient conditions on M to ensure that M is a "canonical modelizer". It seems natural to assume beforehand that in M (where at any rate small direct and inverse limits must exist) internal **Hom**'s exist, and more generally, for X, Y two objects over an object S of M, that $\mathbf{Hom}_S(X, Y)$ – this implies that base change $S' \to S$ in M commutes with small direct limits and a fortiori, that direct sums are universal – we may as well suppose them disjoint too. One feel quite willing too to throw in the total 0-connectedness assumption (cf. section 58), and that every nonempty object has a section over the final object. This preliminary set of conditions on an algebraic structure species is of course highly unusual, however it is satisfied for must "elementary" algebraic structures (by which I mean $M \approxeq A\hat{}$ for some small category A), as well as for n-stacks or ∞-Gr-stacks, for any n between 0 and ∞. The hope now is, in terms of these assumptions, to give a necessary and sufficient condition in order that a) the "canonical" homotopy structure on M be a contractibility structure, and moreover b) the latter structure be
[p. 300] "modelizing", by which we mean that the associated $\mathcal{W}_\infty$-asphericity structure ($\mathcal{W}_\infty$ = usual weak equivalences) be modelizing, which will imply that for *any* basic localizer $\mathcal{W}$, the corresponding $\mathcal{W}$-asphericity structure is modelizing.

Even if I don't look into this question now, it'll turn up soon enough in a similar shape, when it comes to prove modelizing properties for categories of stacks of various kinds. The best we could hope for would be a statement in terms of the category structure of M alone, with no assumption that M be defined in terms of an algebraic structure type. If I try to formulate anything by way of wishful testing conjecture, what comes to mind is: is it enough that there should exist a separating contractible interval? So the first I would try to get an idea, is to see how to make a counterexample to this. . .

3.7.

89 In connection with the left exactness properties of a $f_!$ functor, considered three days ago (section 85), I have been befallen by some doubts whether any subcategory of (Cat) containing the subcategory of standard simplices is strictly generating. I wrote there (p. 284) that as this is true for $\triangle$ itself, it "follows a fortiori" for any subcategory A of (Cat) containing $\triangle$. Assuming A to be full and denoting by i the inclusion functor, this is known to be equivalent (cf. remark 2 same page) to $i^* : (\mathrm{Cat}) \to A^\wedge$ being fully faithful, and in this form, it doesn't look so obvious that when this is true for one full subcategory, A_0 say, it should be true for any larger one A. This thought had been lingering for a second while writing the "a fortiori" and I then brushed it aside, because of the formulation of being generating in terms of strict epimorphisms. Only the next day did it occur to me that it is by no means clear that if a family of maps $X_i \to X$ in a category M is strictly epimorphic, any larger family with same target X should be "a fortiori" strictly epimorphic too – the "a fortiori" is known to apply only in the case of the similar notions of epimorphic, or universally strictly epimorphic, families of maps. After a little perplexity, I found the situation was saved, in the case I was interested in, through the fact that it was known from Giraud's article on descent (Bull. Soc. Math. France, Mémoire 2, 1964, prop. 2.5, p. 28) that $\triangle$ and even the smaller subcategory of simplices of dimension ≤ 2, is even generating by "*universally* strict epimorphisms", a notion which is stable under enlargement of the family of maps, as recalled above. Thus, the statement made on p. 284 does hold true. And I just checked today that, while this stability property by enlargement is surely not always true for a family of maps which is strictly epimorphic, however, it *is* true that if a full subcategory A_0 of a category M is generating by strict epimorphisms (or, as we'll say, is "strictly generating"), then so is any larger full subcategory A. This is seen by an easy direct argument, in terms of the initial definition, as meaning that for any object X in M, the family of maps $a_i \to X$ with target X and source in the given subcategory (A say) should be strictly epimorphic. (For the definition of common variants of the notion of epimorphism, see the "Glossaire" at the end of chapter 1, SGA 4, vol. 1.)

Digression (continued) on left exactness properties of $f_!$ functors.

[Giraud (1964)]

[p. 301]

[Artin, Grothendieck, and Verdier (SGA 4.1)]

It occurred also to me that (as suspected in remark 3, loc. cit.) the functor

$$i_! : \triangle^\wedge \to (\mathrm{Cat})$$

coming from the inclusion functor $i : \triangle \to (\mathrm{Cat})$ is *not* left exact (for another reason though than first contemplated), namely because (Cat) is known *not to be a topos* (for instance, an epimorphism need not be *strict* (or, what amounts here to the same, *effective*) – as stated in the cited result of Giraud). Indeed, we have the following

Proposition (which should belong to section 85!). *Let M be a $\mathfrak{U}$-category stable under small direct limits, A a small full subcategory, $i : A \to M$ the* *inclusion functor, hence a functor*

$$i_! : A^\wedge \to M.$$

If A is strictly generating (i.e., $i^* : M \to A^\wedge$ fully faithful), then $i_!$ is left exact iff M is a topos.

Indeed, the inclusion functor into $A^\wedge$ of M', the essential image of i^* in $A^\wedge$, admits a left adjoint ($i_!$ essentially). By the criterion of Giraud, left exactness of this adjoint, or equivalently of $i_!$, means that M' is the category of sheaves on A for a suitable site structure on A, qed.

Corollary. *If M is* not *a topos, then $i_!$ does* not *commute to fibered products in $A^\wedge$ of diagrams of the type*

$$(*)\qquad \begin{tikzcd} b \arrow[dr] & & c \arrow[dl] \\ & F & \end{tikzcd} ,$$

[p. 302] *with b, c in A and F in $A^\wedge$, while it does commute to finite products, and to fibered products of any two objects of $A^\wedge$ over an object of A.*

The "while" comes from prop. 1 and prop. 2 of section 85, which imply too that, if M is strictly generating and whether or not M is a topos, left exactness of $i_!$ is equivalent with commutation to fibered products of the diagrams (*). Hence the corollary.

This corollary answers also the perplexity raised in remark 1 (p. 283), as to a hypothetical sharper version of part b), concerning fibered products. As anticipated there, it turns out that this sharper version is not valid, – not without additional assumptions at any rate.

90 *Review of questions (continued): 6) Existence of test functors and related questions. Digression on strictly generating subcategories.*

After this digression on exactness properties of $f_!$ functors, let's come back to the review of those questions not yet dealt with, which seem more or less to belong to the present part IV of the notes. We had stopped two days ago with the question 5) of finding some simple characterization of canonical modelizers, comparable maybe in simplicity to the characterization we found for test categories (in part II). This question may well turn out to be related to the following one.

6) This is the question of finding handy existence theorems for test functors, whereas so far our attention to test functors had been turned towards a thorough understanding of the very notion of a test functor and its variants. I have the feeling that, after the reflections of sections 78 and 86 notably, the notion in itself is about understood now, so that time is getting ripe for asking for existence theorems. As all modelizers we have been meeting so far were associated to asphericity structures, it seems reasonable to restrict to these, namely to the case of a given modelizing asphericity structure

$$(M, M_{\mathrm{a}}),$$

and, if need be, even restrict to the case when this structure is associated to a contractibility structure M_{c}. We suppose given moreover a test category A, which we may (if needed) assume to be strict even, or even a contractor (i.e., the objects of A in $A^\wedge$ are moreover contractible, for the homotopy interval structure in $A^\wedge$ defined by all intervals coming from A). The question then is whether there exists a test functor

$$A \to M.$$

This (under the assumptions made) just reduces to the existence of a functor which be M_a-$\mathcal{W}$-aspheric. Here, $\mathcal{W}$ is a given basic localizer, with respect to which we got an asphericity structure. The most important case for us surely is the one when $\mathcal{W} = \mathcal{W}_\infty$, namely usual weak equivalence. It is immediate indeed that an M_a-$\mathcal{W}$-aspheric functor is equally aspheric for the corresponding $\mathcal{W}'$-asphericity structure of M, for any basic localizer $\mathcal{W}' \supset \mathcal{W}$. Thus, if we get an aspheric functor for $\mathcal{W}_\infty$, the finest basic localizer of all, we get ipso facto an aspheric functor for any basic localizer $\mathcal{W}$. (Note also that if an asphericity structure is modelizing for a given $\mathcal{W}$, the corresponding $\mathcal{W}'$-structure is modelizing too, for any $\mathcal{W}' \supset \mathcal{W}$; and the analogous fact holds for the notion of a test category – namely a $\mathcal{W}$-test category is also a $\mathcal{W}'$-test category, and similarly for total asphericity of $A^\wedge$ and hence for the condition of being a *strict* test category.) [p. 303]

In case M is even endowed with a contractibility structure, we will be interested, more specifically still, in aspheric functors factoring not only through M_a, but even through M_c:

$$i : A \to M_c,$$

while replacing the asphericity requirement on this functor, by the stronger one that for any x in M_c, the object $i^*(x)$ in $A^\wedge$ be *contractible* (for the homotopy structure in $A^\wedge$ defined by homotopy intervals coming from objects in A, say). In other words, we are interested in the question of existence of *bimorphisms* of contractibility structures (in the sense of section 86) from (M, M_c) to $(A^\wedge, A^\wedge_c)$. It may be noted that in both cases (working with asphericity structures or with the contractibility structures instead), in this existence question, we may altogether forget M itself, and consider it as an existence question for functors from A into either M_a, or M_c, with the property that for any object x in the target category M_a or M_c, the object $i^*(x)$ in $A^\wedge$ be either aspheric, or contractible. In the second case, we may even restrict x to be in any given subcategory C of M_c generating the contractibility structure – and in the cases met with so far, we can find such a C reduced to just one object I. In the case of asphericity structures, the same holds when taking for C a subcategory generating the asphericity structure, provided however C contains the image of A by i (which gives little hope to have C restricted to just one element!)

The interest of finding criteria for existence of $\mathcal{W}$-aspheric or or more stringently still of "c-*aspheric functor*" (as we may call them) is rather evident, as it gives a way, via i^*, for any homotopy type described in terms of a "model" x in M, to find a corresponding model $i^*(x)$ in $A^\wedge$. The situation would be more satisfactory still if we could find the test [p. 304] functor i such that the corresponding functor

(1) $$i_! : A^\wedge \to M$$

be modelizing too (assuming M to be stable under small direct limits, so that $i_!$ is defined as the left adjoint of

(2) $$i^* : M \to A^\wedge \quad .)$$

In this case, for a homotopy type described by a model K in $A^\wedge$, $i_!(K)$ gives a description of the same by a model in M.

Maybe we should remember though that even if we do not know about any test functor from A to M, still we always can find in three steps a modelizing functor

$$M \to A^\wedge,$$

namely a composition

(*) $$M \xrightarrow{j^*} B^\wedge \xrightarrow{i_B} (\mathrm{Cat}) \xrightarrow{j_A = i_A^*} A^\wedge,$$

where $j : B \to M$ is an M_{a}-aspheric functor from an auxiliary small category, which we may assume to be a test category, by a mild extra assumption on M (cf. cor. 3 p. 253). The modelizing functor we thus get has the disadvantage of not being left exact, whereas the looked-for functor i^* commutes to small inverse limits. Still, the composition (*) is pretty near to being left exact, it commutes to fibered products (because i_B does) which is the next best – we can view it as a left-exact functor from M to $A^\wedge{}_{/E}$, where E is the image in $A^\wedge$ of the final object of M (assuming e_M exists).

There is another advantage still of having a test functor $i : A \to M$, rather than merely using (*), namely it allows us to "enrich" the category structure of M, in such a way as to get "external Hom's" of objects of M, with "values in $A^\wedge$", by defining, for x, y in M, the object $\mathbf{Hom}(A)(x, y)$ of $A^\wedge$ as

(3) $$\mathbf{Hom}(A)(x, y) = \big\{a \mapsto \mathrm{Hom}_M(i(a) \times x, y)\big\}.$$

Such enriched structure, when $A = \triangle$, plays an important part in the second part of Quillen's treatment of homotopical algebra, under the name of (semi-)simplicial categories, especially with the notion of (semi-)simplicial model categories, which looks quite handy indeed. [p. 305] We should of course define composition of the $\mathbf{Hom}(A)$'s, as required too in Quillen's set-up. This is done by relating the $\mathbf{Hom}(A)$'s to the well-known internal $\mathbf{Hom}$'s in $M^\wedge$ – which will show at the same time that for formula (3) to make sense, we do not really have to assume M be stable under binary products, as we can interpret the products $i(a) \times x$ as being taken in $M^\wedge$, as well as the Hom, so as to get

$$\mathrm{Hom}_{M^\wedge}(i(a) \times x, y) \simeq \mathrm{Hom}_{M^\wedge}(i(a), \mathbf{Hom}_{M^\wedge}(x, y)),$$

hence

$$\textbf{Hom}(A)(x,y) \simeq i^*(\textbf{Hom}_{M^\wedge}(x,y)), \tag{4}$$

where i^* in the right hand side is interpreted as a functor

$$i^* : M^\wedge \to A^\wedge,$$

rather than $M \to A^\wedge$. (I leave to the reader the task of enlarging the basic universe, as need may be...) As i^* commutes to products, the evident composition of the internal **Hom**'s in $M^\wedge$ gives rise to the looked-for composition of the **Hom**(A)'s, with the required associativity properties. Of course, in case M is stable under binary products and **Hom**'s, which apparently is going to be the case in all modelizing situations, there is no need in the interpretation (4) to introduce the prohibitively large $M^\wedge$, and we can work in M throughout.

There is an important relation though on the external **Hom**(A)'s which we would like to be true for a satisfactory formalism, namely

$$\Gamma_{A^\wedge}(\textbf{Hom}(A)(x,y)) \xleftarrow{\simeq} \mathrm{Hom}_M(x,y), \tag{5}$$

where $\Gamma_{A^\wedge}$ just means $\mathrm{Hom}_{A^\wedge}(e_{A^\wedge}, \dots)$. This is equivalent to the requirement

$$i_!(e_{A^\wedge}) \simeq e_M \tag{6}$$

(assuming a final object e_M in M to exist), or equivalently

$$i(e_A) \simeq e_M \tag{7}$$

if we assume moreover e_A to exist. Thus, it will be natural to ask for test functors satisfying the extra condition (5) or (6) – and when trying to construct test functors in various situations (even without being aware of constructing test functors, as Mr Jourdain was "doing prose without knowing it"...), the very first thing everybody has been doing instinctively was to write down formula (7), I would bet!

[the reference is of course to Molière's *comédie-ballet*, Le Bourgeois gentilhomme]

The motivation for wanting to find test functors being reasonably clear by now, what kind of existence theorems may we hope for? When A is such a beautiful test category as △, □ or ◎, I would expect that for practically any M endowed with a modelizing contractibility structure say, under mild restrictions (such as the exactness assumptions which are natural in the modelizing story), there should exist a test functor indeed. What I feel less definite about is whether it is reasonable to expect we can find i even such that $i_!$ be modelizing too, in which case we would expect of course that the pair of equivalences of categories

[p. 306]

$$\mathrm{Hot}_A \rightleftarrows \mathrm{Hot}_M \tag{8}$$

defined in terms of $i_!$ and i^* should be quasi-inverse to each other, and the adjunction maps deduced from those between the functors $i_!$ and i^* themselves. This in turn is equivalent with the adjunction morphism

$$F \to i^* i_!(F) \tag{9}$$

being a weak equivalence, for any object F in $A^\wedge$. A test functor satisfying this exacting extra property merits a name of its own, we may call it a *perfect test functor* (or a *perfect aspheric functor*, when not making any modelizing assumptions on A or M). Thus, the existence problem of finding test functors can be sharpened to the one of finding perfect ones. Remember though that the most familiar test functor of all (besides the geometric realization functor $\blacktriangle \to$ (Spaces), namely the inclusion

$$i : \blacktriangle \to (\mathrm{Cat}) \tag{10}$$

giving rise to the nerve functor (introduced for the first time, I believe, in a Bourbaki talk of mine, on passage to quotient by a preequivalence relation in the category of schemes...), is *not* perfect. The most natural perfect test functor from $\blacktriangle$ into (Cat), more generally from any weak test category A into (Cat), is of course i_A – the functor indeed which has been dominating the whole modelizing picture in our reflections from the start. In the case of $A = \blacktriangle$, Thomason discovered another perfect test functor, conceptually less simple surely, namely $i_! \,\mathrm{Sd}^2$, where Sd is the "barycentric subdivision functor". I suspect there must be an impressive bunch of perfect test functors from $\blacktriangle$ with values in more or less any given modelizer, not only the basic one – and the question here
[p. 307] is to get a clear picture of how to get them, and the same of course for test functors which need not be perfect, including (10).

Next question then would be to see whether the existence theorems we may get for $\blacktriangle$, or its siblings $\square$ and $\mathbb{O}$ and the like, still hold true for a more or less arbitrary test category, or contractor. If so, this would be a very strong confirmation of the feeling which has been prompting the reflections in part II, namely that for the purpose of having "all-purpose"-models for homotopy types (insofar as this is feasible), any strict test category, or any contractor at any rate, is just as good as simplices or cubes, which people have kept working with for the last twenty-five years. If not, it will be quite interesting indeed to come a grasp of what the relevant extra features of $\blacktriangle$ and the like are, and how restrictive they are.

I doubt I will dive into these questions, still less come to a clear picture, in the present part of the notes. Still, before leaving the topic now, I would like to write down some hints I came upon while doing my scratchwork on homotopy properties of $A^\wedge$ categories. When looking for functors

$$A \to M$$

having some specified properties (such as being a test functor, or a perfect one, etc.), we may view this question as meaning that we are looking for an object with specified properties in the category

$$M^A = \mathbf{Hom}(A, M).$$

Presumably, this category is endowed with an asphericity or contractibility structure if M is (as we assume), presumably even a modelizing one. This reminds me that as far as the notion of weak equivalence goes, there may be even several non-equivalent ways of finding such structure on M^A, one being modelizing, whereas another, more useful one in some respects, is not. Thus, if M is of the type $B\hat{}$, we may rewrite M^A as

$$M^A \approxeq (A^{\mathrm{op}} \times B)\hat{} \approxeq P\hat{};$$

hence we get the weak equivalence notion coming from $P\hat{}$, disregarding its product structure, which is modelizing indeed if B is a test category and A aspheric, hence $A^{\mathrm{op}} \times B$ a test category. The structure which should be of more relevance though for our purpose should be a considerably finer one (namely with a smaller set of weak equivalences), which we may visualize best maybe by writing [p. 308]

$$M^A = (\mathbf{Hom}(A^{\mathrm{op}}, M^{\mathrm{op}}))^{\mathrm{op}},$$

i.e., interpreting the dual of M^A as the category of A-objects of M^{op}, for instance (if $A = \triangle$) as the dual of the category of ss objects of M^{op}. Now, Quillen has given handy conditions, in terms of projectives of $M^{\mathrm{op}} = N$, namely in terms of injectives of M, for the category $\mathbf{Hom}(\triangle^{\mathrm{op}}, N)$ of ss objects of N to be a closed model category – hence the dual category M^A will turn out as a closed model category too, under suitable conditions involving existence of injective objects in M. These conditions are satisfied for instance when M is a topos, and notably when M is of the type $B\hat{}$ – quite an interesting particular case indeed! Assuming that M is stable under both types of limits, so is M^A, hence there is an initial and final object, and according to Quillen's factorization axiom, the map from the former to the latter can be factored through an object

$$F \text{ in } M^A, \quad \text{i.e.,} \quad F : A \to M,$$

which is *cofibering*, and such that $F \to e$ is a *trivial fibration*. The idea is that these conditions mean more or less, at any rate imply, that F is a test functor.

I hit upon this "way out" while trying to construct test functors from $\triangle$ to any elementary modelizer $B\hat{}$, in order to try and check that $B\hat{}$ is a (semi)simplicial model category in the sense of Quillen. The intuitive idea of constructing inductively the components F_n of F was simple enough, still I got stuck in some messiness and did not try to push through this way, all the less as this naive approach had no chance of generalizing to the case of a more or less general test category A. Of course, for the time being Quillen's theorems, about certain categories $\mathbf{Hom}(A^{\mathrm{op}}, N)$ being closed model categories, is equally restricted to the case when $A = \triangle$, which looks as usual like a rather arbitrary assumption. Thus, to "test" whether the feeling about *any* test category more or less being "just as good" as $\triangle$, a second point would be to see whether Quillen's theorems extend, which presumably is going to be very close to the first point I raised.

I take this occasion to raise a third point – where there is no reason to restrict to an M which is modelizing (neither was there such reason before, when phrasing everything in terms of aspheric or c-aspheric [p. 309] functors, rather than test functors…). Namely, assuming as above that for any test category or contractor A, M^A or $M^{A^{\mathrm{op}}}$ can be endowed with an asphericity structure or a closed model structure, or at any rate with a set of weak equivalences, hence a localization or corresponding "homotopy category". What one would expect now is that up to (canonical) equivalence, the latter does not depend upon the choice of A, and hence is the same for arbitrary A as when using Δ, i.e., simplices. This should be true at any rate for $M^{A^{\mathrm{op}}}$ and when M is a topos – which means that *Illusie's derived category* $\mathrm{D}_\bullet(X)$ *of the category of semisimplicial sheaves on a topos* X*, could be constructed by using, instead of ss objects, A-objects of the category of sheaves on* X*, where* A *is any test category.* This should be one of the main points to settle in part VII of the notes.

There is a slight discrepancy though between the first point, about existence of test functors with values in M, depending on a given asphericity or contractibility structure of M, and the second and the third, which seem to depend only on the category structure of M. This is further evidence that the set of questions raised here is still far from being clear in my mind yet. Stating them now, however confusingly, is a first step towards clarification!

4.7.

Just still two comments about the existence questions for test functors, before going over to the next questions in our present review. One is that for given A, to prove that for rather general modelizing M there exists a test functor from A to M, we are reduced to the case when M is of the type $B^\wedge$, where B is a test category – namely, it is enough to take a B such that there exists a test functor $B \to M$. If we can even find a perfect test functor from some B to M, then likewise the existence question for perfect test functors from A to M is reduced to the case when $M = B^\wedge$. These comments may be useful for applying to the situation Quillen's model theory, as envisioned on the previous page – as his criteria for $\mathbf{Hom}(\Delta^{\mathrm{op}}, N)$ to be a closed model category apply when N is the dual of a topos, for instance the dual of $B^\wedge$. The second comment is about Thomason's result concerning the standard inclusion

$$i : \Delta \hookrightarrow (\mathrm{Cat}),$$

which can be expressed by saying that, although i itself is not a *perfect*
[p. 310] test functor, however, for any integer $n \geq 2$, the composition

$$i_n = i_! \,\mathrm{Sd}^n\, \alpha : \Delta \to \Delta^\wedge \to \Delta^\wedge \to (\mathrm{Cat})$$

is a perfect test functor, where $\alpha : \Delta \to \Delta^\wedge$ is the canonical inclusion. It is tempting to surmise that this result is not special to i alone, but that it holds for a large class, if not all, test functors from Δ to asphericity or contractibility modelizers. Here, Sd^n denotes the n'th iterate of the barycentric subdivision functor Sd (following now the notation in Thomason's paper, which presumably is standard, while I have been using "Bar" in part II of the notes). Presumably, functors analogous to Sd can be defined in any elementary modelizer $A^\wedge$, as suggested by the natural constructions arising in connection with the factorization property for a closed model structure on $A^\wedge$. Thus, possibly there is a general method in view for deducing perfect test functors from ordinary ones. However, it definitely seems to me that the natural place for these existence questions is in part V of the notes, as they seem intimately related to an understanding of the homotopy structures of elementary modelizers, and more specifically to the closed model structures to which such modelizers give rise in various ways.

* * *

Before proceeding, I would like to state still another afterthought to the reflections of section 89, about strictly generating subcategories of a category M. I recall that a family of arrows in M with same target X

$$u_i : X_i \to X$$

is called *strictly epimorphic*, if for every object Y of M, the corresponding map

$$\text{(*)} \qquad \mathrm{Hom}(X,Y) \to \prod_i \mathrm{Hom}(X_i,Y)$$

is injective (which is expressed by saying that the family (u_i) is *epimorphic*), and if *moreover* the following obviously necessary condition for an element (f_i) of the product set of (*) to be in the image of the map (*), is also sufficient:

(Comp) For any two indices i,j (possibly equal) and any commutative square

$$\begin{tikzcd} & T \arrow[dl, "v_i"'] \arrow[dr, "v_j"] & \\ X_i \arrow[dr, "u_i"'] & & X_j \arrow[dl, "u_j"] \\ & X & \end{tikzcd}$$

in M, the relation $f_i v_i = f_j v_j$ holds.

It is immediate that the condition for (u_i) to be strictly epimorphic depends only on the *sieve* X_0 of X in M (namely, the subobject of X, viewed as an object of $M^\wedge$) generated by the u_i's – we'll say also that this *sieve is strictly epimorphic*. One should beware that this does not mean of course that $X_0 \to X$ is epimorphic in $M^\wedge$ (which would imply $X_0 = X$, i.e., that one of the u_i's admits a section, i.e., a right inverse); nor is it true that if a sieve is strictly epimorphic, a large one should be so too – which means that when adding more arrows to a strictly epimorphic family, the family need not stay str. ep. [p. 311]

We'll say that the family (u_i) is *universally strictly epimorphic* if it is strictly epimorphic, i.e., the corresponding sieve X_0 is, and if the latter remains so by arbitrary base change $X' \to X$ in M, i.e., if the corresponding sieve X'_0 of X' is strictly epimorphic too. If the fibered products

$$X'_i = X_i \times_X X'$$

exist in M, this condition also means that the corresponding family of maps

$$u_i' : X_i' \to X'$$

is strictly epimorphic. The condition that (u_i) be univ. str. epimorphic again depends only on the generated sieve, it is moreover *stable under base change*, and equally *stable under adding new arrows*, i.e., replacing a sieve in X by a larger one.

It should be noted that if the fibered products $X_i \times_X X_j$ exist in M, then the compatibility condition (Comp) above is equivalent to the one obtained by restricting to

$$T = X_i \times_X X_j,$$

with v_i and v_j the two projections.

Assume the indices i are objects of a category I, and the X_i are the values of a functor

$$I \to M,$$

and that the family of arrows (u_i) turns X into the direct limit in M of the X_i:

$$X = \varinjlim_i X_i,$$

then it follows immediately that the family (u_j) is strictly epimorphic.

[p. 312] After these terminological preliminaries, we're ready to give the following useful statement, which is lacking in SGA 4 Chap. I (compare loc. cit. prop. 7.2, page 47, giving part of the story):

[Artin, Grothendieck, and Verdier (SGA 4.1)]

Proposition. *Let M be a $\mathfrak{U}$-category, A a small full subcategory, $i : A \to M$ the inclusion functor, hence a functor*

$$i^* : M \to A^\wedge.$$

For any object X of M, we consider the family F_X of all arrows in M with target X, source in A. The following conditions are equivalent:

(i) *For any X in M, the family F_X is strictly epimorphic.*

(ii) *For any X in M, the family F_X is universally strictly epimorphic.*

(iii) *For any X in M, F_X turns X into a direct limit of the composition functor $A_{/X} \to A \to M$, i.e.,*

$$X \xleftarrow{\sim} \varinjlim_{A_{/X}} a.$$

(iv) *The functor i^* is fully faithful.*

Proof left to the reader (who may consult loc. cit. for (i) ⇒ (iii) ⇔ (iv), so that only (i) ⇒ (ii) is left to prove).

Definition. When the equivalent conditions above are satisfied, we'll say that A is a *strictly generating* subcategory of M.

NB The notion makes sense too without assuming A to be small, nor M to be a $\mathfrak{U}$-category, by passing to a larger universe (it is immediate for (i) or (iii) that the condition does not depend on the choice of the universe.)

Corollary. *If A is strictly generating in M, then so is any larger full subcategory B.*

This is clear by criterion (ii) (whereas it isn't by any one of the other two criteria!).

91 I see three more questions to review – presumably they will be a lot shorter than the last!

Review of questions (continued): 7) Homotopy types of finite type, 8) test categories with boundary operations, 9) miscellaneous.

[p. 313]

7) **Description of homotopy types "of finite type",** in terms of an elementary modelizer $A^\wedge$. In terms of the modelizer (Spaces), a natural finiteness condition on a homotopy type is that it may be described (up to isomorphism) as the homotopy type of a *space admitting a finite triangulation*. In terms of ss sets, i.e., of the modelizer $\mathbb{\Delta}^\wedge$, the natural finiteness condition, suggested by the algebraic formalism, is that the homotopy type be isomorphic to one defined by an object *"of finite presentation" in $\mathbb{\Delta}^\wedge$*, namely one which is a direct limit of a *finite* diagram in $\mathbb{\Delta}^\wedge$, made up with simplices, i.e., coming from a diagram in $\mathbb{\Delta}$. It is clear that the first finiteness condition implies the second, by using a total order on the set of vertices of the triangulation. The converse shouldn't be hard either, using an induction argument on the number of simplices occurring in the diagram, and using the fact that any quotient object in $\mathbb{\Delta}^\wedge$ of a simplex is again a simplex, hence also a subobject of a simplex is a union of subsimplices; from this should follow by induction that the geometric realization of a ss set of finite presentation is endowed with a natural compact *piecewise linear structure*, and hence can be finitely triangulated. Presumably, one can even find a canonical triangulation, using twofold barycentric subdivision Sd^2 (again!) on any simplex. All this is surely standard knowledge, and I don't feel like diving into technicalities on this matter, unless I am forced to.

If we start with an arbitrary test category A, the notion of an object of finite presentation in $A^\wedge$ still makes sense. Indeed, in any category M, stable under filtering small direct limits, we may define objects of finite presentation as those for which the corresponding *covariant* functor

$$Y \mapsto \mathrm{Hom}(X, Y)$$

commutes to filtering direct limits. If M is stable under some type of finite direct limits, say under any finite direct limits, then so is the full subcategory M_{fp} of objects of finite presentation of M. In the case when $M = A^\wedge$, A any small category, it is obvious that objects of A, and hence finite direct limits of such, are of finite presentation, and it is not difficult to show that the converse equally holds. As a matter of fact $A^\wedge_{\mathrm{fp}}$ *can be viewed as the solution of the 2-universal problem of "adding finite direct limits to A".*

Coming back to the case when A is a test category, and hence $A^\wedge$ is modelizing, one may ask for conditions upon A which ensure that the homotopy types of finite presentation are exactly those isomorphic to the homotopy types defined by objects of $A^\wedge_{\mathrm{fp}}$. One expects that some
[p. 314] stringent extra condition is needed on A to ensure this. To see this, let's take a finite group G, and an aspherical topological space E_G upon which G operates freely, with quotient B_G, a classifying space of G. If $G \neq 1$, the homotopy type of B_G isn't of finite type, because B_G has non-vanishing cohomology groups in arbitrarily high dimensions, as well known. We could transpose the following construction in either (Cat) or $\triangle^\wedge$ say, but we may as well work in the modelizer $M = $ (Spaces), and take any small full subcategory A of M, containing the unit interval but not the empty space, and stable under finite products – which implies that A is a strict test category. We'll take A large enough to contain E_G, and small enough to be made up with aspheric spaces, hence the inclusion functor

$$i : A \to \text{(Spaces)}$$

is a test functor. Now consider the quotient object *in* $A^\wedge$

$$F = E_G/G,$$

i.e., the presheaf on A

$$F : T \mapsto \mathrm{Hom}(T, E_G)/G \simeq \mathrm{Hom}(T, B_G),$$

where the last isomorphism comes from the fact that the object T of A is aspheric and hence 1-connected. Thus, we get

$$F \simeq i^*(B_G),$$

hence the homotopy type defined by F is the homotopy type of B_G, which is not of finite type, despite the fact that F is of finite presentation.

In order to ensure that the homotopy type defined by any object in $A^\wedge_{\mathrm{fp}}$ be of finite type, it may be useful perhaps to make on A the assumption that any quotient in $A^\wedge$ of an object in A is isomorphic to an object in A, and that the set of all subobjects in $A^\wedge$ of an object a in A (i.e., the set of all sieves on a) is finite – possibly too that for any two objects a and b of A, $\mathrm{Hom}(a, b)$ is finite. As for the opposite inclusion, namely that any homotopy type of finite type can be described by an object of $A^\wedge_{\mathrm{fp}}$, this would follow from the existence of a *perfect* test functor from $\triangle$ into
[p. 315] $A^\wedge$, factoring through $A^\wedge_{\mathrm{fp}}$. Thus, the present question about finiteness conditions, seems to be related (possibly) to the previous one about existence of various types of test functors.

The condition for a strict test category A we are looking at is surely satisfied, besides $\triangle$, by the cubical and hemispherical test categories $\square$ and $\mathbb{O}$, and surely also by any finite products of these. I add this comment, of course, in order to "push through" the point that not any more with respect to finiteness conditions on homotopy types, than (presumably at least, for the time being)in any other essential respect concerning the ability for expressing basic situations and facts in homotopy or cohomology theory, the category of simplices stands singled out by itself from all other test categories. Nor does it seem that the "trinity"

$$\triangle, \square, \bigcirc$$

has this property, with the only exception so far, possibly, of the Dold-Puppe theorem (as no other test category except these is known to me for which a Dold-Puppe theorem in its strict form holds true (compare reflections section 71)).

8) I could make the same point in favor of more general test categories than the trinity above, when it comes to the existence of an algorithm for computing homology and cohomology groups, using suitable *boundary operators*. What is meant by these is clear of course for the three types of complexes, but then it extends in an obvious way to multicomplexes too – which means that for the test categories deduced from the trinity by taking finite products, we still get an algorithm for cohomology via boundary operators. Of course, for any test category A, using a test functor (if we can find one) from one of the three above (say) into $A^\wedge$ will allow us to reduce "computation" of homology and cohomology invariants in terms of a model in $A^\wedge$, to the case of the corresponding type of complexes – hence again an algorithm (similar to the familiar one of computing the cohomology of an object of (Cat) semisimplicially, via the nerve). But this is cheating of course! The question I want to raise here is about existence of "boundary operations" *in A*, similar to the familiar ones used for the three basic types of complexes, and allowing to compute the homology and cohomology groups of an object of $A^\wedge$ in the usual way, involving suitable *signs* $+$ or $-$ associated to the various boundary operations. It shouldn't be hard, I feel, to pin down exactly [p. 316] what is needed for getting such a formalism. The intuitive idea behind it (suggested by the example of standard complexes and multicomplexes) is that *such a formalism should be associated to cellular decompositions of n-cells for variable n*, such that the interior of each n-cell should be an open cell of the subdivision. There may of course be several n-cells for the same n, which are not combinatorially isomorphic. When trying to express this idea in a precise way, we are led to assume, as an extra structure on the would-be test category A, a functor

(1) $$i : A \to (\mathrm{Ord})$$

of A into the category of ordered sets, such that for any a in A, $i(a)$ be a *finite* ordered set, whose geometric realization (cf. section 22) is an n-cell for suitable $n \overset{\text{def}}{=} \dim(a)$. We assume moreover that $i(a)$ has a largest element $e(a)$, and that the geometric realization of $i(a) \setminus \{e(a)\}$ is the bounding $(n-1)$-sphere of the n-cell $|i(a)|$:

(2) $$|i(a)^*| \simeq \mathrm{S}^{n-1}, \quad \text{where} \quad i(a)^* \overset{\text{def}}{=} i(a) \setminus \{e(a)\},$$

which will imply the precedent condition, namely

(3) $$|i(a)| \simeq \mathrm{B}^n,$$

as B^n can be identified to the cone over S^{n-1}. As another condition, we need that

(4) For any a in A and $x \in i(a)$, there exists b in A and an isomorphism

$$i(b) \xrightarrow{\sim} i(a)_{/x} \overset{\text{def}}{=} \left\{ y \in i(a) \;\middle|\; y \leq x \right\} \hookrightarrow i(a),$$

induced by a map

$$b \to a$$

in A.

It is enough to make this assumption for x of codimension 1 in $i(a)$, which will imply that it is true for any x. It seems reasonable on the other hand to make the assumption that for a given x, the object b in (4), viewed as an object of $A_{/a}$, is determined up to a unique isomorphism, we may call it a_x, and ∂_x the canonical map of b into a

(5) $$\partial_x : a_x \to a \qquad (x \in i(a), \text{ of codim. 1 in } i(a), \text{ i.e., } \dim(x) = \dim(a) - 1).$$

As an extra structure, we need for any a in A

(6) ω_a, an *orientation* of the n-cell $i(a)$ $\quad (n = \dim(a))$.

This allows us, for any x as in (5), to define a signature

(7) $$\varepsilon(x) \quad \text{or} \quad \varepsilon_a(x) \in \{+1, -1\},$$

[p. 317] which will be +1 or −1, depending on whether in the inclusion

$$|\partial_x| : |i(a_x)| \to |i(a)|,$$

the orientation $\omega(a_x)$ is induced "à la Stokes" by the orientation $\omega(a)$ of the ambient n-cell, or not. Having the boundary operations (5) with their signatures (7), and the decomposition

(8) $$A = \coprod_{n \geq 0} A_n, \quad \text{where } A_n = \left\{ a \in \operatorname{Ob} A \;\middle|\; \dim a = n \right\},$$

we get in the usual way, for any contravariant functor $K_\bullet$ from A with values in an additive category, a corresponding chain complex in this category, with components

(9) $$K_n = \coprod_{a \in A_n} K_\bullet(a),$$

and boundary operators defined in the usual way via (5) and (7). Applying this to the case of the category (Ab) of abelian groups, or to its dual, and to the abelianization of an object X of $A^\wedge$, we obtain a tentative way for computing the homology and cohomology groups of the homotopy type of $A_{/X}$, and similarly for any system of twisted coefficients on X. The question which arises here is to write down a set of natural extra conditions on the data, which will ensure that we do get a canonical isomorphism between the "homology" and "cohomology" groups thus constructed, and the usual homology and cohomology invariants of the object $A_{/X}$ of (Cat). Moreover, we would like too to have conditions to ensure that A is a test category, or even a strict one.

One difficulty here, if one really wants a test category and not just a weak one (which may not be without any problem either), is that presumably for this, we'll need suitable *degeneracy operations*, which may well turn out a very exacting condition indeed! The skeptical reader may wonder, as I am just doing myself, whether there will be any example within the set-up I propose, which does not reduce to a finite product of test categories in our trinity. [p. 318]

I just spent a while trying to find some convincing example, by using a suitable full subcategory A_0 of (Ord), made up with finite sets satisfying the assumption (2) above for $i(a)$, under some additional assumption on A_0 such as stability under finite products and under passage from a to an object $a_{/x}$, and that A_0 contain the ordered set

$$I = \begin{tikzcd}[row sep=tiny] \bullet \arrow[dr] & \\ & \bullet \\ \bullet \arrow[ur] & \end{tikzcd},$$

whose geometric realization is the segment B^1 with its usual cellular decomposition. In terms of A_0 and introducing "orientations" of object of A_0, the idea was to define another category A (of pairs (a,ω), with a in A_0 and ω an orientation of $|a|$), *stable under finite products* so that $A^\wedge$ is totally aspheric, together with a functor $i : A \to (\mathrm{Ord}) \hookrightarrow (\mathrm{Cat})$ such that $i^*(I)$ should be representable, and hence furnish the homotopy interval needed to ensure that A is a test category. The first idea that comes to mind, namely define a map from (a,ω) to (a',ω') as merely a map from a to a' in A_0, is nonsense unfortunately, as in the data (6), the orientations will not be stable under isomorphisms, a condition which I forgot to state before, and which is visibly needed in order to be able to define the differential between the K_n's. If we try to define A taking into account this compatibility condition, we loose existence of products, anyhow $i^*(I)$ isn't representable anymore, so why should it be aspheric over the final object, so why should the functor i be a test functor?

The difficulty I find in carrying through any explicit example for a "test category with boundary operations", except those which stem from our trinity, is rather intriguing I feel. The question is whether maybe in this direction, one might get at an intrinsic description of the trinity, in terms of the rather natural structure species of a "test category with boundary operations". This is the second instance where the thought arises that the three standard test categories Δ, $\square$ and $\bigcirc$ may be distinguished in some respects – the first instance was in relation to the Dold-Puppe theorem.

9) **Miscellaneous residual questions from part II.** One of these was about the category Δ^{f} of simplices without degeneracies being a weak test category (cf. section 43) – while it is definitely *not* a test category. It seems worth while to write down a proof for this, maybe too for the analogous statements for $\square^{\mathrm{f}}$ and $\bigcirc^{\mathrm{f}}$. This reminds me too that I never got [p. 319]

around to introducing formally the hemispherical test category, which presumably will be very useful when it comes to studying stacks – this too could be done in part IV, as well as proving of course that ◎ is a strict test category indeed, or better still, a contractor. It may be fun too constructing test functors from any one of the three basic test categories in the trinity, to the category of complexes defined by the two others – six cases altogether to consider! But as I am not in the process of writing the "Elements d'Algèbre Homotopique", maybe I will skip this!

[pity!]

In the same section 43, I raised the question as to whether the ordered set of all non-empty finite subsets of a given infinite set, viewed as a category in the usual way for an ordered set, was a weak test category (on page 78 it was seen not to be a test category, and it is immediate then that it is not totally aspheric either). One interesting application, as noticed there, would be to the effect that (Ord), the subcategory of (Cat) defined by ordered sets, is a modelizer (for the induced notion of weak equivalences). Now the question arises moreover whether this modelizing structure comes from an asphericity structure, or even from a contractibility structure – and the same question arises in the more general situation described in the proposition of page 74.

A last question along these lines I would like to clear up, is the relation of total $\mathcal{W}$-asphericity for an asphericity structure, for variable $\mathcal{W}$, when $\mathcal{W} \subset \mathcal{W}'$. Assuming the localizers satisfy the condition Loc 4), is it true that total $\mathcal{W}$-asphericity is equivalent to total $\mathcal{W}'$-asphericity – or equivalently, is it equivalent to total 0-connectedness?

[p. 320] 5.7.

92 *Short range working program, and an afterthought on abelianization of homotopy types: a handful of questions around the Whitehead and Dold-Puppe theorems.*

The review on "pending questions and topics" related to part IV of the notes has taken pretty much longer than expected. It was quite useful though, to get a clearer view of what those questions are about, and to get a feeling for what to include and develop, and where. As I do not intend to spend my life on the task, not even one year, it is becoming clear that I am not going to get the whole picture of all the questions touched – and some presumably I am going to leave just aside, as they do not seem indispensable for a comprehensive overall picture of what I'm after. This seems to me to be the case for the questions 7) and 8), concerned with finiteness conditions for homotopy types in terms of models, and with test categories with boundary operations. At the opposite side, it seems that the questions 2), part of 3), and 9), about the notion of contractor, induced asphericity and contractibility structures on a category $M_{/a}$, and "miscellaneous" left-overs from part II, should be dealt with in part IV – whose end now is in sight after all! On the other hand, questions 1), another part of 3), 5) and 6), about morphisms of asphericity structures and related problems, about an autodual treatment of asphericity and contractibility notions, about a handy criterion for canonical modelizers, and about existence theorems for various kinds of test functors or aspheric functors, while I feel that I should come at least to a considerably clear understanding of these matters than now, the adequate place for developing such reflection is definitely *not* in the present part IV, but belong to one or the other of the three parts still ahead in our overall reflection on the modelizing story.

During our review, we came a number of times upon situations when the question arose as to whether one point I like to make, namely that a more or less arbitrary (strict) test category "is just as good" as the sacrosanct test category $\triangle$, or its twin brothers $\square$ and $\bigcirc$, is a valid one or not. I would like to list here these situations, with a view of coming back to it later:

a) Existence theorems for test functors (cf. section 90).

b) $A^\wedge$ and various other categories constructed in terms of A, such as $\mathbf{Hom}(A^{\mathrm{op}}, M)$, are closed model categories (under suitable assumptions...).

c) Independence of the derived category of $\mathbf{Hom}(A^{\mathrm{op}}, M)$ on the choice of test category, notably when M is a topos or the dual of a topos (with suitable assumptions on $A, M \dots$). [p. 321]

d) Possibility of expressing finite type of a homotopy type in terms of $A^\wedge_{\mathrm{fp}}$, for suitable test categories A.

e) Possibility of defining boundary operations within a test category – and/or getting Dold-Puppe type relations.

* * *

Before resuming more technical work with the matters left over for part IV, I would like still to write down some afterthoughts, concerning the question of boundary operations in a test category (question 8) in our review). It occurred to me that perhaps it isn't such a good idea, to try at all costs to subordinate this question to a question of cellular decompositions of spheres, however natural this idea may be in view of the examples of the standard types of complexes and multicomplexes. In this connection, I remember that among my first thoughts when starting unwittingly on the modelizing story, was that a "test category" A (namely one such that $A^\wedge$ should be "modelizing") should more or less correspond to such decompositions. Soon after it came as a big surprise that so little was needed in fact for A to merit the name of a test category – and that the relevant conditions had nothing to do with cellular decompositions of this or that. The same may well turn out, when looking for a generalization of the standard simplicial, cubical or hemispherical chain complexes, giving rise to the homology and cohomology invariants of a given "complex". The kind of set-up I proposed in yesterday's notes, for a formalism of boundary operations in a test category A, now looks to me in some respects somewhat "étriqué", and I'll try another start in a different spirit. ["étriqué" can again be translated as "narrow-minded"]

In order not to get involved in irrelevant technicalities, I assume that the basic localizer $\mathcal{W}$ is $\mathcal{W}_\infty$ = usual weak equivalence. It seems that one basic fact for writing down a relationship between homotopy types and "homology types", is the existence of a canonical "abelianization functor”

$$(\mathrm{Hot}) \to \mathrm{D}_\bullet((\mathrm{Ab})) \quad (\overset{\mathrm{def}}{=} \mathrm{Hotab}), \tag{1}$$

where (Ab) is the abelian category of abelian groups, and $\mathrm{D}_\bullet$ designates the “derived category” of the category $\mathrm{Ch}_\bullet((\mathrm{Ab}))$ of chain complexes of abelian groups, namely its localization with respect to “weak equivalences”, i.e., quasi-isomorphisms:

[p. 322]

$$\mathrm{D}_\bullet(\mathrm{Ab}) = W^{-1}\mathrm{Ch}_\bullet(\mathrm{Ab}), \tag{2}$$

where W means “quasi-isomorphisms”, i.e., maps inducing isomorphisms for all homology groups. The most common way for defining the canonical functor (1), where as usual here (Hot) is defined as $\mathcal{W}^{-1}(\mathrm{Cat})$, is via the test category $\mathbb{\Delta}$, as the composition in the bottom row of

$$\begin{tikzcd}
(\mathrm{Cat}) \arrow[r, "i^*"] \arrow[d] & \mathbb{\Delta}^\wedge \arrow[r, "\mathrm{Wh}_{\mathbb{\Delta}}"] \arrow[d] & \mathbb{\Delta}^\wedge_{\mathrm{ab}} \arrow[r, "\approx^{\mathrm{DP}}"] \arrow[d] & \mathrm{Ch}_\bullet(\mathrm{Ab}) \arrow[d] \\
(\mathrm{Hot}) \arrow[r, "\approx"] & \mathrm{Hot}_{\mathbb{\Delta}} \arrow[r] & \mathrm{Hotab}_{\mathbb{\Delta}} \arrow[r, "\approx"] & \mathrm{D}_\bullet(\mathrm{Ab})
\end{tikzcd} \quad , \tag{3}$$

which is deduced from the top row by passing to the localized categories. The subscript ab in $\mathbb{\Delta}^\wedge_{\mathrm{ab}}$ denotes the category of abelian group objects in $\mathbb{\Delta}^\wedge$, the functor

$$\mathrm{Wh}_{\mathbb{\Delta}} : \mathbb{\Delta}^\wedge \to \mathbb{\Delta}^\wedge_{\mathrm{ab}} \tag{4}$$

is the “*abelianization functor*” obtained by composing a presheaf $\mathbb{\Delta}^{\mathrm{op}} \to$ (Sets) with the abelianization functor

$$(\mathrm{Sets}) \to (\mathrm{Ab}), \quad X \mapsto \mathbb{Z}^{(X)}.$$

We call this functor $\mathrm{Wh}_{\mathbb{\Delta}}$ also the “Whitehead functor”, as its main property is expressed in *Whitehead’s theorem*, namely that it is compatible with weak equivalences (where weak equivalences in $\mathbb{\Delta}^\wedge_{\mathrm{ab}}$ are defined in terms of the underlying semisimplicial sets, forgetting the addition laws). The localized category of $\mathbb{\Delta}^\wedge_{\mathrm{ab}}$ with respect to the latter notion of weak equivalence is denoted by $\mathrm{Hotab}_{\mathbb{\Delta}}$, the functor

$$\mathrm{Hot}_{\mathbb{\Delta}} \to \mathrm{Hotab}_{\mathbb{\Delta}} \tag{4'}$$

induced by $\mathrm{Wh}_{\mathbb{\Delta}}$ may be equally (and still more validly) be designated by $\mathrm{Wh}_{\mathbb{\Delta}}$. The two subscripts $\mathbb{\Delta}$ (in Wh and in Hotab) refer to the fact that the notions make still a sense when $\mathbb{\Delta}$ is replaced by an arbitrary small category A, cf. below.

The functor DP in the top row is the well-known *Dold-Puppe functor*, which is an equivalence of categories. As for i^*, it is defined in terms of an arbitrary test functor

$$i : \mathbb{\Delta} \to (\mathrm{Cat}),$$

which may be either the standard inclusion (which is the more commonly used one) or the canonical functor $a \mapsto \triangle_{/a}$, called $i_\triangle$. The functors i^* corresponding to different choices of i are of course in general non-isomorphic, however (as follows from section 77) the corresponding functor [p. 323]

$$\overline{i^*} : (\mathrm{Hot}) \to \mathrm{Hot}_\triangle$$

is independent of such choice, up to canonical isomorphism.

Instead of $\triangle$, we could have worked with $\square$ or $\bigcirc$ instead, as these give rise to a Dold-Puppe functor (which is still an equivalence), and (almost certainly, see below) to a corresponding variant of the "Whitehead theorem". We thus get two other ways for defining a canonical "abelianization functor" (1) for homotopy types, and it should be an easy and pleasant exercise to show these three functors are canonically isomorphic, using the fact that a product of test categories is again a test category.

Remark. Of course, when concerned mainly with defining a functor (1) we don't really need Whitehead and Dold-Puppe theorems – indeed, instead of taking the functor

$$\mathrm{DP} \circ \mathrm{Wh}_\triangle : \triangle\hat{} \to \mathrm{Ch}_\bullet(\mathrm{Ab}),$$

we could have taken directly (using the standard boundary operations between the components of a semisimplicial abelian group) the standard chain complex structure of $\mathbb{Z}^{(K_\bullet)}$ (for $K_\bullet$ in $\triangle\hat{}$), without taking the trouble and normalizing it à la Dold-Puppe – and it is a lot more trivial than either Whitehead's or Dold-Puppe's theorem, that the latter functor transforms weak equivalences into quasi-isomorphisms; moreover, it gives rise to the same functor

$$\mathrm{Hot}_\triangle \to \mathrm{D}_\bullet(\mathrm{Ab}) \quad (= \mathrm{Hotab}) \tag{5}$$

as $\mathrm{DP} \circ \mathrm{Wh}_\triangle$.

Let now A be *any* small category, we are interested in the functor

$$\mathrm{Hot}_A \to \mathrm{D}_\bullet(\mathrm{Ab}) \quad (= \mathrm{Hotab}) \tag{6}$$

obtained as the composition

$$\mathrm{Hot}_A \to (\mathrm{Hot}) \to \mathrm{D}_\bullet(\mathrm{Ab}), \tag{7}$$

where the second functor is the abelianization functor (1), and the first is the canonical functor, deduced by localization from

$$i_A : A\hat{} \to (\mathrm{Cat}), \quad a \mapsto A_{/a}. \tag{8}$$

We see immediately that for $A = \triangle$, the functor (6) reduces to (5) up to canonical isomorphism – and the same of course when A is either $\square$ or $\bigcirc$. In these three cases, the functor (6) can be factorized in a natural way through the category [p. 324]

$$\mathrm{Hotab}_A = W^{-1} A\hat{}_{\mathrm{ab}},$$

where now W stands for the set of "weak equivalences" in $A^{\wedge}_{\mathrm{ab}}$, defined in the same way as above in the case $A = \Delta$. The question then arises, for any given A, as to whether such a factorization can be still obtained, and how exactly.

This formulation is inspired by the description of abelianization of homotopy types via the (slightly sophisticated diagram) (3). When following the more naive approach of the remark above, this leads us to the closely related question of defining (6) via a composition

$$A^{\wedge} \xrightarrow{\mathrm{Wh}_A} A^{\wedge}_{\mathrm{ab}} \xrightarrow{L} \mathrm{Ch}_{\bullet}(\mathrm{Ab}), \tag{9}$$

(for a suitable functor L, cf. below), by passing to localizations.

Both approaches seem to me of interest. The first one, to make sense at all as stated, relies on the existence of a canonical functor

$$\mathrm{Hot}_A \to \mathrm{Hotab}_A, \tag{9'}$$

induced by the abelianization functor

$$\mathrm{Wh}_A : A^{\wedge} \to A^{\wedge}_{\mathrm{ab}}, \quad X \mapsto \mathbb{Z}^{(X)},$$

i.e., on the *validity of Whitehead's theorem, with Δ replaced by A.* This looks like an interesting question, whose answer should be in the affirmative. At any rate, if we can find an aspheric functor

$$j : \Delta \to A^{\wedge} \tag{10}$$

(with respect to the standard asphericity structure of A), then the answer is affirmative, as we are immediately reduced to the known case A is replaced by Δ. Thus, the answer is possibly tied up with the question of existence of test functors, which we'll deal with presumably in part V. It should be noted though that if such a functor (10) exists, then necessarily A is aspheric, and even totally aspheric – a substantial restriction indeed.

More generally, let

$$j : B \to A^{\wedge} \tag{10'}$$

any aspheric functor with respect to the standard asphericity structure
[p. 325] of $A^{\wedge}$, where B is any small category. The corresponding functor

$$j^* : A^{\wedge} \to B^{\wedge} \tag{11}$$

then satisfies

$$(j^*)^{-1}(\mathcal{W}_B) = \mathcal{W}_A,$$

and the corresponding functor for the localizations gives rise to a commutative diagram

$$\begin{tikzcd} \mathrm{Hot}_A \arrow[rr, "\overline{j^*}"] \arrow[dr, "\overline{i_A}"'] & & \mathrm{Hot}_B \arrow[dl, "\overline{i_B}"] \\ & (\mathrm{Hot}) & \end{tikzcd} ,$$

and hence the corresponding diagram

(12)
$$\begin{tikzcd} \mathrm{Hot}_A \arrow[rr, "\overline{j^*}"] \arrow[dr] & & \mathrm{Hot}_B \arrow[dl] \\ & \mathrm{D}_\bullet(\mathrm{Ab}) = \mathrm{Hotab} & \end{tikzcd}$$

is commutative, where the vertical arrows are the canonical functors (6). Coming back to the base $B = \blacktriangle$ say, this shows that (6) can be viewed as the composition

$$\mathrm{Hot}_A \xrightarrow{\overline{j^*}} \mathrm{Hot}_{\blacktriangle} \xrightarrow{\overline{\mathrm{Wh}_{\blacktriangle}}} \mathrm{Hotab}_{\blacktriangle} \xrightarrow[\approx]{\overline{\mathrm{DP}}} \mathrm{Hotab},$$

and hence it can be inserted in the commutative diagram

(13)
$$\begin{tikzcd} \mathrm{Hot}_A \arrow[r] \arrow[d] & \mathrm{Hot}_{\blacktriangle} \arrow[d] & \\ \mathrm{Hotab}_A \arrow[r] & \mathrm{Hotab}_{\blacktriangle} \arrow[r, "\approx"] & \mathrm{Hotab} \end{tikzcd},$$

where the functor

$$\mathrm{Hotab}_A \to \mathrm{Hotab}_{\blacktriangle}$$

is induced by j^*_{ab}. Thus, we get the wished for factorization of (6) via Hotab_A, *provided we can find an aspheric functor* (10). It should not be hard moreover to see that the factorizing functor obtained from (13), namely

(14) $$\mathrm{Hotab}_A \to \mathrm{Hotab},$$

does not depend up to canonical isomorphism on the choice of j, at least in the case when A is a contractor, using the end remarks of section 82 (p. 272) concerning products of aspheric functors.

The question remains whether we can define (14) for (more or less) any small category A, without having to rely upon the existence of an aspheric functor (10), in such a way that it factors the canonical functor (6) (granting Whitehead's theorem holds for A), and moreover that for an aspheric functor (10') $j : B \to A^\wedge$, giving rise to [p. 326]

$$\overline{j^*_{\mathrm{ab}}} : \mathrm{Hotab}_A \to \mathrm{Hotab}_B,$$

then corresponding diagram

(15)
$$\begin{tikzcd} \mathrm{Hotab}_A \arrow[rr] \arrow[dr] & & \mathrm{Hotab}_B \arrow[dl] \\ & \mathrm{Hotab} & \end{tikzcd}$$

should commute, where the vertical arrows are the functors (14).

For defining (14) in this general case, recalling that

$$\mathrm{Hotab} \xleftarrow{\approx} \mathrm{Hotab}_{\Delta},$$

we can't help it and have to use diagram (3) and the functor

$$i_A : A^{\wedge} \to (\mathrm{Cat}),$$

or rather ((Cat) serving only as an intermediary) the functor

$$u \overset{\mathrm{def}}{=} i^* i_A : A^{\wedge} \to \Delta^{\wedge},$$

so that (6) can be viewed as deduced by localization of $A^{\wedge}$ from the composition

$$(*) \qquad A^{\wedge} \xrightarrow{u} \Delta^{\wedge} \xrightarrow{\mathrm{Wh}_{\Delta}} \Delta^{\wedge}_{\mathrm{ab}} \to \mathrm{Hotab}_{\Delta} \quad (\xrightarrow{\approx} \mathrm{Hotab}).$$

One difficulty here is that u does not commute to finite products, and hence doesn't induce a functor

$$A^{\wedge}_{\mathrm{ab}} \to \Delta^{\wedge}_{\mathrm{ab}},$$

it would seem. Now this difficulty, I just noticed, can be overcome, using the fact that i_A and hence also *u commutes to fibered products*, or, what amounts to the same, induces an *exact* functor

$$u_0 : A^{\wedge} \to \Delta^{\wedge}{}_{/F} \xrightarrow{\approx} (\Delta_{/F})^{\wedge},$$

where

$$F = u(e_{A^{\wedge}}) = i^*(A)$$

is a suitable object in $\Delta^{\wedge}$. A fortiori, u_0 commutes to finite products, hence transforms abelian group objects into same, i.e., induces

$$u_{0\mathrm{ab}} : A^{\wedge}_{\mathrm{ab}} \to (\Delta_{/F})^{\wedge}_{\mathrm{ab}},$$

[p. 327] on the other hand, we do have too a natural functor

$$(16) \qquad \alpha^{\mathrm{ab}}_{!} : (\Delta_{/F})^{\wedge}_{\mathrm{ab}} \to \Delta^{\wedge}_{\mathrm{ab}},$$

defined as the left adjoint of the evident functor

$$\alpha^*_{\mathrm{ab}} : \Delta^{\wedge}_{\mathrm{ab}} \to (\Delta_{/F})^{\wedge}_{\mathrm{ab}}$$

induced by the left exact functor α^*, where α is the "localization morphism of topoi"

$$\alpha : \Delta_{/F} \to \Delta_F$$

defined by the object F of $\Delta^{\wedge}$. We thus get a diagram

$$(17) \qquad \begin{tikzcd}
A^{\wedge} \arrow[r, "\mathrm{Wh}_A"] \arrow[d, "u_0"'] \arrow[dd, bend right=60, "u"'] & A^{\wedge}_{\mathrm{ab}} \arrow[d, "u_{0\mathrm{ab}}"] & & \\
(\Delta_{/F})^{\wedge} \arrow[r, "\mathrm{Wh}_{\Delta_{/F}}"] \arrow[d, "\alpha_!"'] & (\Delta_{/F})^{\wedge}_{\mathrm{ab}} \arrow[d, "\alpha^{\mathrm{ab}}_!"] & & \\
\Delta^{\wedge} \arrow[r, "\mathrm{Wh}_{\Delta}"] & \Delta^{\wedge}_{\mathrm{ab}} \arrow[r] & \mathrm{Hotab}_{\Delta} \arrow[r] & \mathrm{Hotab}
\end{tikzcd},$$

containing (*) above as the composition of maps in the left-hand vertical column and in the bottom row. The lower square in this diagram commutes (up to natural isomorphism) – which is a general fact surely for morphisms of topoi $f : X \to Y$ such that $f_!$ exists, which allows to define too a functor $f_!^{\mathrm{ab}}$ as (16) above. The upper square though doesn't look at all commutative, too bad! The only hope left now is that the natural compatibility arrow for this square (there is bound to be one, isn't there!), when composed with the lower square (so as to give a compatibility map for the composite rectangle) *should give rise to a weak equivalence in* $\mathbb{\Delta}^{\wedge}_{\mathrm{ab}}$, for any choice of an object X in $A^{\wedge}$.

It would seem to me that a reasonable functor (14) will exist, without an existence assumption of a test functor (10), exactly in those cases when the composite rectangle in (17) is "commutative up to weak equivalence". I have no idea whether or not this is true for any small category A, not even (I confess) when there *is* an aspheric functor (10) – as a matter of fact, I don't feel like going any further now in this direction, and trying to check anything whatsoever.

I was a little rash in the definite statement I made about the "exact" assumption to make for a "reasonable" functor (14) to exist; another seems needed still, namely that the functor [p. 328]

$$A^{\wedge}_{\mathrm{ab}} \to \mathbb{\Delta}^{\wedge}_{\mathrm{ab}} \tag{18}$$

we obtain by composing the two arrows in the right hand vertical column, transforms weak equivalences into same, which is needed in order to deduce (14) by passing to the localized categories. When the two assumptions are satisfied, then the functor (14) obtained from (18) does factorize (6) as required, and it should be clear too that it satisfies the compatibility (15).

Thus, it seems there are good prospects for getting canonical functors

$$\mathrm{Hot}_A \to \mathrm{Hotab}_A \to \mathrm{Hotab},$$

whose composition is (6), i.e., inserting into the commutative diagram

$$\begin{tikzcd} \mathrm{Hot}_A \arrow[r] \arrow[d] & \mathrm{Hotab}_A \arrow[d] \\ \mathrm{Hot} \arrow[r] & \mathrm{Hotab} \end{tikzcd} \quad . \tag{19}$$

The next question then which arises is *whether the second vertical arrow* (namely (14)) *is an equivalence, whenever the first one is*, i.e., when A is a pseudo-test category, or whether this is true if we make some familiar extra assumption on A, such as being an actual test category say.

I feel I am getting gradually back into thin air conjecturing, I wouldn't go on too long this way! This whole Hotab_A business was just a digression, which then took me longer than expected, it doesn't seem to have much to do with what I have been out for in this section, namely afterthoughts about *boundary operations in a test category*, which are designed to gave a "computational" description of the canonical functor (6), the latter being defined without any restriction nor difficulty for *any* small category A. More accurately still, we want to describe the composition of (6) with the canonical functor $A^\wedge \to \mathrm{Hot}_A$, namely $A^\wedge \to \mathrm{Hotab}$, via a suitable functor $A^\wedge \to \mathrm{Ch}_\bullet(\mathrm{Ab})$, with the expectation that the latter should factor through $A^\wedge_{\mathrm{ab}}$ via the abelianization functor Wh_A. In other words, we are looking for commutative diagrams

(20)
$$\begin{tikzcd} & A^\wedge \arrow[r] \arrow[dl] \arrow[d, "K"] & \mathrm{Hot} \arrow[d] \\ A^\wedge_{\mathrm{ab}} \arrow[r, "L"'] & \mathrm{Ch}_\bullet(\mathrm{Ab}) \arrow[r] & \mathrm{Hotab} \end{tikzcd},$$

[p. 329] where all functors in the diagram, except for K and L, are the canonical ones familiar to us. The question then is how to define a suitable L, such that the corresponding square (where $K = L \circ \mathrm{Wh}_A$) should commute up to (canonical?) isomorphism.

There *is* such an L, whenever we have an aspheric functor (10) (where $\triangle$ may be replaced by one of its twins), using corresponding semisimplicial chain complexes – but, as already remarked yesterday, taking things this way is "cheating"! Conceivably too, there are quite general theorems asserting that a functor from a category $A^\wedge$ say to a derived additive category such as Hotab can be lifted to the category of models (here abelian chain complexes) it comes from, and possibly even in a way factoring through $A^\wedge_{\mathrm{ab}}$? I don't intend to dive into these questions either, but rather, make a comment on a general method for constructing certain functors

(21) $$L : A^\wedge_{\mathrm{ab}} \to \mathrm{Ch}_\bullet(\mathrm{Ab}),$$

(maybe not in a way to give rise to a commutative diagram (20)), as suggested by the standard chain complexes associated to the three types of complexes, using simplices, cubes or hemispheres, or multicomplexes (using products of the standard test categories). Writing

$$A^\wedge_{\mathrm{ab}} \simeq \mathbf{Hom}(A^{\mathrm{op}}, (\mathrm{Ab})),$$

we remark that the standard constructions of chain complexes associated to semisimplicial (say) complexes of abelian groups, makes sense not only for such abelian complexes, but more generally for *complexes with values in* any *additive category*, M say. This induces us to look more generally, for any such M, for a functor

(22) $$L_M : \mathbf{Hom}(A^{\mathrm{op}}, M) \to \mathrm{Ch}_\bullet(M),$$

in a way *compatible with additive functors*

$$M \to M'$$

(in the obvious sense of the word).

Now, for any category B (here A^{op}) one can define an "*enveloping additive category*" $\mathrm{Add}(B)$, together with a canonical functor

(23) $$B \to \mathrm{Add}(B),$$

which is "2-universal" for all possible functors of B into any additive category M. More specifically, for any such M, the corresponding functor "composition with (23)" is an *equivalence* [p. 330]

(24) $$\mathbf{Homadd}(\mathrm{Add}(B), M) \xrightarrow{\approx} \mathbf{Hom}(B, M).$$

This condition defines (22) "up to canonical equivalence" – but we'll give an explicit description in a minute. Before doing so, let's just remark that the universal property of (23) implies that to give a system of functors L_M as above, "amounts to the same" as giving a chain complex $L_\bullet$ in $\mathrm{Add}(B)$. More accurately, the category of all systems L_M (where maps are defined in an evident way) is *equivalent* to the category $\mathrm{Ch}_\bullet(\mathrm{Add}(B))$, where $B = A^{\mathrm{op}}$. The functors L (21) we are specifically interested in, are those which are associated to some chain complex in $\mathrm{Add}(A^{\mathrm{op}})$,

(25) $$L_\bullet \in \mathrm{Ob}(\mathrm{Ch}_\bullet(\mathrm{Add}(A^{\mathrm{op}}))),$$

by the formula

(26) $$L(X) = \widetilde{X}(L_\bullet) \quad \text{for any } X \text{ in } A^\wedge_{\mathrm{ab}} \simeq \mathbf{Hom}(A^{\mathrm{op}}, \mathrm{Ab})$$

where
$$\widetilde{X} : \mathrm{Add}(A^{\mathrm{op}}) \to (\mathrm{Ab})$$
is the *additive* functor corresponding to X.

We are thus led to the question: if A is any small category, does there exist a chain complex $L_\bullet$ in $\mathrm{Add}(A^{\mathrm{op}})$, the additive envelope of A^{op}, giving rise to a functor (21) via (26) and hence to a diagram (20), such that the square in (20) commutes up to isomorphism? And when this is so, what kind of unicity statement, if any, can be made for $L_\bullet$ (such as being unique up to chain homotopy say), and what about the structure of the category of all pairs $(L_\bullet, \lambda)$, where λ is a compatibility isomorphism making the square in (20) commute?

We are far here from the rather narrow set-up in yesterday's notes, and as far as existence goes, if no extra conditions are put upon $L_\bullet$, it seems likely that for a rather large class of small categories A (if not all) it should hold true. At any rate, the functor L obtained from a functor (10), i.e., "by cheating", is visibly associated to an $L_\bullet$. Sorry, we have to assume that the functor j factors even through A itself, i.e., is just an aspherical functor between the small categories $\mathbb{A}$ and A, a much more stringent condition on A to be sure – and which implies that there is an induced functor [p. 331]

(27) $$\mathrm{Add}(j^{\mathrm{op}}) : \mathrm{Add}(\mathbb{A}^{\mathrm{op}}) \to \mathrm{Add}(A^{\mathrm{op}}),$$

hence we get an $L_\bullet$ as the image of the canonical chain complex $L_\bullet^{\mathbb{A}}$ we got in $\mathrm{Add}(\mathbb{A}^{\mathrm{op}})$.

6.7.

The afterthought continued: abelianizators, and "standard" abelianizators for categories with boundary operators.

93 It is time to give the promised construction of $\mathrm{Add}(B)$, the additive envelope of B, for any given category B. The obvious idea is to enlarge the sets $\mathrm{Hom}(a,b)$, for a and b in B, by taking linear combinations with coefficients in $\mathbb{Z}$, i.e., writing

(1) $$\mathrm{Hom}_{\mathrm{Add}(B)}(a,b) = \mathbb{Z}^{(\mathrm{Hom}(a,b))},$$

and composing these $\mathrm{Hom}_{\mathrm{Add}}$ in the obvious way. This is not quite enough though, as we still have to add new objects, namely direct sums of objects in B. The most convenient way for doing so seems by defining an object of $\mathrm{Add}(B)$ to be defined by a finite set I (in the given universe), namely the indexing set for taking the direct sum, and a map

$$I \to \mathrm{Ob}\, B,$$

in other words, the new objects are just *families* of objects of B

$$(b_i)_{i\in I},$$

indexed by finite sets. We'll however denote by

(2) $$\bigoplus_{i\in I} b_i$$

the corresponding object of $\mathrm{Add}(B)$, as this will turn out to be the direct sum indeed of the images of the b_i's in $\mathrm{Add}(B)$ – but of course we'll ignore the possible existence of direct sums in B itself, when they exist, and not confuse (2) with a direct sum taken in B. Writing Homadd instead of $\mathrm{Hom}_{\mathrm{Add}(B)}$ for the sake of abbreviation, the maps between objects (2) are defined by matrices in the obvious way

(3) $$\mathrm{Homadd}((a_i)_{i\in I},(b_j)_{j\in J}) = \\ \left\{(u_{ij})_{(i,j)\in I\times J} \,\middle|\, u_{ij} \in \mathrm{Homadd}(a_i,b_j) = \mathbb{Z}^{\mathrm{Hom}(a_i,b_j)}\right\},$$

while composition of maps is defined by the composition of matrices. We thus get a new category $\mathrm{Add}(B)$ and a functor

(4) $$B \to \mathrm{Add}(B),$$

[p. 332] it is immediately checked that $\mathrm{Add}(B)$ is an additive category and that the functor (4) has the 2-universal property for functors from B into any additive category, stated in yesterday's notes (p. 330).

Remark. The same construction essentially applies when considering the universal problem of mapping B into any k-*additive* category M (where k is any commutative ring with unit), i.e., an additive category M endowed with a ring homomorphism

$$k \to \mathrm{End}(\mathrm{id}_M),$$

replacing $\mathbb{Z}$ by k in formulas (1) and (3). The "abelianization" questions touched at in yesterday's notes still make sense in terms of "k-linearization" – a notion much in the spirit of our introduction of a general basic localizer $\mathcal{W}$, as the very notion of k-linearization will give rise to a corresponding basic localizer $\mathcal{W}_k$...

Let's come back to the case when $B = A^{\mathrm{op}}$, A being a small category, and to our question about chain complexes

$$L_\bullet \quad \text{in} \quad \mathrm{Ch}_\bullet(\mathrm{Add}(A^{\mathrm{op}}))$$

giving rise to a commutative diagram (20) (p. 328), up to isomorphism. A pair

$$(L_\bullet, \lambda),$$

where $L_\bullet$ is a chain complex as above, and λ a compatibility isomorphism for the square in (20), could be suggestively called an *abelianizator* for the small category A. The question of existence, and uniqueness up to homotopy say, of an abelianizator for A seems especially relevant when A is a test category say, and hence $A^\wedge$ modelizes homotopy types. In any case, in terms of an abelianizator we get an additive functor

$$L : A^\wedge_{\mathrm{ab}} \to \mathrm{Ch}_\bullet(\mathrm{Ab}), \tag{5}$$

and the question arises whether this is compatible with weak equivalences and quasi-isomorphisms; maybe even if this is not automatic, we should insist it holds when defining the notion of an abelianizator. When this is OK, then by passing to localizations we deduce from (5) a functor

$$\mathrm{Hotab}_A \to \mathrm{Hotab}, \tag{6}$$

i.e., a functor (14) as looked for in yesterday's notes, giving rise to the commutative diagram (19) (p. 328) – whereas commutativity of diagrams of the type (15) (p. 326) looks less obvious.

When A is a finite product of copies taken from among the three standard test categories $\triangle$, $\square$, $\circledcirc$, the standard chain complex structure on multicomplexes does furnish us with a *"canonical" abelianizator* for A, which we may denote by $L^A_\bullet$ (as we did yesterday for $A = \triangle$). This "standard" abelianizator has some very remarkable extra features which I would like to pin down, which had caused our rather narrow focus in the notes of two days ago (section 91). [p. 333]

a) There is a "dimension map"

$$\dim : \mathrm{Ob}\, A \to \mathbb{N}. \tag{7}$$

It can be described (in the particular case above at any rate) in terms of the intrinsic category structure of A, by associating to every a in A the ordered set

$$i(a) = \text{set of subobjects of } a \text{ in } A. \tag{8}$$

(NB not to be confused with subobjects of a in $A^\wedge$, namely sieves in a). This is an ordered set with a largest object (namely a itself), and which turns out to be finite (in the particular cases considered), hence of finite combinatorial dimension (equal to the dimension of the geometrical realization $|i(a)|$), and we have

$$\dim(a) = \dim i(a). \tag{9}$$

b) The n'th component L_n of $L_\bullet^A = L_\bullet$ is given by

$$L_n = \bigoplus_{\dim(a)=n} a \tag{10}$$

(where the direct sum of course is taken in $\mathrm{Add}(A^{\mathrm{op}})$ as in (2) above), which makes sense when we assume (as is the case in our example) that the map (7) is "finite", i.e., has finite fibers. For instance, in all our "standard" cases, there is just *one* object of A which is of dimension 0, and this is also the final object.

c) The differential operator

$$d_n : L_n \to L_{n-1} \tag{11}$$

can be obtained in the following way. We have only to describe d_n on each summand a of L_n, i.e., by (10) on each a in

$$A_n = \left\{ a \in \mathrm{Ob}\,A \;\middle|\; \dim a = n \right\}. \tag{12}$$

In view of (3), this restriction $d_n \mid a$ can be described as a linear combination of elements in the disjoint sum of the sets

$$\mathrm{Hom}(b, a), \quad \text{with} \quad b \in A_{n-1} \quad (a \in A_n \text{ fixed}).$$

[p. 334] This being clear, the non-zero coefficients which occur in this linear combination are all ± 1, and moreover the maps which target a

$$b \to a \quad (b \in A_{n-1})$$

which occur with non-zero coefficient are exactly *all monomorphisms* from objects b in A_{n-1} into a. Thus, the differential operators are known, when we know, for all monomorphisms in A

$$\partial : b \to a, \quad \text{with} \quad \dim a = \dim b + 1 \tag{13}$$

(the so-called *"boundary maps"*), the corresponding coefficients

$$\varepsilon_\partial \in \{\pm 1\}. \tag{14}$$

Instinct tells us, at this point, that we may get into trouble, when trying to define (in a more or less general case) boundary operations in such a way, because of the ambiguity in the definition of subobjects, namely, because of possible *existence of isomorphisms which may not be identities*. But precisely, in the standard cases we are copying from, *any isomorphism is an identity*!

d) For describing the "signatures" (14), in one of the "standard" cases, we still need to remark that for any a in A, we have

(15) $$i(a) \text{ is an } n\text{-cell, with } n = \dim(a),$$

and the choice of the signatures will be determined by a choice of orientations

(16) $$\omega_a \text{ an } \textit{orientation} \text{ of the } n\text{-cell } |i(a)|,$$

(a notion which could be given a purely combinatorial definition, by induction on the dimension of a given ordered set whose geometrical realization is a variety...). We then get a the *"Stokes rule"*

(17) For a boundary map $\partial : b \to a$, $\varepsilon_\partial = +1$ iff ω_b is "induced" à la Stokes by ω_a, via the induced orientation on the boundary of $i(a)$ (which is the union of the images of all $i(b)$'s, for all boundary operations with target a).

Whether or not we are in a "standard" case, if A is any category such that for any object a of A, the ordered set $i(a)$ of its subobjects in A is finite, and its geometrical realization is an n-cell (call n the "dimension" of a), and if moreover for a given n, the set A_n of objects with dimension n is finite, and also (to be on the safe side!) assuming that all isomorphisms are identities, then for *any* choice of orientations (16), giving rise to a system of signatures (13) by the "Stokes rule" (17), [p. 335]
the corresponding operators (11) do turn the family (L_n) into a chain complex, namely we have the relations

(18) $$d_{n-1} d_n = 0.$$

This follows immediately from the well-known anti-commutativity property of (twofold) induction of orientation on boundaries.

Things are a little more delicate if we don't assume that isomorphisms are identities, even if (by compensation) we should insist that two distinct objects are never isomorphic. To define d_n, we then must *choose*, for any subobject b of dimension $n-1$ of an object a of dimension n, just *one* representative monomorphism (13) of b. The coherence condition then needed in order to get (18) is that any square diagram

(19) $$\begin{tikzcd} & c \arrow[dl] \arrow[dr] & \\ b \arrow[dr] & & b' \arrow[dl] \\ & a & \end{tikzcd}$$

made up with such restricted boundary maps, should commute – a somewhat delicate condition, presumably hard to ensure, for the choices involved for defining the "strict" boundary maps in A.

In one case as in the other, we are very close of course to the set-up envisioned in section 91 – it wouldn't be hard even to fit the case considered here into this set-up, if we make the slight extra assumption that any map in A factors into an epimorphism-with-section, followed by a monomorphism (which is true indeed in the "standard" cases), which will ensure that for varying a, $i(a)$ is indeed a functor with values in (Ord), as stated in loc. cit. But from the point of view of construction of abelianizators, it would seem that the existence of the functor

$$i : A \to (\mathrm{Ord}) \tag{20}$$

is irrelevant.

Our main question now, of course, is about *the chain complex $L_\bullet$ being an abelianizator or not*. The question is interesting even in the standard cases, by choosing the orientations (16) in a way different from the standard one. Are the corresponding chain complexes in $\mathrm{Add}(A^{\mathrm{op}})$ necessarily chain-homotopic?

[p. 336] It just occurs to me that indeed, between the chain complexes

$$L_\bullet^\omega = ((L_n)_{n\in\mathbb{N}}, (d_n^\omega)_{n\in\mathbb{N}}) \tag{22}$$

associated to all possible systems of orientations

$$\omega = (\omega_a)_{a\in\mathrm{Ob}(A)} \tag{23}$$

of the various cells $i(a)$, there is a canonical *transitive system* of isomorphisms, by defining the isomorphism

$$u_{\omega,\omega'} : L_\bullet \xrightarrow{\sim} L'_\bullet \tag{24}$$

for two different choices ω, ω' of systems of orientations, by

$$u_{\omega,\omega'} \mid a = \varepsilon_a^{\omega,\omega'} \operatorname{id}_a, \quad \varepsilon_a^{\omega,\omega'} \in \{\pm 1\}, \tag{25}$$

where the sign $\varepsilon_a^{\omega,\omega'}$ is equal to $+1$ or $-!$, *according to whether ω_a and ω'_a are equal or not*. It is immediate that (24) then is an isomorphism componentwise, respecting degrees, and commuting to the respective differential operators. Transitivity of the isomorphisms (24) for a triple $(\omega, \omega', \omega'')$ is equally immediate. This implies that by this transitive system of isomorphisms, *we may identify all the chain complexes $L_\bullet^\omega$* in $\mathrm{Add}(A^{\mathrm{op}})$ to a single chain complex, canonically isomorphic to each $L_\bullet^\omega$, and which we may just designate by $L_\bullet$. This chain complex now is defined intrinsically in terms of the category structure of A (up to canonical isomorphism), in the "safe" case at any rate when every isomorphism in A is an identity, so that in the construction of $L_\bullet^\omega$ there enters no other choice besides ω. Otherwise as seen above (precedent page), we must still suitably choose the so-called "strict" boundary operators (13), among all monomorphisms $b \hookrightarrow a$ in A such that $\dim a = \dim b + 1$.

In the first case say (isomorphisms being identities), all conditions considered for A are stable under finite products, that's why in terms of the three standard cases of $\triangle$, $\square$ and $\bigcirc$, we could construct others by taking finite products. The three standard test categories may be viewed as particularly "economic" of skillful ways of "cutting out" a suitable bunch of cellular decompositions, and of eliminating automorphisms (by total ordering of vertices and the like...), so as to ensure: a) that isomorphisms in A are identities, b) the canonical chain complex $L_\bullet$ in $\mathrm{Add}(A^{\mathrm{op}})$ is an abelianizator, and c) A moreover is a strict test category, and even a contractor. On the other hand, as all these conditions (plus the condition (15) of course about the $i(a)$'s representing n-cells) are stable under taking products (of finite non-empty families of categories A_i), hence in terms of the three standard cases, the possibility of satisfying them too by the "multistandard" test categories. I wonder if there are any other ways (up to equivalence). If we take categories such as $\widetilde{\mathbb{\Delta}}$ (non-ordered simplices), we still get contractors, but objects have non-trivial automorphisms, and if we take categories such as $\mathbb{\Delta}^{\mathrm{f}}$ (ordered simplices without degeneracy operations, only boundary maps), it is true that isomorphisms are identities, but the category is no longer a test category but only a weak one. [p. 337]

If we do not insist on the rigidity assumption (isomorphisms are identities), but on suitable choice of so-called "*strict* boundary operations" within A, then it would seem after all that we do have a lot more elbow freedom than it seemed by the end of our reflections on that matter two days ago (cf. p. 318), where the picture of the relevant data and corresponding construction of chain complexes was still a little confused. Let now A be the category called A_0 in loc. cit. We don't have to modify it in order to introduce orientations of cells $i(a)$ as extra structure and take account of this in defining a new notion of maps. Therefore, it is clear that *A just as it is, is a strict test category* (presumably *not* a contractor though). There is problem of course of isomorphisms which are not identities, and particularly of non-trivial automorphisms – for instance the object I (playing the part of the unit segment) has a non-trivial automorphism, the elimination of which does not look so trivial! However, there *is* a rather evident way of cutting out *strict* boundary maps, in a way as to satisfy the transitivity condition of p. 335 – namely by taking boundary maps (13) $\partial : b \to a$ which are *inclusions* in the strict sense, namely the inclusion map of a *subset* of a, endowed with the induced order relation.

Thus, there are many other cases still than just multi-standard test categories for getting a canonical chain complex $L_\bullet$ in $\mathrm{Add}(A^{\mathrm{op}})$, and for which now the question makes sense as to whether $L_\bullet$ is an abelianizator. In the construction above, we were careful to assume that the full subcategory A of (Ord), besides containing I, was stable under finite products, so as to make sure it comes out as a test category. The silly thing is that this condition is not satisfied by any one among the standard test categories – thus, it seems reasonable to try and replace it by a suitable substitute, such as the existence, of any two objects a and b in A, of a cellular subdivision of $|i(a)| \times |i(b)|$, made up with cells of the type $|i(c)|$, and inducing on the latter the given cellular structure of $|i(c)|$. The problem is now (besides getting or not an abelianizator $L_\bullet$) *whether A is at any rate a weak test category* (in view of the example $\mathbb{\Delta}^{\mathrm{f}}$, we can't expect now of course to get an actual test category). Maybe I'll [p. 338] come back to this later, when writing down a proof for $\triangle^{\mathrm{f}}$ being a weak test category, i.e., a more general result along these lines should come out alongside.

Remarks. I feel the canonical chain complex $L_\bullet$ in $\mathrm{Add}(A^{\mathrm{op}})$ constructed in this section, under suitable assumptions on the small category A, merits a name of its own. We may call it the *standard abelianizator* of A – but this is reasonable only if it turns out that in all cases when it can be constructed, it is an abelianizator indeed. Another convenient name may be the *Dold-Puppe chain complex*, as in the three standard cases, the standard Dold-Puppe construction of the "normalization" of an abelian complex (ss say) can be viewed as being performed in the "universal" case, namely for $A^{\mathrm{op}} \to \mathrm{Add}(A^{\mathrm{op}})$, and the corresponding "full" chain complex, namely $L_\bullet$ – with this grain of salt though that we still have to enlarge $\mathrm{Add}(A^{\mathrm{op}})$ slightly, so as to make stable under taking direct summands corresponding to projectors. But then it occurs to me that the name of Dold-Puppe chain complex is much more suitable for the *result of normalization* applied to $L_\bullet$, which (if I got it right) is the "new" complex discovered by Dold-Puppe, together with the inverse construction, whereas $L_\bullet$ had already been known for ages (even if not under its universal disguise. . .).

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Afterthought (continued): retrospective on the "De Rham complex with divided powers" and on some wishful thinking about linearization of homotopy types and arbitrary ground-ring extension in homotopy types.

In the last section, as in the two preceding days, our emphasis with abelianization of homotopy types has been to look at it in terms of more or less arbitrary test categories and the corresponding elementary modelizers, and even in terms of arbitrary small categories. This has causes as spinning a kind of dream for a while, with the Whitehead and Dold-Puppe theorems and generalized boundary maps as our main thread. Now this reminds me of a rather different line of thoughts tied up with abelianization, quite independently of playing around with variable modelizers – a question which has been intriguing me for a very long time now, ever since I got acquainted a little with the very notion of homotopy types, and the corresponding homology and cohomology invariants. This is the question of *how far a homotopy type can be expressed in terms of homology or cohomology invariants* (or both together), plus *some relevant extra structure*, the most important surely being cup-products in cohomology (or, dually, "interior" operation of cohomology on homology). Once the notion of derived categories of various kinds had become familiar, in the early sixties, the question would appear as expressing, or recovering, a homotopy type, namely an object in the (highly non-abelian) "derived category" (Hot), in terms of its abelianization in $\mathrm{Hotab} = \mathrm{D}_\bullet(\mathrm{Ab})$, *endowed with suitable extra structure*. It was about clear that this extra structure had to include, as its main non-commutative item, the fundamental group π, so as to allow for description of homology and cohomology invariants with twisted coefficients. The most natural candidate for expressing this would be the chain complex associated to the universal covering, viewed as an

[p. 339]

[I wonder whether AG knew of the Steenrod algebra. . . yes, see below] object in the derived category

(26) $$\mathrm{D}_\bullet(\mathbb{Z}(\pi))$$

of chain complexes of modules over the group ring $\mathbb{Z}(\pi)$. Another important structural item, giving rise to all cup-products with non-twisted coefficients, is the diagonal map for the abelianization

$$L_\bullet \quad \text{in} \quad \mathrm{Hotab} = \mathrm{D}_\bullet(\mathrm{Ab}),$$

namely a map

(27) $$L_\bullet \to L_\bullet \overset{\mathrm{L}}{\otimes} L_\bullet,$$

where $\overset{\mathrm{L}}{\otimes}$ is the "total" left derived functor of tensor product. This map is subjected to suitable conditions, concerning mainly commutativity and associativity. In case of a non-1-connected space, i.e., $\pi \neq 1$, it shouldn't be hard combining the two structural items so as to get a structure embodying at any rate cup-products with arbitrary twisted coefficients. One key question in my mind, which I never really looked into, was whether these two structures were enough in order to reconstruct entirely (up to canonical isomorphism) the (pointed, 0-connected) homotopy type giving rise to it, and hence also any other homotopy invariants, such as "operations" on cohomology and the like, K-invariants, etc. [aka the Tor functor]

If I got it right, it has been known now for quite a while that even for a 1-connected homotopy type, so that the relevant structure reduces to (27), that this is *not* quite enough for recovering the homotopy type, maybe not even the rational homotopy type. I believe I first got this from Sullivan, namely that what was needed for recovering a 1-connected rational homotopy type was not merely (27) (where now $L_\bullet$ is an object of $\mathrm{D}_\bullet(\mathbb{Q})$ rather than of $\mathrm{D}_\bullet(\mathbb{Z}) = \mathrm{D}_\bullet(\mathrm{Ab})$), which reduces more or less (under suitable finiteness assumption) to knowing the rational cohomology ring, but an anti-commutative and associative *differential graded algebra* over $\mathbb{Q}$ (giving rise to (27) by duality). Thus, 1-connected rational homotopy types are expressible as objects of the derived category defined in terms of such algebras, and the obvious notion of quasi-isomorphism for these. To any space or ss set, Sullivan associates a corresponding *"De Rham complex"* with rational coefficients, in order to get a functor from rational homotopy types to the derived category obtained from those algebras – and (if I remember it right) this is an equivalence of categories, provided one restricts to 1-connected homotopy types, and correspondingly to 1-connected algebras. Probably somebody must have explained to me by then (it was in 1976 more or less) why not every eligible differential algebra could be recovered (up to isomorphism in the derived category) by the corresponding cohomology algebra, namely why it was not necessarily isomorphic to the latter, endowed with zero differential operator; I am afraid I forgot it since! Also, it was well-known by the informed people (as I was told too) that there where obstructions against expressing the multiplicative structure in cohomology with (say) integer coefficients, in terms of an [p. 340] anti-commutative differential graded $\mathbb{Z}$-algebra; so there was no hope, I was informed, for defining something like a "De Rham complex with integer coefficients" for an arbitrary topological space.

All this was very interesting indeed – still, I found it hard to believe that, while succeeding in constructing De Rham complexes with rational coefficients for arbitrary spaces, by looking at the algebraic De Rham complex on the enveloping affine space for the various singular simplices of a space, that the same could not be achieved with integral coefficients. Of course, the basic Poincaré lemma for algebraic differential forms was no longer true, however this reminded me strongly of a similar difficulty met with in algebraic geometry, and which is overcome by working with suitable "divided power structures" – as Poincaré's lemma becomes valid when replacing usual polynomials (as coefficients for differential forms) by "polynomials with divided powers". Then I got quite excited and involved in a formalism of De Rham complexes with divided powers for arbitrary semisimplicial sets, which took me a few weeks to work out and alongside getting back into homotopy and cohomology formalism again. I had the feeling that this structure, or the technically more adequate dual "coalgebra" structure, might well turn out to be the more refined version of (27) needed for recovering homotopy types – or at any rate 1-connected ones. I gave a talk about the matter at IHES while things were still hot in my mind – but it doesn't seem it went really through. It doesn't seem this structure (which was worked out independently by someone else too, I understand) has become a familiar notion to topologists. Maybe one reason is that most topologists and homotopy theorists never really got acquainted with the formalism of derived categories – and it seems that moreover, by the mid-seventies, it had even become altogether unfashionable and "mal vu" to make any mention of them, let alone work with them, also among some of the people who during some time had been helping develop it. Now one of the main points I was making in that talk was a somewhat delicate property of derived categories of abelian categories, with respect to binomial coefficients – too bad!

[p. 341]

[I'm guessing the reference is to Cartan (1976) and Miller (1978)... anyhow, there has been many developments since concerning Witt vectors, crystalline cohomology, etc....]

I have not heard since about any work done in this direction I am reflecting about now (somewhat retrospectively) – namely recovering homotopy types from their abelianization, *plus* extra structure. For all I know, *the* relevant structure may well be the differential algebra with divided powers structure embodied by the De Rham complex (with a bigraduation however instead of just a graduation), or its coalgebra version – viewed as defining an object of a suitable derived category. (Of course, when there is a fundamental group π around, one will have to look at a slightly more complex structure still, involving operations of π, by looking at the De Rham complex of the universal covering.) If it is just the matter of describing homotopy types in terms of other models than semisimplicial complexes, it must be admitted that the new models are of incomparably more intricate description than the complexes! There *is* however one feature of it which greatly struck me by that time, and still seems to me quite intriguing, namely *that this structure, although definitely not "abelian" anymore* (due to multiplication as well as to divided power structure), *makes a sense over any commutative ground ring* (or even scheme, etc.). When this ring is $\mathbb{Q}$, the "models" we get modelize rational homotopy types, which was the starting point of my reflections about seven years ago. Replacing $\mathbb{Q}$ by a more general ring, [p. 342] this suggests that *there might exist a notion of "homotopy types" over any ground ring k* – and a corresponding notion of ground ring extension for homotopy types. For abelianizations of homotopy types, this is particularly "obvious", as being just the functorial dependence of the derived category $\mathrm{D}_\bullet(k)$ with respect to the ground ring k, corresponding to ring extension in a chain complex. For a week or two I played around with this idea, which on the semisimplicial level tied in with expressing homotopy types of some simple spaces (such as standard $K(\pi, n)$ spaces and fibrations between these) in terms of some simple *semisimplicial schemes* (affine and of finite type over Spec($\mathbb{Z}$)), by taking $\mathbb{Z}$-valued points of these; ring extension $\mathbb{Z} \to k$ was interpreted in the scheme-theoretic sense.

I didn't go on very long, as soon after I was taken by personal matters and never took up the matter later – and maybe it was an altogether unrealistic or silly attempt. If I remember it right, the idea lurking was something of this kind, that *there was a functor from* (Hot) *to* (if not an equivalence of (Hot) with…) *a suitable derived category of some category of semisimplicial schemes over* Spec($\mathbb{Z}$), and that the base change intuition, as suggested by the abelianized theory or by the subtler "divided power De Rham theory", would reflect in naive base change $\mathbb{Z} \to k$ for schemes.

I was then looking mainly at 1-connected structures, but there was an idea too that nilpotent fundamental groups might fit into the picture, with the hope that such a group (under suitable restrictions, finite presentation and torsion freeness say) could be expressed in a canonical way in terms of an affine nilpotent group scheme of finite type over Spec($\mathbb{Z}$), by taking the integral points of the latter. It seems (if I remember right) that this is not quite true though – that one couldn't hope for much better than getting a nilpotent algebraic group scheme *over* $\mathbb{Q}$ – and that one would recover the discrete group one started with only "up to commensurability". Possibly, there may be *an equivalence between localization of the category of nilpotent groups as above* (*with respect to monomorphisms with image of finite index*) *and affine nilpotent connected algebraic group schemes over* $\mathbb{Q}$, or equivalently, group schemes whose underlying scheme is isomorphic to standard affine space.

7.7. [p. 343]

95 *Contractors*

After this cascade of "afterthoughts" on abelianization of homotopy types, it is time now to resume some more technical work, and get through with this unending part IV, in accordance with the short range working program I had come to four days ago (section 92, p. 320). I'll take up the three topics stated there – namely contractors, induced structures, and "miscellaneous" – in that order, as reviewed previously. Thus, we'll start with contractors. I have in mind now mainly the definition of contractors, and a few basic facts following easily from what is already known to us.

The first thought that comes to my mind is to define a contractor as a category A such that the set $\mathrm{Ob}(A)$ of all objects of A is a contractibility structure on A, i.e., that there exists a contractibility structure on A for which every object in A is contractible. The trouble with this definition is that it makes the implicit assumption that A is stable under finite products – as the notion of a contractibility structure was defined only in a category satisfying this extra assumption (cf. section 51, D)). Now, this assumption is *not* satisfied by the three standard test categories, including $\blacktriangle$, which surely we do want to consider as contractors! The next thought then, suggested by this reflection, is to embed A into $A^\wedge$ to supply the products which may be lacking in A, and demand there exist a contractibility structure on $A^\wedge$, such that the objects in A be contractible and moreover generate; or, what amounts to the same, that for the homotopy interval structure on $A^\wedge$ defined by intervals coming from A as a generating family, the objects of A are contractible (which implies that this structure "is" indeed a contractibility structure). This condition (in the more general case, when A appears as a full subcategory of any larger category M) has been restated in wholly explicit terms as the *"basic assumption"* (Bas 4) on a set of objects, in order that it generate a contractibility structure (section 51, p. 118). It is immediate that in the case when A itself is stable under finite products in the ambient category, that this condition is intrinsic to A and just amounts to the first definition we had in mind.

[p. 344] Still, we will call a category A satisfying the condition (Bas 4) with respect to the embedding

$$A \hookrightarrow A^\wedge \tag{1}$$

a *precontractor*, as we'll expect something more still from a contractor, which will be automatically satisfied in the particular case when A is stable under finite products. Roughly speaking, we want to have a satisfactory relation between contractibility and asphericity in $A^\wedge$ – we'll make this more precise below. For the time being, let's dwell just a little more on the notion of a precontractor.

A second thought about contractors, coming alongside with the first, is that for any full embedding of A into a larger category

$$f : A \to M, \quad M \text{ stable under finite products,} \tag{2}$$

$f(A)$ should generate in M a contractibility structure. In the particular case when A is stable under finite products (and hence the notion of a precontractor, already defined, coincides with the notion of a contractor), this is indeed so provided A is a (pre)contractor, and moreover f commutes to finite products. When A is just assumed to be a precontractor (without an assumption about stability of A under products), we'll assume in compensation that M is stable under small direct limits, which allows to take the canonical extension $f_!$ of f to $A^\wedge$ in a way commuting to direct limits

$$f_! : A^\wedge \to M, \tag{3}$$

and we can now state: *if $f_!$ commutes to finite products* (cf. prop. 1, a), p. 281), *then $f(A)$ generates a contractibility structure in M*. This statement is true even without assuming that the functor f is fully faithful (and follows immediately from the criterion (Bas 4) of p. 118); however, in the particular case when f is fully faithful, we have a handy criterion (prop. 2, p. 283) for $f_!$ to commute to finite products, namely that $f(A)$ be a strictly generating subcategory of M, or equivalently, that the functor

$$f^* : M \to A^\wedge \tag{4}$$

(right adjoint to $f_!$) be fully faithful. In this case, we may identify M (up to equivalence) to a full subcategory of $A^\wedge$ containing A, and the fact that $f(A)$ generates a contractibility structure in M follows immediately directly (without having to rely on existence of direct limits in M, nor even existence of $f_!$). To sum up:

Proposition 1. *Let A be a small category, M a category stable under finite products, $f : A \to M$ a functor, we assume A is a* precontractor. *Then $f(A)$ generates a contractibility structure in M in each of the following three cases:* [p. 345]

a) *A stable under finite products, and f commutes to these.*
b) *There exists a functor $f_! : A^\wedge \to M$ extending f, and commuting to final object and binary products in $A^\wedge$ of objects in A.*
c) *The functor f is fully faithful and strictly generating.*

Of course, the validity of the conclusion in either case b) or c), for fixed A and variable M and f, *characterizes* the property for A of being a precontractor, and the same for a) if we assume beforehand that A is stable under finite products. Thus, we may view the proposition 1 as the most comprehensive statement of the meaning of this property.

Proposition 2. *Let A be a precontractor. Let $A^\wedge_{\mathrm{c}}$ be the set of contractible objects in $A^\wedge$ for the contractibility structure generated by the subcategory A, $A^\wedge_{\mathrm{as}}$ (resp. $A^\wedge_{\mathrm{loc.as}}$) the set of aspheric (resp. locally aspheric – cf. p. 250) objects of $A^\wedge$, h the homotopy structure on $A^\wedge$ associated to the contractibility structure $A^\wedge_{\mathrm{c}}$, i.e., generated by the intervals in $A^\wedge$ coming from A. As usual, $\mathcal{W}_A$ denotes the set of weak equivalences in $A^\wedge$ – it is understood here that the basic localizer $\mathcal{W} \subset \mathrm{Fl}(\mathrm{Cat})$ is $\mathcal{W}_\infty$ = usual weak equivalence. The following conditions on A are equivalent:*

(i) *$A^\wedge$ is totally aspheric (i.e., $\mathrm{Ob}\,A \subset A^\wedge_{\mathrm{loc.as}}$).*
(ii) *The asphericity structure $A^\wedge_{\mathrm{as}}$ on $A^\wedge$ is generated by the contractibility structure $A^\wedge_{\mathrm{c}}$.*
(ii') $A^\wedge_{\mathrm{c}} \subset A^\wedge_{\mathrm{as}}$.
(ii'') $A^\wedge_{\mathrm{c}} \subset A^\wedge_{\mathrm{loc.as}}$.

(iii) *Any h-homotopism is in* $\mathcal{W}_A$ (*i.e.,* $\mathcal{W}_A$ *is "strictly compatible" with the homotopy structure h,* (*cf. section* 54), *i.e.,* $h \le h' = h_{\mathcal{W}_A}$).

(iv) *The homotopy structure h is equal to the homotopy structure* $h' = h_W$ *associated to* $W = \mathcal{W}_A$ (*cf. section* 54).

Proof of proposition. Immediate from what is known to us, via

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(ii')} \Leftrightarrow \text{(ii'')} \Rightarrow \text{(i)} \quad \text{and} \quad \text{(ii)} \Rightarrow \text{(iv)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)}.$$

[p. 346] **Definition 1.** A small category A is called a *contractor* if it is a precontractor, and if moreover it is totally aspheric or, equivalently, satisfies one of the equivalent condition (i) to (iv) of prop. 2.

Equivalently, this also means that

(5) $$\operatorname{Ob} A \subset A^{\wedge}_{\mathrm{c}} \cap A^{\wedge}_{\mathrm{loc.as}},$$

i.e., every object in A is contractible and locally aspheric, where the set $A^{\wedge}_{\mathrm{c}}$ of "contractible" objects of $A^{\wedge}$ is defined in terms of the homotopy interval structure h generated by all intervals in $A^{\wedge}$ coming from objects in A. (Thus structure is not necessarily a contractibility structure, but it is when A is a precontractor, namely $\operatorname{Ob} A \subset A^{\wedge}_{\mathrm{c}}$.)

The most trivial example of a contractor is the final category Δ_0, and more generally, any category equivalent to it (NB a category equivalent to a precontractor resp. to a contractor is again a precontractor resp. a contractor). Such a contractor will be called *trivial*. For a trivial contractor A, we get an equivalence

$$A^{\wedge} \approxeq (\mathrm{Sets}).$$

If

$$(M, M_{\mathrm{c}})$$

is a contractibility structure, and $A \subset M$ any small full subcategory of M generating the contractibility structure, then A is a precontractor, hence a contractor iff A is totally aspheric, which will be the case if A is stable in M under binary products, a fortiori if it is stable under finite products, i.e., contains moreover a final object of M. Thus, *the contractibility structure of M can always be generated by a full subcategory A of M which is a contractor*.

Apart from these two examples, the most interesting examples of contractors are of course the three standard test categories Δ, $\square$ and $\bigcirc$, and also their finite products. Note that the notion of a precontractor or of a contractor is clearly stable under finite products.

Proposition 3. *Let A be a precontractor, assume A non-trivial, i.e., non-equivalent to the final category. Then A contains a separating interval, and hence it is a* (*strict*) *test category if A is tot. asph., i.e., is a contractor.*

Proof. As any object of A has a section (over $e_{A^\wedge}$), it follows immediately that any non-empty object of $A^\wedge$ has a section too, hence *any non-empty subobject of $e_{A^\wedge}$ is equal to $e_{A^\wedge}$*. This implies that for any interval [p. 347]

$$\mathbb{I} = (I, \delta_0, \delta_1)$$

in $A^\wedge$, either $\mathbb{I}$ is separating, i.e., $\operatorname{Ker}(\delta_0, \delta_1)$ is the empty object of $A^\wedge$, or $\delta_0 = \delta_1$. If no interval coming from A was separating, this then would just mean that any two sections (over $e_{A^\wedge}$) of an object I of A are equal. By the definition of the homotopy structure in $A^\wedge$ generated by these intervals, this would imply that any homotopism in $A^\wedge$ is an isomorphism, and hence that any contractible object for this structure is isomorphic to the final object $e_{A^\wedge}$. As by assumption on A all objects of A are contractible, this would mean that A is trivial, which is against our assumptions, qed.

Corollary 1. *Let (M, M_c) be a contractibility structure.*

a) *The following conditions are equivalent (and will be expressed by saying that this given contractibility structure is* trivial*):*
   (i) *Two maps in M which are homotopic are equal.*
   (ii) *Any homotopism in M is an isomorphism.*
   (iii) *Any homotopy interval in M is "trivial", i.e., any two homotopic sections of an object of M are equal.*
   (iv) *Any contractible object of M is a final object, i.e., M_c is just the set of all final objects of M.*
   (v) *Any two sections of a contractible object are equal.*

b) *Assume the contractibility structure M_c non-trivial, i.e., there exists an interval* [p. 348]

$$\mathbb{I} = (I, \delta_0, \delta_1) \quad \textit{with} \quad I \in M_c, \delta_0 \neq \delta_1.$$

Then for any small category A and any functor

$$i : A \to M$$

factoring through M_c, the interval $i^*(\mathbb{I})$ in $A^\wedge$ is separating. Hence if (for a given basic localizer $\mathcal{W}$) i is totally $\mathcal{W}$-aspheric (cf. theorem 1 cor. 1 p. 252), hence $i^*(I)$ is totally aspheric in $A^\wedge$, then A is a $\mathcal{W}$-test category. In particular, if A is totally $\mathcal{W}$-aspheric and i is M_a-$\mathcal{W}$-aspheric, then A is a strict $\mathcal{W}$-test category.

Proof. Part a) is a tautology in terms of section 51. For part b), to prove that $i^*(\mathbb{I})$ is separating, we only have to check that for any a in A, the two compositions

$$a \to i^*(e_M) \simeq e_{A^\wedge} \overset{i^*(\delta_0), i^*(\delta_1)}{\rightrightarrows} i^*(I)$$

are distinct, or what amounts to the same by the definition of i^*, that the compositions

$$i_!(a) \overset{\text{def}}{=} x \to e_M \rightrightarrows I$$

are distinct. As x is in M_{c}, it has a section over e_M, so it is enough to check that the compositions with $e_M \to x$ are distinct, which just means that $\delta_0 \neq \delta_1$, qed.

Remarks. 1) Part b) of the corollary replaces cor. 3 on page 253, which is a little monster of incongruity (as I just discovered) – namely, two of the assumption on $\mathbb{I}$ made there (namely that $\mathbb{I}$ be a multiplicative interval, and $I \in \operatorname{Ob} C$) are useless if we assume just I contractible, moreover the awkward separation assumption made there just reduces by the trivial argument above to the assumption $\delta_0 \neq \delta_1$!

2) It should be noted that the homotopy structure on $A^\wedge$ envisioned in th. 1 of section 79 (p. 252) is *not* defined as in the present section, in terms of intervals in $A^\wedge$ coming from A (call this structure h), but as

$$h' = h_{\mathcal{W}_A}$$

defined in terms of intervals in $A^\wedge_{\mathrm{loc.as}}$; this depends a priori on the choice of $\mathcal{W}$, as it has to because th. 1 gives a criterion for the functor i to be M_{a}-$\mathcal{W}$-aspheric which does depend on $\mathcal{W}$. (It surely won't be the same if we take $\mathcal{W} = \mathcal{W}_\infty =$ usual weak equivalence, or $\mathcal{W} = \mathrm{Fl}((\mathrm{Cat}))$
[p. 349] hence $\mathcal{W}_A = \mathrm{Fl}(A)$ and the condition that i be M_{a}-$\mathcal{W}$-aspheric is always satisfied!) However, let's assume A to be totally $\mathcal{W}$-aspheric and every object of A has a section (over the final object $e_{A^\wedge}$ of $A^\wedge$) or, what amounts to the same, every "non-empty" object of $A^\wedge$ has a section – we'll say in this case A is *"strictly totally $\mathcal{W}$-aspheric"* (compare section 60, p. 149, in the particular case $\mathcal{W} = \mathcal{W}_\infty$, and with $A^\wedge$ replaced by an arbitrary topos). Let's assume moreover that $\mathcal{W}$ satisfied Loc 4). In this case, the homotopy structure $h' = h_{\mathcal{W}_A}$ does not depend on the choice of $\mathcal{W}$, namely it is the so-called "canonical homotopy structure"

$$h'' = h_{A^\wedge}$$

of the (strictly totally 0-connected) category $A^\wedge$ (cf. section 57), which in the special case of a category $A^\wedge$ can also be defined as the homotopy structure $h_{\mathcal{W}_0}$ associated to $\mathcal{W}_{0A}$, where $\mathcal{W}_0$ is the coarsest basic localizer satisfying Loc 4), i.e.,

$$\mathcal{W}_0 = \left\{ f \in \mathrm{Fl}(\mathrm{Cat}) \,\middle|\, \pi_0(f) \text{ bijective} \right\}.$$

The proof of this fact $h' = h''$, i.e.,

$$h_{\mathcal{W}_A} = h_{A^\wedge} \quad (= h_{\mathcal{W}_0})$$

is essentially the same as for the similar prop. (section 60, p. 149). The condition Loc 4) on $\mathcal{W}$, i.e., $\mathcal{W} \subset \mathcal{W}_0$ clearly implies

$$h_{\mathcal{W}_A} \subset h_{\mathcal{W}_{0A}},$$

and to get the opposite inequality, for which we'll use the assumption on A, we only have to prove that for any 0-connected object K of $A^\wedge$, any two sections are ($h' = h_{\mathcal{W}_A}$)-homotopic, a fortiori (as $h \leq h'$ by the assumption of total $\mathcal{W}$-asphericity of A) it is enough to prove they are h-homotopic. Now this follows from lemma 2, p. 268, applied to $\mathcal{C} = A^\wedge$, $C = A$.

The preceding reflections thus prove the following afterthought to theorem 1 of section 79: [p. 350]

Proposition 4. *Let (M, M_c) be a contractibility structure, A a small category, $i : A \to M$ a functor factoring through M_c. Let moreover $\mathcal{W}$ be a basic localizer satisfying Loc 4). We assume A* strictly totally *$\mathcal{W}$*-aspheric*, i.e., totally $\mathcal{W}$-aspheric and moreover any object of A has a section (over $e_{A^\wedge}$).*

a) *The homotopy structure $h_{\mathcal{W}_A}$ on $A^\wedge$ is equal to the canonical homotopy structure $h_{A^\wedge}$ defined by* 0*-connected intervals, and equal also to the homotopy structure h defined by intervals coming from A:*

$$h = h_{\mathcal{W}_A} = h_{A^\wedge}. \tag{6}$$

In what follows, we assume $A^\wedge$ endowed with this homotopy structure, and denote by $A^\wedge_c$ the set of all contractible object in $A^\wedge$. We equally endow $A^\wedge$ with its canonical $\mathcal{W}$-asphericity structure, and M with the $\mathcal{W}$-asphericity structure associated to its contractibility structure M_c. With these conventions:

b) *The following conditions on i are equivalent, where*

$$i^* : M \to A^\wedge$$

is the functor defined as usual in terms of i:

(i) *i^* is compatible with the homotopy structures (cf. criteria on pages 251–252), which can be expressed also by*

$$i^*(M_c) \subset A^\wedge_c$$

(a condition independent from $\mathcal{W}$, in view of a)).

(ii) *i is $\mathcal{W}$-aspheric, i.e.,*

$$M_{\mathcal{W}} \subset (i^*)^{-1}(A^\wedge_{\mathcal{W}})$$

(where $M_{\mathcal{W}}$ and $A^\wedge_{\mathcal{W}}$ are the sets of $\mathcal{W}$-aspheric objects in M and in $A^\wedge$).

(iii) *(For a given full subcategory C of M generating the contractibility structure M_c):*

$$i^*(C) \subset A^\wedge_{\mathcal{W}_0} = \text{set of } 0\text{-connected objects of } A^\wedge.$$

c) *Assume these conditions hold, and moreover that the contractibility structure of M is non-trivial. Then A is a strict $\mathcal{W}$-test category.*

Proof. Part a) has been proved in remark 2) above, and in view of th. 1, p. 252, the equivalence of (i) and (ii) is clear, hence also the equivalence with (iii) by applying loc. cit. to $\mathcal{W}_0$ instead of $\mathcal{W}$. Part c) now follows from prop. 3 cor. 1 b). [p. 351]

Corollary. *Under the conditions of c) above, if M_c is $\mathcal{W}$-modelizing, then i is a $\mathcal{W}$-test functor, and induces an* equivalence

$$\mathrm{Hot}_{M,\mathcal{W}} \stackrel{\text{def}}{=} \mathcal{W}_M^{-1}M \xrightarrow{\simeq} \mathrm{Hot}_{A,\mathcal{W}} \stackrel{\text{def}}{=} \mathcal{W}_A{}^{-1}A^\wedge.$$

8.7.

96 *Vertical and horizontal topoi…* (*afterthought on terminology*).

[I'm pretty sure that "to follow one's nose" is an English idiom in the sense of following one's instinct, while the meaning of going straight ahead is shared with the German "der Nase nach gehen"]

Yesterday's notes have proceeded very falteringly, to my surprise, while everything seemed ready for smooth sailing. A number of times, after going on for a page or two "following my nose" (as they say in German), or for half a page, it turned out it just wasn't right that way and I would feel quite stupid and put the silly pages away as scratchpaper and have another start. There wouldn't have been any point dragging the poor reader (if there is still one left…) along on my stumbling path, where it was a matter merely of getting some technical adjustments right. Maybe it is just that attention was distracted, perhaps precisely through this (partly mistaken, and anyhow not too inspiring) feeling that everything was kind of cooked already, and what was left to do was just swallow! What came out in the process was that finally things were not so clear yet in my mind as I thought they were. It is a frequent experience that whenever one wants to go ahead too quickly, one finds oneself dispersing stupidly a hell of a lot of energy…

There occurred to me some inadequacies with terminology. One is about the property of certain categories (contractor or precontractors for instance) that every object of A has a section (over $e_{A^\wedge}$), which can be viewed also as a property of the topos $A^\wedge = \mathcal{A}$, namely that any "non-empty" object of the topos has a section. This is immediately seen (for any given topos $\mathcal{A}$) to imply the property that the final object $e_{\mathcal{A}}$ has only the two trivial subobjects, the "empty" and the "full" one – or equivalently, that any subtopos of the topos is either the empty of the full one, – a property, too of obvious geometric significance. In case of a topos of the type $A^\wedge$, one immediately sees the two properties are [p. 352] equivalent – but this is not true for an arbitrary topos: for instance the classifying topos B_G of a discrete group G has the second property, but visibly not the first unless G is the unit group. (I recall that the category of sheaves on B_G is the category (G-sets) of sets on which G operates.) I feel both properties for a topos merit a name. The first (every sheaf has a section) can be viewed as the strongest conceivable (I would think) global asphericity property for a topos, as far as H^1 goes at any rate, as the H^1 of X with coefficients in *any* group object will be zero. (But I confess I didn't try and look if any precontractor, say, is aspheric…) The second property (every subtopos is trivial) comes with a rather different flavor, it suggests the image of just one "point" – and as a matter of fact, the étale topos of a scheme, say has this property iff it is reduced to a point. Such a topos may called "punctual" (not to be confused though with some other meanings suggested by this word, such as being equivalent to the topos defined by a one-point topological space, namely $\mathcal{A}$ being equivalent to (Sets)) or "atomic" (which has rather unpleasant connotations though nowadays!), or maybe "vertical" (this image is suggested by the B_G above) – the "base", i.e., the final object of $\mathcal{A}$ being very "small" (in terms of harboring subobjects), so the inner structure is expressed like a kind of tower, related (in the case of B_G) to the "Galois tower" of subgroups of G…The corresponding notion of a "horizontal" topos is visibly the one when $\mathcal{A}$ admits the subobjects of the final object as a generating family. In terms of these definitions, a topos is horizontal and vertical iff it is either the "empty" or the "final" (or "one-point") topos. This brings to mind that in the notion of verticality, we should exclude the "empty" topos (which formally satisfies the condition – every sheaf has a section). This brings to my attention too that I certainly do not want to consider an empty category A (defining the "empty" topos $A^\wedge$) as a precontractor, although formally (in terms of yesterday's definition) it is. Thus, I suggest I'll introduce the following

Definition. A topos is called *vertical* if it is not an "empty" topos (i.e., the category of sheaves on it is not equivalent to the final category Δ_0), and if moreover any open subtopos is either the "empty" or the "full" one (hence the same for any subtopos, whether open or not). A small category A is called vertical, if the associated topos (with category of sheaves $A^\wedge$) is, or equivalently, if A is non-empty and any object of A has a section (over $e_{A^\wedge}$). A topos is called *horizontal* if the family of all subobjects of the final object in the category of sheaves $\mathcal{A}$ is generating. [p. 353]

For instance, the topos associated to a topological space is horizontal – in particular, an "empty" topos is horizontal. A topos is both horizontal and vertical iff it is a (2-)final topos, i.e., equivalent to the topos defined by a one-point topological space (i.e., the category of sheaves is equivalent to (Sets)).

The property of verticality, I feel, is of interest in its own right, as exemplified notably by lemma 2 p. 268 (which we used yesterday), and the related proposition of section 60 (p. 149). It does not seem at all subordinated to notions such as total asphericity or total 0-connectedness, and goes in an entirely different direction – thus, the terminology "*strictly* totally aspheric" (or totally 0-connected), which I still used yesterday (hesitatingly, I should say), is definitely inadequate. I would rather say "totally aspheric (or totally 0-connected) *and* vertical".

Another point is about the terminology of *totally aspheric* and *locally aspheric* objects in a category $A^\wedge$ (with respect to a given basic localizer $\mathcal{W}$), introduced in section 79 (p. 250), and still used yesterday. This terminology does not seem inadequate by itself, I introduced it because it struck me as suggestive (and the notions it refers to do deserve a name, in order to be at ease). The trouble here is that it conflicts with another possible meaning, in accordance with the principle insisted upon forcefully in the reflections of section 66 – namely that for objects or arrows in (Cat), or within a category $A^\wedge$, the terminology used for naming properties for these should be in accordance with the terminology used for the corresponding topoi or maps of topoi. Now, we do have already the notions of a locally aspheric and totally aspheric topos, which therefore should imply automatically the meaning of these notions for an object of (Cat) (which was done satisfactorily months ago), or for an object of a category $A^\wedge$. But in the latter case, there is definitely conflict with the terminology introduced on p. 252. This conflict has not manifested itself yet in any concrete situation, while the unorthodox terminology has been used quite satisfactorily a number of times. Therefore, I would like to keep it, as long as I am not forced otherwise.

* * *

[p. 354] We were faced yesterday with three different homotopy structures h, h', h'' on a category $A^\wedge$, for a given small category A, which make sense for any A, and which in case A is a contractor all coincide. The exact relationship between these structures in more general cases has remained somewhat confused, and in order to dispel the resulting feeling of uneasiness, I took finally the trouble today to write it out with some case. One of these structures, h', depends on the choice of a basic localizer $\mathcal{W}$, whereas the two others don't…

12.7.

97 *"Projective" topoi. Morphisms and bimorphisms of contractors.*

I was interrupted in my notes by visiting friends arriving in close succession – then since yesterday I have been busy mainly with letter writing. Now, I am ready to take up the thread where I left it – namely some afterthoughts to the reflections of section 95 on contractors.

First an afterthought to the afterthoughts! I had introduced the name "*vertical* topos" for a topos admitting only the two trivial open subtopoi (page 352), whereas the stronger property that every "non-empty" sheaf has a section remained unnamed (which is no real drawback as long as we are restricting to topoi of the type $A^\wedge$, where indeed the two notions coincide). Now, the latter property can be viewed as the property that every sheaf F *such that* $F \to e$ *be epimorphic*, should admit a section. It is this last property which does merit to "be viewed as the strongest conceivable asphericity property for a topos" as I commented on it last Friday (p. 352). After I had written this down as a kind of selfevidence, a doubt turned up though and I qualified the comment by added "as far as H^1 goes at any rate, as the H^1 with coefficients in *any* group object will be zero". I didn't pause then to see if the doubt was founded – quite evidently it isn't, except for H^0, as it is clear by the usual shift argument, using embedding of an abelian sheaf into an injective one, that if for given k (here $k = 1$) $\mathrm{H}^k(X, F) = 0$ for any abelian sheaf F, then the same holds for H^n with any $n \geq k$ – i.e., the global cohomological dimension of X is $< k$. This implies that any small vertical category (a fortiori any *precontractor*) is aspheric, provided it is 0-connected, indeed its cohomology variants with values in *any* sheaf of coefficients (not necessarily commutative as far as H^1 goes) are trivial.

[p. 355] The property for a topos X, with category of sheaves $\mathcal{A}$, that any object F in $\mathcal{A}$ covering the final object $e_{\mathcal{A}}$ should have a section, can be expressed by saying that the latter is a *projective* object in the category $\mathcal{A}$. Following the principle to use the same names for properties of a topos, and corresponding properties of the final sheaf on it, we may call a topos with the above property a *projective topos*. Thus, the "non-empty" topoi such that every "non-empty" sheaf has a section, are exactly the topoi which are both vertical and projective.

* * *

Here is the promised "exact relationship" between the three standard homotopy structures h, h', h'' on $A^\wedge$, where A is any small category (cf. end of section 96, p. 354):

$$
(*) \qquad \begin{array}{ccccccc}
h & \overset{(A \text{ tot. } \mathcal{W}\text{-asph.})}{\leq} & h' & \overset{(\mathcal{W}\subset\mathcal{W}_0)}{\leq} & h'' & \overset{(A \text{ vertical})}{\leq} & h \\
\| & & \| & & \| & & \\
h_{A^\wedge, A} & & h_{\mathcal{W}_A} & & h_{\mathcal{W}_{0A}} = h_{A^\wedge} & & ,
\end{array}
$$

where $\mathcal{W}$ is a given basic localizer. Above each one of the three conditional inequalities between homotopy structures h, h', h'' I wrote the natural assumption on A or $\mathcal{W}$ validating it, and in the diagram I have recalled the definition of the three homotopy structures. Apropos the description $h = h_{A^\wedge, A}$, the notation used here is $h_{M,A}$ when M is a category stable under finite products and A a full subcategory, for designating the homotopy structure on M generated by intervals in M coming from A. Apropos $h'' = h_{A^\wedge}$, I recall the notation h_M for designating the canonical homotopy structure on a category M satisfying suitable conditions (section 57). Also, I recall

$$\mathcal{W}_0 = \left\{ f \in \mathrm{Fl}(\mathrm{Cat}) \;\middle|\; \pi_0(f) \text{ bijective} \right\}.$$

The diagram implies that if $\mathcal{W} \subset \mathcal{W}_0$, i.e., $\mathcal{W}$ satisfies Loc 4), and if moreover A is vertical and totally $\mathcal{W}$-aspheric, then all three homotopy structures coincide. Also, taking $\mathcal{W} = \mathcal{W}_0$, we see that $h = h''$ if A is vertical and totally 0-connected, which is lemma 2 of p. 268 for $A^\wedge$, A.

From (*) it follows of course that if A is a contractor, then (for any $\mathcal{W}$ satisfying Loc 4)) the three homotopy structures h, h', h'' on $A^\wedge$ coincide. In case the contractor A is not trivial, hence A is a strict test category and $A^\wedge$ is $\mathcal{W}$-modelizing, it follows that $A^\wedge$ is even a *canonical modelizer* (with respect to $\mathcal{W}$), i.e., defined in terms of the $\mathcal{W}$-asphericity structure associated to the "canonical" homotopy structure $h'' = h_{A^\wedge}$ on $A^\wedge$ (cf. prop. 2 (ii) p. 345). These, for the time being, together with the modelizers (Cat) and (Spaces), are the main examples we got of canonical modelizers. Presumably, stacks should give another sizable bunch of canonical modelizers, not of the type $A^\wedge$. [p. 356]

* * *

We still have to say a word about morphisms between contractors A, B. The first thing that comes to my mind is that this should be a functor

$$(0) \qquad f : A \to B$$

such that the corresponding functor

$$(1) \qquad f^* : B^\wedge \to A^\wedge$$

should be compatible with the homotopy structures, which can be expressed, as we know, in manifold ways, the most natural one here being the following two

$$f^*(B) \subset A^\wedge_{\mathrm{c}} \tag{2}$$

or

$$f^*(B^\wedge_{\mathrm{c}}) \subset A^\wedge_{\mathrm{c}}, \tag{3}$$

which are both implied by the apparently weaker one

$$f^*(B) \subset A^\wedge_{\mathcal{W}_0} \overset{\text{def}}{=} \text{set of 0-connected objects of } A^\wedge, \tag{4}$$

and equivalently still, as $A^\wedge_{\mathrm{c}} \subset A^\wedge_{\mathcal{W}} \subset A^\wedge_{\mathcal{W}_0}$ (where $\mathcal{W}$ is a basic localizer satisfying Loc 4)), to the condition

$$f^*(B) \subset A^\wedge_{\mathcal{W}} \quad (\overset{\text{def}}{=} \text{set of } \mathcal{W}\text{-aspheric objects of } A^\wedge). \tag{5}$$

Thus, the condition for f to be a "morphism of contractors" just boils down to the long familiar $\mathcal{W}$-*asphericity* of f, and implies the following relation, apparently stronger than (5):

$$B^\wedge_{\mathcal{W}} = (f^*)^{-1}(A^\wedge_{\mathcal{W}}). \tag{5'}$$

This comes almost as a surprise (after a four day interruption in contact with the stuff!) – but it occurs to me now that we got already a more general statement with prop. 4 of section 95 (p. 350), which includes
[p. 357] the situation when instead of $f : A \to B$, we got a functor

$$f : A \to B^\wedge, \quad \text{factoring through } B^\wedge_{\mathrm{c}} \tag{6}$$

(which need not factor through B), or equivalently a functor

$$f_! : A^\wedge \to B^\wedge \tag{7}$$

commuting with small direct limits, or equivalently still, a functor f^* in opposite direction, commuting with small inverse limits, but in the last two cases with the extra condition that $f_!(A) \subset B^\wedge_{\mathrm{c}}$. We may want to extend the notion of morphism of contractors to include this situation, hence expressed by the two following conditions on a functor f (6) or $f_!$ (7), or on the pair $(f_!, f^*)$ of adjoint functors

$$\begin{cases} f_!(A) \subset B^\wedge_{\mathrm{c}} \\ f^*(B) \subset A^\wedge_{\mathrm{c}} \end{cases} \quad . \tag{8}$$

However, in order for this notion to be stable under composition, we should strengthen the first of the relations (8) into

$$f_!(A^\wedge_{\mathrm{c}}) \subset B^\wedge_{\mathrm{c}}, \tag{9}$$

which follows automatically whenever $f_!$ commutes to finite products (cf. section 85), but may not follow from (8) in general, even in the case when f factors through B, i.e., in the case we start with a functor

(0) $f : A \to B$. Thus, we get *two* plausible notions of a morphism of contractors, neither of which implies the other, and I feel unable to predict which one will prove the more useful. As far as terminology goes, it seems reasonable to reserve the name "morphism of contractors" to the first notion, as the second is adequately characterized as a *bimorphism* between the contractibility structures $(A^\wedge, A^\wedge_{\mathrm{c}})$ and $(B^\wedge, B^\wedge_{\mathrm{c}})$ defined by the contractors A and B (cf. section 86).

16.7.

98 Next point on my provisional program is induced structures (asphericity of contractibility structures) on a category $M_{/a}$, when one is given on M – but finally I decided to skip this, as there was no urgent need for clarifying this and I am not writing a treatise, thanks Gods! I felt more interested writing down the proof that the category $\triangle^{\mathrm{f}}$ of standard ordered simplices without degeneracies is a weak test category, as announced months ago, in section 43. It then seemed to come out rather simply, but I didn't keep notes of the proof I thought I found, which caused me spending now a day or two feeling a little stupid, as the stuff was resisting while I felt it shouldn't! It did come out in the end I guess – and still I feel a little stupid, with the impression of having bypassed definitely some very simple argument which had presented itself as a matter of evidence by the end of March. On the other hand, I was led to reflect on some other noteworthy features of the situation, so I don't feel I altogether have been loosing my time.

Sketch of proof of $\triangle^{\mathrm{f}}$ being a weak test category – and perplexities about its being aspheric!

[p. 358]

There are four main variants of categories of standard simplices, inserting into a diagram

(1) $$\begin{array}{ccc} \triangle & \xrightarrow{\beta} & \widetilde{\triangle} \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\tilde\alpha} \\ \triangle^{\mathrm{f}} & \xrightarrow{\beta^{\mathrm{f}}} & \widetilde{\triangle}^{\mathrm{f}} \end{array} ,$$

where $\widetilde{\triangle}$ denotes the category of non-ordered standard simplices $\triangle_n$, and where the exponent f in $\triangle^{\mathrm{f}}$ and $\widetilde{\triangle}^{\mathrm{f}}$ denotes restriction to maps which are injective, namely compositions of boundary maps (plus symmetric in the non-ordered case). I recall that $\triangle$ and $\widetilde{\triangle}$ are contractors, wheres $\triangle^{\mathrm{f}}$ and $\widetilde{\triangle}^{\mathrm{f}}$ are not even test categories. We'll see however that $\triangle^{\mathrm{f}}$ is a *weak* test category, and presumably the same kind of argument should apply to prove that $\widetilde{\triangle}^{\mathrm{f}}$ is a weak test category too. On the other hand, from the point of view of the modelizing story, the main common property of the four functors in (1) should be asphericity. However, I checked this for α and β only, as this was all I needed for getting the desired result on $\triangle^{\mathrm{f}}$. As a matter of fact, β is even better than being aspheric, it is a *morphism of contractors*; more precisely still, for any object E in $\widetilde{\triangle}$, namely essentially a finite non-empty set, choosing one point a in E, one easily constructs an elementary homotopy for $\beta^*(E)$ from the identity map to the constant map defined by $\beta^*(a)$. I do not know on the other hand whether β defines a bimorphism between the canonical contractibility structures on $\Delta^\wedge$ and $\widetilde{\Delta}^\wedge$, namely whether $\beta_!$ transforms contractible elements into contractible ones (which would be clear if we knew that $\beta_!$ commutes to finite products).

We'll come back upon proof of asphericity of α, and show at once how this implies that Δ^{f} is a weak test category, or equivalently, that the
[p. 359] canonical functor

$$i_{\Delta^{\mathrm{f}}} : \Delta^{\mathrm{f}} \to (\mathrm{Cat})$$

is aspheric, for the canonical asphericity structure on (Cat). (I should have noted that the asphericity statements are meant here in the strongest possible sense, namely with respect to $\mathcal{W} = \mathcal{W}_\infty =$ usual weak equivalence.) Now, for any Δ_n in Δ^{f}, the category

$$i_{\Delta^{\mathrm{f}}}(\Delta_n) \overset{\mathrm{def}}{=} \Delta^{\mathrm{f}}{}_{/\Delta_n}$$

is canonically isomorphic to the category associated to the ordered set of all non-empty subsets of Δ_n, hence we get a canonical isomorphism

$$i_{\Delta^{\mathrm{f}}} \simeq \widetilde{i}(\beta\alpha), \tag{2}$$

where

$$\widetilde{i} : \widetilde{\Delta} \to (\mathrm{Cat}) \tag{3}$$

is the standard test functor, associating to any non-ordered simplex E the category associated to the ordered set of all non-empty subsets of E. We know already (section 34) that $\widetilde{i}$ is aspheric, i.e., $\widetilde{i}^*$ transforms aspheric objects of (Cat) into aspheric ones, hence the same holds for its composition with the aspheric functor $\beta\alpha$, hence also for $i_{\Delta^{\mathrm{f}}}$, qed.

Thus, we are left with proving that α is aspheric, i.e., that the categories

$$\Delta^{\mathrm{f}}{}_{/\alpha^*(\Delta_n)} \tag{*}$$

are aspheric. Now, let's denote by Δ'^{f} the category deduced from Δ^{f} by adding an initial object $\varnothing$ (which we may view as being the empty simplex), which defines an "open subcategory" U, namely as sieve in Δ'^{f}, in such a way that Δ^{f} appears as the "closed subcategory", i.e., cosieve in Δ'^{f} complementary to U. One immediately checks that the category (*) is canonically isomorphic to

$$(\Delta'^{\mathrm{f}})^{n+1} \setminus U^{n+1}, \tag{**}$$

where U^{n+1} is the open subcategory defined by the initial object of the ambient category $(\Delta'^{\mathrm{f}})^{n+1}$. Now, asphericity of (**) and hence of (*) follows from the following two lemmas:

Lemma 1. *The category Δ^{f} is aspheric.*

Lemma 2. *Let $(X_i, U_i)_{i\in I}$ be a finite non-empty family of pairs (X_i, U_i), where X_i is a small category, U_i an open subcategory. We assume that for any i in I, X_i and the closed complement Y_i of U_i in X_i are aspheric.*
[p. 360] *Let X be the product of the X_i's, U the products of the U_i's, then (X and) $X \setminus U = Y$ are aspheric too.*

Proof of lemma 2: by an immediate induction, we are reduced to the case when I has just two elements, $I = \{1,2\}$, but then $X \setminus U$ can be viewed as the union of the two closed subcategories $X_1 \times Y_2$ and $X_2 \times Y_1$, whose intersection is $Y_1 \times Y_2$. As all three categories are aspheric (being products of aspheric categories), it follows by the well-known Mayer-Vietoris argument that so is $X \times U$, qed.

Thus, we are left with proving that $\triangle^{\mathrm{f}}$ is aspheric. Somewhat surprisingly, that's where I spent a number of hours not getting anywhere and feeling foolish! There is a very simple heuristic argument though involving the standard calculation of the cohomology invariants of any semisimplicial "complex", in terms of the standard boundary operations: if we admit that the same calculations are valid when working with semisimplicial "face complexes", i.e., objects of $\triangle^{\mathrm{f}\,\hat{}}$, then it is enough to apply this to the final object of $\triangle^{\mathrm{f}\,\hat{}}$ (including for computation of the non-commutative H^1 with constant coefficients) to get asphericity of $\triangle^{\mathrm{f}}$. As a matter of fact, this argument would give directly asphericity of α, bypassing altogether the categories (**) and lemmas 1 and 2. Apparently, I got a block against the down-to-earth computational approach to cohomology via semisimplicial calculations, and have been trying to bypass it at all price – and not succeeding! Then, curiously enough, when finding no other way out than look at those boundary operations and try to understand what they meant (something I remember vaguely have been doing once ages ago!), this brought me back again to the abelianization story of sections 92 and 93, and to a more comprehensive way for looking at "abelianizators", and get an existence and unicity statement for these. (At any rate, for a suitably strengthened version of these.) This seems to me of independent interest, and worth being written down with some care.

99 *Afterthoughts on abelianization IV: Integrators.*

When writing down (in sections 92 and 93) some rambling reflections about "abelianization" and "abelianizators", there has been a persistent feeling of uneasiness, which I kept pushing aside, as I didn't want to spend too much thought on this "digression". This uneasiness had surely something to do with the way abelianization (of an object X say of an elementary modelizer $A\hat{}$) was handled, so that it was designed in a more or less exclusive way for embodying information about the "homology structure" of the homotopy type modelized by X, or equivalently, to describe its cohomology invariants with arbitrary *constant* coefficients. [p. 361]
Now, among the strongest reflexes I acquired in the past while working with cohomology, was systematically to look at coefficients which are arbitrary sheaves (abelian say), and to view constant or locally constant coefficients as being just particular cases. This reflex has been remaining idle, not to say repressed, during nearly all of the reflections of the last four months, due to the fact that in the whole modelizing story woven around weak equivalence, there was a rather exclusive emphasis on constant and locally constant coefficients, disregarding any other coefficients throughout. Probably, while reflecting on abelianization, a more or less underground reminiscence must have been around of

the semisimplicial boundary operators having a meaning for computing cohomology of "something", with coefficients in arbitrary sheaves – and also that to get it straight, one had to be careful not to get mixed up in the variances. But I just didn't want to dive into all this again if I could help it – and now it is getting clear, after a day or two of feeling silly, that it can't be helped, and I'll have to write things down at last, however "well-known" they may be.

Let A be a small category. In section 93 we defined an "abelianizator" for A to be a chain complex $L_\bullet$ in the additive envelope $\mathrm{Add}(A^{\mathrm{op}})$ of the category A^{op} opposite to A, satisfying a suitable condition of commutativity (in the diagram (20) of p. 328), and endowed with a mild extra structure λ, expressing this commutativity. The function of an abelianizator in loc. cit. was essentially to allow for a simultaneous "computational" description of the homology structure of the homotopy types stemming from a variable object X in $A^\wedge$, or equivalently, to describe cohomology of such X (as an object of a suitable derived category say, to get it at strongest) with coefficients in any (*constant*) ring or abelian group. Introducing by an independent symbol the opposite category

$$B = A^{\mathrm{op}}, \tag{1}$$

I want now to establish a relationship between this property or function of a chain complex $L_\bullet$ in $\mathrm{Add}(B)$, involving objects in $A^\wedge$ and their abelianizations in $A^\wedge_{\mathrm{ab}}$, with an apparently different one, in terms of a variable object of $B^\wedge_{\mathrm{ab}}$ (*not* $A^\wedge_{\mathrm{ab}}$ this time!), namely expressing *cohomology of B* (i.e., of the topos $B^\wedge$ defined by B) with coefficients in an *arbitrary*
[p. 362] abelian presheaf F on B, i.e., an arbitrary object of $B^\wedge_{\mathrm{ab}}$. I will first describe this property of (possible) function of a chain complex in $\mathrm{Add}(B)$, forgetting for the time being the category $A = B^{\mathrm{op}}$ and the homotopy types defined by objects X in $A^\wedge$. Once this property is well understood, it will be time to show it implies the previous one relative to A and objects of $A^\wedge$, and presumably is even equivalent with it.

First, we'll have to interpret the category $\mathrm{Add}(B)$, which was constructed somewhat "abstractly" in section 93 (as the solution of a universal problem stated in section 92), as a full subcategory of the category $B^\wedge_{\mathrm{ab}}$ of abelian presheaves on B. It will be useful to keep in mind the following diagram of canonical functors

$$\begin{tikzcd} B \arrow[r, "\alpha_B"] \arrow[d] \arrow[dr, "\beta_B"] & B^\wedge \arrow[d, "\mathrm{Wh}_B"] \\ \mathrm{Add}(B) \arrow[r, "\gamma_B"'] & B^\wedge_{\mathrm{ab}} \end{tikzcd}, \tag{2}$$

where α_B is the canonical inclusion, Wh_B is the abelianization functor, β_B the composition of the two, and γ_B the *additive* functor factoring β_B, in virtue of the universal property of $\mathrm{Add}(B)$. This functor is defined up to canonical isomorphism, and the lower triangle of (2) is commutative, up to a given commutativity isomorphism. Also, we'll use the composition of the following sequence of canonical equivalences of categories:

$$\begin{aligned}B^\wedge_{\mathrm{ab}} = \mathbf{Hom}(B^{\mathrm{op}}, (\mathrm{Ab})) &\xrightarrow{\sim} \mathbf{Hom}(B, (\mathrm{Ab})^{\mathrm{op}})^{\mathrm{op}} \\ &\xrightarrow{\sim} \mathbf{Homadd}(\mathrm{Add}(B), (\mathrm{Ab})^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{Homadd}(\mathrm{Add}(B)^{\mathrm{op}}, (\mathrm{Ab})),\end{aligned}$$

i.e., a canonical equivalence of category

(3) $$B^\wedge_{\mathrm{ab}} \xrightarrow{\sim} \mathbf{Homadd}(\mathrm{Add}(B)^{\mathrm{op}}, (\mathrm{Ab})), \quad F \mapsto \widetilde{F},$$

which is a particular case of

(3') $$\mathbf{Hom}(B^{\mathrm{op}}, M) \xrightarrow{\sim} \mathbf{Hom}(\mathrm{Add}(B)^{\mathrm{op}}, M),$$

where M is any additive category. If F is an abelian presheaf on B, i.e., an object in the left-hand side of (3), we'll denote by

(4) $$\widetilde{F} : \mathrm{Add}(B)^{\mathrm{op}} \to (\mathrm{Ab})$$

the corresponding additive functor. Now, this functor can be interpreted very nicely in terms of the functor γ_B in (2), by the canonical isomorphism of abelian groups

(5) $$\widetilde{F}(L) \xrightarrow{\sim} \mathrm{Hom}_{B^\wedge}(\gamma_B(L), F),$$

functorial with respect to the pair (F, L) in $B^\wedge_{\mathrm{ab}} \times \mathrm{Add}(B)^{\mathrm{op}}$. This formula [p. 363]
in turn implies easily that the functor γ_B is *fully faithful*. Thus, we can interpret $\mathrm{Add}(B)$ as the full subcategory of $B^\wedge_{\mathrm{ab}}$ whose objects are all finite direct sums (in $B^\wedge_{\mathrm{ab}}$) of objects of the type $\mathrm{Wh}_B(b)$, with b in B. In terms of this interpretation, γ_B is just an inclusion functor, and on the other hand, for F in the ambient category $B^\wedge_{\mathrm{ab}}$, $\widetilde{F}$ is just the restriction to the subcategory $\mathrm{Add}(B)$ of the contravariant functor on $B^\wedge_{\mathrm{ab}}$ represented by F.

This situation is the exact "additive" analogon of the situation of B embedded in $B^\wedge$ as a full subcategory, the functor on B defined by an object F of $B^\wedge$ being the restriction to B of the contravariant functor on $B^\wedge$ represented by F, i.e., an object of $F(b)$ or $\widetilde{F}(b)$ can (often advantageously) be interpreted as a map in $B^\wedge$, $b \mapsto F$. Moreover, the fact that

$$\alpha^*_B : F \mapsto \widetilde{F} : B^\wedge \xrightarrow{\sim} \mathbf{Hom}(b^{\mathrm{op}}, (\mathrm{Sets}))$$

is an equivalence (in fact, an isomorphism even), is paralleled by the equivalence (3), which can likewise be interpreted as γ^*_B, or more accurately as the canonical factorization of the purely set-theoretic γ^*_B : $B^\wedge_{\mathrm{ab}} \to \mathbf{Hom}(\mathrm{Add}(B)^{\mathrm{op}}, (\mathrm{Sets}))$ through $\mathbf{Homadd}(\mathrm{Add}(B)^{\mathrm{op}}, (\mathrm{Ab}))\ldots$

The objects L of the full subcategory $\mathrm{Add}(B)$ of $B^\wedge_{\mathrm{ab}}$ have a very strong common property, namely they are *projectives*, and they are of *finite presentation* ("small" in Quillen's terminology), namely for variable F in $B^\wedge_{\mathrm{ab}}$, the functor

$$F \mapsto \mathrm{Hom}_{B^\wedge_{\mathrm{ab}}}(L, F)$$

commutes with filtering direct limits. Both properties are immediate, and they nearly characterize the objects in $\mathrm{Add}(B)$ – more accurately, it is immediately checked that the projectives of finite presentation in $B^{\wedge}_{\mathrm{ab}}$ are exactly those which are isomorphic to *direct factors* of objects in $\mathrm{Add}(B)$. It shouldn't be hard to check that the full subcategory of $B^{\wedge}_{\mathrm{ab}}$ made up with the projectives of finite presentation can be identified up to equivalence to the "Karoubi envelope" of the category $\mathrm{Add}(B)$ (obtained by formally adding images of projectors), or equivalently, can be described as the solution of the 2-universal problem defined by sending B into categories which are, not only additive, but moreover stable under taking images of projectors (i.e., endomorphisms u of objects, such that $u^2 = u$).

[p. 364] We'll henceforth identify $\mathrm{Add}(B)$ to a full subcategory of $B^{\wedge}_{\mathrm{ab}}$ (by replacing the solution of the universal problem, constructed in section 95, by the essential image in $B^{\wedge}_{\mathrm{ab}}$ say), and rewrite (5) simply as

(5') $$\widetilde{F}(L) \simeq \mathrm{Hom}(L, F),$$

the Hom being taken in $B^{\wedge}_{\mathrm{ab}}$, category of abelian presheaves on B. Accordingly, if $L_\bullet$ is a chain complex in $\mathrm{Add}(B)$, hence in $B^{\wedge}_{\mathrm{ab}}$, the corresponding *cochain* complex $\widetilde{F}(L_\bullet)$ in (Ab) can be interpreted as

(6) $$\widetilde{F}(L_\bullet) \simeq \mathrm{Hom}^\bullet(L_\bullet, F),$$

where the symbol $\mathrm{Hom}^\bullet$ means taking Hom's componentwise.

What we're after here is to find a *fixed* chain complex $L_\bullet$ in $\mathrm{Add}(B)$, such that for *any* abelian presheaf F on B, the cochain complex (6) in (Ab) should be isomorphic (in the derived category $\mathrm{D}^\bullet(\mathrm{Ab})$ of cochain complexes in (Ab) with respect to quasi-isomorphisms) to the "integration" of F over the topos $B^{\wedge}$, i.e., to $\mathrm{R}\Gamma_B(F)$:

(*) $$\mathrm{Hom}^\bullet(L_\bullet, F) \simeq \mathrm{R}\Gamma_B(F)\ ? \quad (\text{isom. in } \mathrm{D}^\bullet(\mathrm{Ab})),$$

namely to the total right derived functor $\mathrm{R}\Gamma_B$ (taken for the argument F) of the "sections" functor

(7) $$\Gamma_B(F) \stackrel{\text{def}}{=} \varprojlim_{B^{\mathrm{op}}} F.$$

Now, using the fact that the components of the chain complex $L_\bullet$ are projective, hence $\mathrm{Ext}^i(L_n, F) = 0$ for $i > 0$ (any n, any F), we get at any rate a canonical isomorphism in $\mathrm{D}^\bullet(\mathrm{Ab})$, or in $\mathrm{D}(\mathrm{Ab})$:

(8) $$\mathrm{Hom}^\bullet(L_\bullet, F) \simeq \mathrm{R}\mathrm{Hom}(L_\bullet, F),$$

i.e., an interpretation of (6) as a "hyperext". Now, let's remember that $\mathrm{R}\Gamma_B(F)$ (as on any topos) can be interpreted equally as

(9) $$\mathrm{R}\Gamma_B(F) \simeq \mathrm{R}\mathrm{Hom}(\mathbb{Z}_B, F),$$

where $\mathbb{Z}_B$ denotes the constant presheaf on B with value $\mathbb{Z}$. Thus, the wished-for isomorphism (*) will follow most readily from a corresponding isomorphism in $\mathrm{D}_\bullet(B^{\wedge}_{\mathrm{ab}})$ between $L_\bullet$ and $\mathbb{Z}_B$. But using again the fact that the components of $L_\bullet$ are projective, we see that to give a map in the derived category of $L_\bullet$ into $\mathbb{Z}_B$ amounts to the same as to give an *augmentation*

(10) $$L_\bullet \to \mathbb{Z}_B,$$

and the map is an isomorphism in $\mathrm{D}_\bullet(B^\wedge_{\mathrm{ab}})$ iff the augmentation (10) turns $L_\bullet$ into a (projective) *resolution* of $\mathbb{Z}_B$. [p. 365]

We now begin to feel in known territory again! Let's call "*integrator*" on B any projective resolution of $\mathbb{Z}_B$, and let's call the integrator "*special*" (by lack of a more suggestive name) if its components L_n are in $\mathrm{Add}(B)$, or what amounts to the same, if it can be viewed as a chain complex in $\mathrm{Add}(B)$, endowed with the extra structure (10). Of course, $\mathbb{Z}_B$ is no longer in $\mathrm{Add}(B)$ in general, and therefore the data (10) has to be interpreted as a map $L_0 \to \mathbb{Z}_B$ external to $\mathrm{Add}(B)$, or equivalently (via (5')) as an object

(11) $$\lambda \quad \text{in} \quad \widetilde{\mathbb{Z}}_B(L_0) = \mathbb{Z}^{(I_0)},$$

where I_0 is the set of indices used in order to express L_0 as the direct sum in $B^\wedge_{\mathrm{ab}}$ of elements of B. We know, by the general principles of homological algebra, that any two integrators must be chain homotopic, hence, if they are special, as $\mathrm{Add}(B)$ is a full additive subcategory of $B^\wedge_{\mathrm{ab}}$, they must be chain homotopic in $\mathrm{Add}(B)$.

As for existence of integrators, it follows equally from general principles, as we know that $B^\wedge_{\mathrm{ab}}$ has "enough projectives" (which is a very special feature indeed of $B^\wedge_{\mathrm{ab}}$, coming from the fact that the topos $B^\wedge$ has enough projectives, namely the objects of B...). It isn't clear though that there exists a special integrator, because when trying inductively to construct the resolution $L_\bullet$ of $\mathbb{Z}_B$ with components in $\mathrm{Add}(B)$, it isn't clear that the kernel of $L_n \to L_{n-1}$ is "of finite type", namely is isomorphic to a quotient of an object of $\mathrm{Add}(B)$ (or, equivalently, is a quotient of a projective of finite presentation). If we take for instance B to be the one-object groupoid defined by a group G, an integrator on B is just a resolution of the constant G-module $\mathbb{Z}$ by projective $\mathbb{Z}[G]$-modules, and the integrator is special off the components are even free modules of finite type – I doubt such a resolution exists unless G itself is finite. This example seems to indicate that the existence of a special integrator for B is a very strong condition on B, of the nature of a (homological) finiteness condition. Maybe this condition, more than most others, singles out the three standard test categories and their finite products, from arbitrary test categories (even strict ones and contractors...).

Even in case a strict integrator doesn't exist for B, there is a rather evident way out to get "the next best" in terms of computations, namely replacing the very much finitely restricted category $\mathrm{Add}(B)$ by a larger category [p. 366]

(12) $$\mathrm{Addinf}(B) \hookrightarrow B^\wedge_{\mathrm{ab}}$$

deduced from B by adding, not merely finite direct sums (and linear combinations of maps), but equally infinite ones. The construction can be given "formally" as in section 93, and it can be checked that this category satisfies the obvious 2-universal property with respect to functors

$$f : B \to M$$

from B to infinitely additive categories M (namely additive categories stable under direct sums), and functors $M \to M'$ which are not merely additive, but commute to small direct sums. Moreover, it is checked that the category $\mathrm{Addinf}(B)$ thus constructed embeds by a fully faithful functor into $B^{\wedge}_{\mathrm{ab}}$ as indicated in (12), and hence can be identified up to equivalence to a full subcategory of $B^{\wedge}_{\mathrm{ab}}$. The formulas (5) and (5') are still valid, when L is in $\mathrm{Addinf}(B)$ only instead of $\mathrm{Add}(B)$. The objects of $\mathrm{Addinf}(B)$ in $B^{\wedge}_{\mathrm{ab}}$ are still projective (as direct sums of projectives), but of course no longer of finite presentation. In compensation, any element in $B^{\wedge}_{\mathrm{ab}}$ is now a quotient of an object in $\mathrm{Addinf}(B)$. As a consequence, the projectives in $B^{\wedge}_{\mathrm{ab}}$ can be characterized as the direct factor of objects of $\mathrm{Addinf}(B)$, and presumably the full subcategory of $B^{\wedge}_{\mathrm{ab}}$ made up with all projectives can again be described (up to equivalence) as the Karoubi envelope of $\mathrm{Addinf}(B)$, or equivalently, as the solution of the 2-universal problem of sending B into infinitely additive karoubian categories (karoubian = every projective has an image, i.e., corresponds to a direct sum decomposition). We may call an integrator $L_\bullet$ for B with components in $\mathrm{Addinf}(B)$ "*quasi-special*". We did just what was needed in order to be sure now that there always exist quasi-special integrators; moreover, these integrators are unique up to chain homotopy in $\mathrm{Addinf}(B)$. The interpretation (11) of the augmentation structure (10) on $L_\bullet$ is still valid in the quasi-special case, with the only difference that now the indexing set I_0 need not be finite anymore.

[p. 367] 17.6.

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Abelianization V: Homology versus cohomology.

Yesterday I introduced the notion of an *integrator* for any small category B, to be just a projective resolution of $\mathbb{Z}_B$ in the category $B^{\wedge}_{\mathrm{ab}}$ of all abelian presheaves on B, where $\mathbb{Z}_B$ denotes the constant presheaf with value $\mathbb{Z}$. Such an object in $\mathrm{Ch}_\bullet(B^{\wedge}_{\mathrm{ab}})$ exists, due to the existence of sufficiently many projectives in $B^{\wedge}_{\mathrm{ab}}$, and it is unique up to homotopism of augmented chain complexes, which encourages us to denote it by a canonizing symbol, namely

$$L^B_\bullet \to \mathbb{Z}_B. \tag{1}$$

As will become clear in the sequel, $L^B_\bullet$ can be viewed as embodying *homology properties* of B, i.e., of the topos associated to B (whose category of sheaves of sets is $B^{\wedge}$). The way we hit upon it though was in order to obtain a "computational" way for computing *cohomology* of B (i.e., of the associated topos) with coefficients in any abelian presheaf F in $B^{\wedge}_{\mathrm{ab}}$, by a canonical isomorphism

$$\mathrm{R}\Gamma_B(F) \xrightarrow{\sim} \mathrm{Hom}^\bullet(L^B_\bullet, F) \tag{2}$$

in the derived category $\mathrm{D}^\bullet(\mathrm{Ab})$, where $\mathrm{Hom}^\bullet$ denotes the cochain complex obtained by applying Hom componentwise. Passing to the cohomology groups of both members, this gives rise to

$$\mathrm{H}^i(B,F) \simeq \mathrm{H}^i \operatorname{Hom}^\bullet(L_\bullet^B, F). \tag{3}$$

The designation "computational" takes a rather concrete meaning, when we choose $L_\bullet^B$ to have its components in the infinitely additive envelope $\mathrm{Addinf}(B)$ of B, which (as we saw yesterday) can be viewed as a full subcategory of $B^\wedge_{\mathrm{ab}}$, made up with projectives, and such that any object in $B^\wedge$ is quotient of an object coming from $\mathrm{Addinf}(B)$; this ensures that there exist indeed integrators which are "*quasi-special*", i.e., are made up with objects of $\mathrm{Addinf}(B)$, and hence can be interpreted as chain complexes of this additive category. Thus, any component L_n can now be written, in an essentially canonical way, as

$$L_n = \bigoplus_{i\in I_n} \mathbb{Z}^{(b_i)}, \tag{4}$$

where

$$(b_i)_{i\in I_n} \tag{5}$$

is a family of objects of B indexed by I_n (NB for simplicity of notations, we assume the I_n's mutually disjoint, otherwise we should write the general object in the family (5) b_i^n rather than b_i). Thus, the n'th component of the cochain complex of the second member of (2) can be explicitly written as [p. 368]

$$\mathrm{Hom}^n(L_\bullet^B, F) = \mathrm{Hom}(L_n^B, F) \simeq \bigoplus_{i\in I_n} F(b_i), \tag{6}$$

and the coboundary operators between these components can be made explicit in a similar way, by means of (possibly infinite) matrices, whose entries are $\mathbb{Z}$-linear combinations of maps from some b_i^n to some b_j^{n-1} ($i \in I_n$, $j \in I_{n-1}$). We feel a little happier still when the direct sums (4) yielding the components L_n are finite, i.e., the sets I_n are finite, which also means that $L_\bullet^B$ can be interpreted as a chain complex in the additive envelope $\mathrm{Add}(B)$ of B, as contemplated in the first place – in which case the integrator will be called "*special*".

The formula (2) immediately generalizes when F is replaced by a complex of presheaves $F^\bullet$, with degrees bounded from below (NB as the notation indicates, the differential operator is of degree $+1$), to

$$\mathrm{R}\Gamma_B(F^\bullet) \xrightarrow{\sim} \mathrm{Hom}^{\bullet\bullet}(L_\bullet^B, F^\bullet), \tag{7}$$

where now the left-hand side designates hypercohomology of B (i.e., of the corresponding topos), viewed as an objects of the right derived category $\mathrm{D}^+(\mathrm{Ab})$ of the category of abelian groups, and where $\mathrm{Hom}^{\bullet\bullet}$ designates the double complex obtained by taking Hom's componentwise, or more accurately, the object in $\mathrm{D}^+(\mathrm{Ab})$ defined by the associated simple complex.

An interesting special case of (7) is obtained when starting with a complex of abelian groups $K^\bullet$ bounded from below, i.e., defining an object of the right derived category $\mathrm{D}^+(\mathrm{Ab})$, and taking

(8) $$F^\bullet = K^\bullet_B = p^*_B(K^\bullet),$$

the corresponding *constant complex of presheaves* on B, which may be viewed equally as the inverse image of $K^\bullet$ by the projection

(9) $$p_B : B \to \mathbb{A}_0 \quad \text{(the final category)},$$

which geometrically interprets as the canonical morphism of the topos associated to B to the final (or "one-point") topos. The second member of (7) can be rewritten componentwise, using the adjunction formula for the pair $(p^{\mathrm{ab}}_!, p^*)$ (where the qualifying B is omitted now in the notation p):

$$\mathrm{Hom}(L_n, p^*(K^m)) \simeq \mathrm{Hom}(p^{\mathrm{ab}}_!(L_n), K^M),$$

[p. 369] so that (7) can be rewritten as

(10) $$\mathrm{R}\Gamma_B(K^\bullet_B) \simeq \mathrm{Hom}^{\bullet\bullet}(p^{\mathrm{ab}}_{B!}(L^B_\bullet), K^\bullet),$$

where this time the Hom's in the right-hand side of (10) are taken in (Ab), not in $B^\wedge_{\mathrm{ab}}$.

This formula very strongly suggests to view the chain complex of abelian groups

(11) $$p^{\mathrm{ab}}_{B!}(L^B_\bullet),$$

which is in fact a complex of projective (hence free) abelian groups defined up to chain homotopy, as embodying the global homology structure of B (or of the corresponding topos), more accurately still, as embodying the homology structure of the corresponding homotopy type. It is easily seen that the corresponding object of $\mathrm{D}_\bullet(\mathrm{Ab})$ depends covariantly on B when B varies in the category (Cat), so that we get a functor

$$(\mathrm{Cat}) \to \mathrm{D}_\bullet(\mathrm{Ab}) \overset{\mathrm{def}}{=} (\mathrm{Hotab}),$$

which in view of (10) (an isomorphism functorial not only with respect to $K^\bullet$, but equally with respect to B) factors through the localization (Hot) of (Cat), thus yielding a canonical functor

(12) $$(\mathrm{Hot}) \to (\mathrm{Hotab}),$$

which deserves to be called the *abelianization functor*, from homotopy types to "abelian homotopy types". This cannot be of course anything else (up to canonical isomorphism) but the functor (1) of section 92 (p. 321), but obtained here in a wholly "intrinsic" way, without having to pass through the particular properties of a particular test category such as $\mathbb{A}$ or one of its twins. One possible way to check this identity would be by proving that an isomorphism (10) is valid when replacing (for a given B in (Cat) and $K^\bullet$ in $\mathrm{D}^+(\mathrm{Ab})$) the chain complex (11) by the corresponding one deduced from the map (1) defined p. 321 (via the diagram (3) on p. 322), and checking moreover that an object $\ell_\bullet$ of $\mathrm{D}^-(\mathrm{Ab})$ is known up to canonical isomorphism, when we know the corresponding functor

$$K^\bullet \mapsto \mathrm{Hom}_{\mathrm{D}(\mathrm{Ab})}(\ell_\bullet, K^\bullet) \tag{13}$$

on $\mathrm{D}^+(\mathrm{Ab})$. Presumably, this latter statement holds when replacing (Ab) by any abelian category, but I confess I didn't sit down to check it, nor do I remember having seen it stated somewhere – as I don't remember either having seen anywhere a comprehensive treatment about the relationship between homology and cohomology. So maybe my present reflections do fill a gap, or at any rate give some indications as to how to fill it...

I played around some yesterday and today with the formalism of integrators, notably with respect to maps

$$f : B' \to B$$

between small categories, and the corresponding *integration functor*

$$f_!^{\mathrm{ab}} : B'^{\wedge}_{\mathrm{ab}} \to B^{\wedge}_{\mathrm{ab}},$$

and its left derived functor $\mathrm{L}f_!^{\mathrm{ab}}$. Thus, the chain complex in $B^{\wedge}_{\mathrm{ab}}$

$$L_\bullet^{B'/B} \quad \text{or} \quad L_\bullet^f \overset{\text{def}}{=} f_!^{\mathrm{ab}}(L_\bullet^{B'}), \tag{14}$$

which has projective components (and even is a chain complex in $\mathrm{Addinf}(B)$ resp. in $\mathrm{Add}(B)$, if $L_\bullet^{B'}$ is quasi-special resp. is special), and is defined up to chain homotopism, embodies the relative homology properties of B' over B, i.e., of f, in much the same way as (11) embodies the global homology properties of B (i.e., of B over one point). When the functor f is "coaspheric", i.e., the functor

$$f^{\mathrm{op}} : B'^{\mathrm{op}} \to B^{\mathrm{op}}$$

between the opposite categories is aspheric, then $L_\bullet^{B'/B}$ is again an integrator on B, and the converse should hold too provided we take the meaning of "coaspheric" and "aspheric" with respect to a suitable basic localizer $\mathcal{W} = \mathcal{W}_\infty^{\mathbb{Z}}$ – presumably, we'll come back upon this in part V or part VI of the notes. For the time being, it seems more interesting to give now the precise relationship between the notion of an *integrator* for B, and the notion of an *abelianizator* for the dual category $A = B^{\mathrm{op}}$, introduced in section 93.

Remarks. 1) It is a familiar fact that when working in Čech-flavored contexts, such as general topoi, or étale topoi for schemes and the like, one has throughout and from the start a good hold upon *cohomology* notions, whereas it is a lot more subtle to squeeze out adequate homology notions, which (to my knowledge) can be carried through only indirectly via cohomology, and using suitable finiteness and duality statements within the cohomology formalism. Historically however, homology was introduced before cohomology via cellular decompositions of spaces, with a more direct appeal to geometric intuition. This preference for homology rather than cohomology seems to be still prevalent among most homotopy theorists, who have a tendency to view a topological space (however wild it may be) as being no more no less than its singular complex. A comprehensive statement establishing, in a suitable wide enough context, essential equivalence between the two viewpoints, [p. 371] seems to be still lacking, as far as I know – although a fair number of partly overlapping results in this direction are known, among the oldest being the relevant "universal coefficients formulæ" relating homology and cohomology (reducing all to a formula of the type (2) or (10) above), or Cartan's old seminar on Leray's sheaf theory, introducing singular homology with coefficients in a sheaf and proving that on a topological variety, this was (up to dimension shift and twist by the twisted integers) essentially the same as singular cohomology (with coefficients in sheaves too). It is not sure that an all-inclusive statement of equivalence between homology and cohomology (in those situations when such equivalence is felt hold indeed) does at all exist – at any rate, according to what kind of coefficients one wants to consider, and what kind of extra structures one is interested in when dealing with homology and cohomology invariants, it seems that each of the two points of view has an originality and advantages of its own and cannot be entirely superseded by the other. From the contexts I have been mainly working in, there definitely was no choice, namely cohomology (including non-commutative one) was the basic data, while sheaves and their generalizations (such complexes of sheaves, or stacks) were the coefficients. I don't remember of any moment where I would have paused and asked myself *why* in most contexts where I was working in (whose common denominator was topoi), there wasn't any direct hold on anything like homology invariants. The reason for this inertness of mine, probably, is that the cohomology formalisms I hit upon were self-contained enough, so as to leave no regret for the absence of a homology formalism, or at any rate of a more or less direct description of it independently of cohomology. Another reason, surely, is that I didn't have too much contact with topologists and homotopists and their everyday tools, such as Steenrod operations, homology of the symmetric group, and the like. This question of "why this reluctance of homology to show forth" has finally surfaced only during these very last days, when the answer for it (or one possible answer at any rate) is becoming evident: namely, that *for a general topos*, embodied by a category of sheaves (of sets) $\mathcal{A}$, *there are not enough projectives in* $\mathcal{A}$, *and not even enough projectives in* $\mathcal{A}_{\mathrm{ab}}$, the category of abelian sheaves. It is becoming apparent (what surely everybody has known ages) that in technical terms, *doing "homology" is working with projectives, while doing "cohomology" is working with injectives*. As there are enough injectives in $\mathcal{A}_{\mathrm{ab}}$ but not enough projectives, cohomology is around and homology not, period!

There is however a rather interesting class of topoi admitting sufficiently many projective sheaves of sets, and hence sufficiently many projective abelian sheaves – namely the topoi $B^\wedge$ defined in terms of small categories B. They include the topoi which can be described in terms of semisimplicial complexes and the like, and can be viewed equally as the topoi which are "closest to algebra" or "purely algebraic" in a suitable sense – for instance, definable directly in terms of arbitrary presheaves, without any reference to the notion of site and of localization. (The intuition of localization remaining however and indispensable guide even in the so-called "algebraic" set-up.) Moreover, the morphisms which arise most naturally among such topoi, namely those associated to maps [p. 372]

$$f : B' \to B$$

in (Cat), besides the traditional adjoint pair (f^*, f_*) of functors between sheaves of sets, gives rise equally to a functor

$$f_! : B'^\wedge \to B^\wedge \tag{15}$$

left adjoint to f^* (i.e., f^* commutes to small inverse limits, not only to small direct limits and to finite inverse limits), inserting in a triple of mutually adjoint functors (from left to right)

$$(f_!, f^*, f_*). \tag{16}$$

The functors f^* and f_* induce corresponding adjoint functors on abelian sheaves (due to the fact that they commute to finite products), f^*_{ab} and f_*^{ab}, whereas $f_!$ does not in general transform group objects into group objects; however, as f^*_{ab} commutes to small inverse limits, it does admit again a left adjoint $f_!^{\mathrm{ab}}$, so as to give again a triple

$$(f_!^{\mathrm{ab}}, f^*_{\mathrm{ab}}, f_*^{\mathrm{ab}}) \tag{17}$$

of mutually adjoint functors. Now, whereas the derived functors

$$f^* \text{ or } \mathrm{L}f^*_{\mathrm{ab}}, \quad \mathrm{R}f_* \text{ or } \mathrm{R}f_*^{\mathrm{ab}}$$

of f^*_{ab} and f_*^{ab} have been extensively used in the every-day cohomology formalism of topoi, the existence in certain cases (such as the one we are interested in here) of a functor $f_!^{\mathrm{ab}}$ and of its left derived functor

$$\mathrm{L}f_! \text{ or } \mathrm{L}f_!^{\mathrm{ab}} : \mathrm{D}^-(B'^\wedge_{\mathrm{ab}}) \to \mathrm{D}^-(B^\wedge_{\mathrm{ab}}), \tag{18}$$

seems to me to have been widely overlooked so far, except in extremely particular cases such as inclusion of an open subtopos; at any rate, I have been overlooking it till lately, when it came to my attention through the writing of these notes. (Namely, first in connection with my reflections on derivators (cf. section 69), and now in connection with the reflections on abelianization.) In view of my reflections on derivators, I would like to view the functor (18) as an operation of "integration", whereas the traditional functor [p. 373]

$$\mathrm{R}f_* : \mathrm{D}^+(B'^\wedge_{\mathrm{ab}}) \to \mathrm{D}^+(B^\wedge_{\mathrm{ab}}) \tag{19}$$

is viewed as "cointegration" (which I prefer to my former way of calling it an "integration"). The first should be viewed as expressing *homology* properties of the map f in (Cat) (or between the corresponding topoi), just as the latter expresses *cohomology* properties of f. This does check with the corresponding qualifications "integration" – "cointegration" – as well as with the intuition, when B is reduced to a point, identifying the first to a kind of direct sum (= integration), whereas the latter is viewed as a kind of direct product (= *co*integration). The idea behind the terminology will go through maybe when looking at the particular case when B' is a sum of copies of B, namely a product of B by a discrete category I, and

$$f : B' = B \times I \to B$$

the projection.

The point I want to make here, mainly to myself, is that in the present context *when* (18)*, namely integration, exists, this operation presumably is by no means less meaningful and important than the familiar* $\mathrm{R}f_*$ *or cointegration* – or *equivalently stated, that the homology properties of* f *are just as meaningful and deserving close attention, as the cohomology properties*, which so far have been the only ones I have been looking at. Presumably, when following this recommendation, a few unexpected facts and relationships should come out, such as various "duality" relationships between homology properties of f, and cohomology properties of the map f^{op} between the opposite categories. (This is suggested by some of the scratchwork I made on derivators and cohomology properties of maps in (Cat).) The only trouble is that such change or broadening of emphasis as I am now suggesting will require a certain amount of extra attention, which I am not too sure to be willing to invest in the subject, namely algebraic topology. Thus presumably, my main emphasis will remain with cohomology, rather than homology. I am no longer convinced though that this point of view is technically more adequate than the dual one.

2) All the reflections of yesterday's notes as well as today's can be extended, when replacing throughout abelian presheaves by presheaves [p. 374] of k-modules, and additive envelopes by k-linear ones, where k is any given commutative ring. Of course, the category (Ab) and its various derived categories will have to be replaced accordingly by the category (k-Mod) of k-modules etc. The same holds for the relationship I am going to write down between integrators for B and abelianizators for $A = B^{\mathrm{op}}$. For simplicity of notations, I am going to keep the exposition in the (Ab)-framework I have started with, and leave the necessary adjustments to the reader.

18.7.

101 Finally with yesterday's non-technical reflections on homology versus cohomology, it was getting prohibitively late, and there could be no question to deal with the relationship between integrators (for B) and abelianizators (for $B^{\mathrm{op}} = A$). Also, I feel I should give some "computational" details about the functor $f_!^{\mathrm{ab}}$ associated to a map in (Cat)

Abelianization VI: The abelian integration operation $\mathrm{L}f_!^{\mathrm{ab}}$ *defined by a map* f *in* (Cat) (*versus abelian cointegration* $\mathrm{R}f_*$).

$$f : B' \to B,$$

namely

$$f_!^{\mathrm{ab}} : B'^{\wedge}_{\mathrm{ab}} \to B^{\wedge}_{\mathrm{ab}},$$

which is a lot less familiar to me than its right adjoint and biadjoint f^* and f_*. One way to get a "computational hold" upon it is by noting that $f_!^{\mathrm{ab}}$ commuting to small direct limits and a fortiori being right exact, and moreover any object F' in $B'^{\wedge}_{\mathrm{ab}}$ being a cokernel of a map between "*special projectives*" in $B'^{\wedge}_{\mathrm{ab}}$, i.e., between objects in $\mathrm{Addinf}(B')$, namely inserting into an exact sequence

$$L'_1 \xrightarrow{d} L'_0 \to F' \to 0 \quad \text{with } L'_0, L'_1 \text{ in } \mathrm{Addinf}(B'),$$

the functor $f_!^{\mathrm{ab}}$ (via its values on any F' say) is essentially known, when we know its restriction to the subcategory $\mathrm{Addinf}(B')$, as we'll get a corresponding exact sequence in $B^{\wedge}_{\mathrm{ab}}$

$$f_!^{\mathrm{ab}}(L'_1) \to f_!^{\mathrm{ab}}(L'_0) \to f_!^{\mathrm{ab}}(F') \to 0,$$

describing $f_!^{\mathrm{ab}}(F')$ as a cokernel of a map $f_!^{\mathrm{ab}}(d)$ corresponding to a map in $\mathrm{Addinf}(B')$. The relevant fact now is that we have a commutative diagram of functors (up to can. isomorphism as usual)

(1)
$$\begin{array}{ccc} \mathrm{Addinf}(B') & \longrightarrow & B'^{\wedge}_{\mathrm{ab}} \\ {\scriptstyle \mathrm{Addinf}(f)}\downarrow & & \downarrow{\scriptstyle f_!^{\mathrm{ab}}} \\ \mathrm{Addinf}(B) & \longrightarrow & B^{\wedge}_{\mathrm{ab}} \end{array} ,$$

where the horizontal arrows are the canonical inclusion functors, and $\mathrm{Addinf}(f)$ is the "tautological" extension of $f : B' \to B$ to the infinitely additive envelopes, defined computationally as [p. 375]

(2)
$$\mathrm{Addinf}(f)(L') \simeq \bigoplus_{i \in I} \mathbb{Z}^{f(b'_i)}$$

for an object of $\mathrm{Addinf}(B')$ written canonically as

(3)
$$L' = \bigoplus_{i \in I} \mathbb{Z}^{(b'_i)}.$$

Here, for an object b in a small category B, we denote by the more suggestive symbol $\mathbb{Z}^{(F)}$ the abelianization $\mathrm{Wh}_B(F)$ of an object F of $B^{\wedge}$, and accordingly of F is an object b in B. The fact that (2) is equally an expression for $f_!^{\mathrm{ab}}$ follows immediately from commutation of $f_!^{\mathrm{ab}}$ to small direct sums, and from the canonical isomorphism

(4) $$f_!^{\mathrm{ab}}(\mathbb{Z}^{F'}) \simeq \mathbb{Z}^{(f_!(F'))},$$

i.e., commutation up to canonical isomorphism of the diagram

(5) $$\begin{tikzcd} B'^{\wedge} \arrow[r, "\mathrm{Wh}_{B'}"] \arrow[d, "f_!"'] & B'^{\wedge}_{\mathrm{ab}} \arrow[d, "f_!^{\mathrm{ab}}"] \\ B^{\wedge} \arrow[r, "Wh_B"] & B^{\wedge}_{\mathrm{ab}} \end{tikzcd} ,$$

the verification of which is immediate. (For a generalization to sheaves endowed with arbitrary "algebraic structures" and taking free objects, see SGA 4 I 5.8.3, p. 30.)

[Artin, Grothendieck, and Verdier (SGA 4.1)]

Of course, (1) and (2) imply that $f_!^{\mathrm{ab}}$ maps $\mathrm{Add}(B')$ into $\mathrm{Add}(B)$, and induces the tautological extension $\mathrm{Add}(f)$ of f to the additive envelopes. Thus, (1) and (5) can be inserted into a beautiful commutative diagram (up to canonical isomorphism)

(6) $$\begin{tikzcd} B' \arrow[rr, hook] \arrow[d, "f"'] \arrow[ddrrr, hook] & & \mathrm{Add}(B') \arrow[rr, hook] \arrow[d, "\mathrm{Add}(f)"] & & \mathrm{Addinf}(B') \arrow[rr, hook] \arrow[d, "\mathrm{Addinf}(f)"'] & & B'^{\wedge}_{\mathrm{ab}} \arrow[d, "f_!^{\mathrm{ab}}"] \\ B \arrow[rr, hook] \arrow[dddrrr, hook] & & \mathrm{Add}(B) \arrow[rr, hook] & & \mathrm{Addinf}(B) \arrow[rr, hook] & & B^{\wedge}_{\mathrm{ab}} \\ & & & B'^{\wedge} \arrow[uurrr, "\mathrm{Wh}_{B'}"] \arrow[d, "f_!"] & & & \\ & & & B^{\wedge} \arrow[uurrr, "\mathrm{Wh}_B"'] & & & \end{tikzcd} .$$

The formula (1) (or equivalently, (2)) can be viewed as giving a computational description of the left derived functor

(7) $$\mathrm{L}f_!^{\mathrm{ab}} : \mathrm{D}^-(B'^{\wedge}_{\mathrm{ab}}) \to \mathrm{D}^-(B^{\wedge}_{\mathrm{ab}}).$$

[p. 376] Indeed, by general principles of homological algebra, for any small category B, from the fact that $\mathrm{Addinf}(B)$ is made up with projective objects of $B^{\wedge}_{\mathrm{ab}}$ and that any object in $B^{\wedge}_{\mathrm{ab}}$ is isomorphic to a quotient of an object in this subcategory, it follows that

(8) $$\mathrm{D}^-(B^{\wedge}_{\mathrm{ab}}) \approxeq W_B^{-1}\,\mathrm{Comp}^-(\mathrm{Addinf}(B)),$$

i.e., the left derived category $\mathrm{D}^-(B^{\wedge}_{\mathrm{ab}})$ is equivalent with the category obtained by localizing, with respect to the set W_B of homotopy equivalences, the category $\mathrm{Comp}^-(\dots)$ of differential complexes in the additive category $\mathrm{Addinf}(B)$, with degrees bounded from above (the differential operator being of degree $+1$, according to my preference for cohomology notation, sorry!). An object of $\mathrm{D}^-(B^{\wedge}_{\mathrm{ab}})$ may thus be viewed as being essentially the same as a differential complex in Addinf(B) with degrees bounded from above, and given "up to homotopism". The similar description holds for $\mathrm{D}^-(B'^{\wedge}_{\mathrm{ab}})$, and in terms of these descriptions, the "integration functor" (in the abelian context) (7) can be described by

$$\mathrm{L}f_!^{\mathrm{ab}}(L'_\bullet) = \mathrm{Addinf}(L'_\bullet), \tag{9}$$

i.e., by applying componentwise the tautological extension Addinf(f) of f to the differential complexes in Addinf(B'). This very concrete description applies notably to the complex

$$L_\bullet^{B'/B} \quad \text{or} \quad L_\bullet^f \overset{\mathrm{def}}{=} f_!^{\mathrm{ab}}(L_\bullet^{B'}) \tag{10}$$

introduced yesterday, whenever a (quasi-special) integrator $L_\bullet^{B'}$ for B' has been chosen. Applying this to the case of the map

$$p_B : B \to \mathbb{A}_0,$$

we get (for a given integrator $L_\bullet^B$ for B) an explicit description of the abelianization of the homotopy type of B in terms of the chain complex $p_{B!}^{\mathrm{ab}}(L_\bullet^B)$ in (Ab), with the n'th component given by

$$\left(L_\bullet^{B/\mathrm{pt}}\right)_n = \mathbb{Z}^{(I_n)}, \tag{11}$$

where I_n is the set of indices used for describing L_n^B as the direct sum of objects of the type $\mathbb{Z}^{(b_i)}$.

Returning to the case of a general map $f : B' \to B$, maybe I should still write down the formula generalizing (2) or (10) of yesterday's notes (pages 367 and 369), relating $L_\bullet^{B'/B}$ to the cohomology properties of the map f, i.e., to cointegration relative to f. The formula expresses cointegration $\mathrm{R}f_*$ with coefficients coming from downstairs, namely $f^*(K^\bullet)$, where $K^\bullet$ is any differential complex in $B^{\wedge}_{\mathrm{ab}}$ with degrees bounded from below (thus defining an object in $\mathrm{D}^+(B^{\wedge}_{\mathrm{ab}})$). The relevant formula is [p. 377]

[see section 139, bottom of p. 588, for corrections to this formula and (12'), (13) below...]

$$\mathrm{R}f_*(f^*(K^\bullet)) \simeq \mathbf{Hom}^{\bullet\bullet}(L_\bullet^{B'/B}, K^\bullet), \tag{12}$$

an isomorphism in $\mathrm{D}^+(B^{\wedge}_{\mathrm{ab}})$, where $\mathbf{Hom}^{\bullet\bullet}$ designates the double complex in $B^{\wedge}_{\mathrm{ab}}$ obtained by applying **Hom** componentwise, more accurately the associated simple complex, and where **Hom** is the internal **Hom** in the category $B^{\wedge}_{\mathrm{ab}}$, namely the (pre)sheaf of additive homomorphisms of a given abelian (pre)sheaf (L_n say) into another (K^m say). The proof of (12) is essentially trivial, it is just the computational interpretation, in terms of using projective resolutions, of the adjunction formula "localized on B

$$\mathrm{R}f_*(\mathrm{L}f^*(K^\bullet)) \simeq \mathrm{R}\mathbf{Hom}(\mathrm{L}f_!^{\mathrm{ab}}(\mathbb{Z}_{B'}), K^\bullet), \tag{12'}$$

which is a particular case of the more general "adjunction formula"

$$\mathrm{R}f_*(\mathrm{R}\mathbf{Hom}(F'_\bullet, \mathrm{L}f^*(K^\bullet)) \simeq \mathrm{R}\mathbf{Hom}(\mathrm{L}f_!^{\mathrm{ab}}(F'_\bullet), K^\bullet), \tag{13}$$

valid for

$$F'_\bullet \text{ in } \mathrm{D}^-(B'^{\wedge}_{\mathrm{ab}}), \quad K^\bullet \text{ in } \mathrm{D}^+(B^{\wedge}_{\mathrm{ab}}),$$

(12') following from (13) by taking $F'_\bullet = \mathbb{Z}_{B'}$.

Remarks. We may view (13), and its particular case (12) or (12'), as the main formula relating the *homology* and *cohomology* invariants for a map f in (Cat), or equivalently, the (abelian) *integration* and *cointegration* operations defined by f. It now occurs to me that this formula, and the variance formalism in which it inserts, is valid more generally whenever we have a map f between two ringed topoi, such that $f_!$ exists for sheaves of sets, hence there exists too a corresponding functor $f_!^{\mathrm{mod}}$ for sheaves of modules. The fact that we have been restricting to the case of the constant sheaves of rings defined by $\mathbb{Z}$ isn't relevant, and (in the case of topoi defined by objects in (Cat), hence with sufficiently many projective sheaves of sets) the formalism of the subcategories $\mathrm{Add}(B)$ and $\mathrm{Addinf}(B)$ in $B^{\wedge}_{\mathrm{ab}}$ can be generalized equally to arbitrary sheaves of rings on B. At present, I don't see though any striking particular case where this generalization would seem useful. [p. 378]

102 *Abelianization VII: Integrators (for A^{op}) are abelianizators (for A).*

We now focus attention upon the pair of mutually dual small categories

(1) $$(A,B),\quad \text{with } B = A^{\mathrm{op}},\ \text{i.e., } A = B^{\mathrm{op}},$$

and recall the equivalence of section 93 following from the universal property of $\mathrm{Add}(B)$

(2) $$A^{\wedge}_{\mathrm{ab}} = \mathbf{Hom}(A^{\mathrm{op}},(\mathrm{Ab})) \approx \mathbf{Homadd}(\mathrm{Add}(A^{\mathrm{op}}),(\mathrm{Ab})),$$

which we parallel with the formula (3) of section 99 (p. 362), which reads when replacing in it B by A

$$A^{\wedge}_{\mathrm{ab}} \simeq \mathbf{Homadd}(\mathrm{Add}(A)^{\mathrm{op}},(\mathrm{Ab}));$$

this immediately suggests a canonical equivalence of categories

(3) $$\mathrm{Add}(A^{\mathrm{op}}) \approx \mathrm{Add}(A)^{\mathrm{op}},$$

following immediately indeed from the 2-universal properties of these categories. We complement (2) by the similar formula

(4) $$F \mapsto \widetilde{F}:\quad A^{\wedge}_{\mathrm{ab}} \xrightarrow{\approx} \mathbf{Homaddinf}(\mathrm{Addinf}(B),(\mathrm{Ab})),\quad B = A^{\mathrm{op}},$$

where **Homaddinf** denotes the category of infinitely additive functors between two infinitely additive categories. In view of the emphasis lately on chain complexes in $\mathrm{Addinf}(B)$ rather than in $\mathrm{Add}(B)$, in order to reconstruct say the derived category $\mathrm{D}_\bullet(B^{\wedge}_{\mathrm{ab}})$ of chain complexes in $B^{\wedge}_{\mathrm{ab}}$, and get existence of "integrators" with components in $\mathrm{Addinf}(B)$ (whereas there may be none with components in $\mathrm{Add}(B)$), it is formula (4) rather than (2) which is going to be relevant for our homology formalism. Using (4), we get a canonical biadditive pairing

(*) $$A^{\wedge}_{\mathrm{ab}} \times \mathrm{Addinf}(B) \to (\mathrm{Ab}),$$

which visibly is exact with respect to the first factor, and which we may equally interpret as a functor

(5) $$L \mapsto \widetilde{L} : \mathrm{Addinf}(B) \to \mathbf{Homex}(A^{\wedge}_{\mathrm{ab}},(\mathrm{Ab})),$$

where **Homex** denotes the category of *exact* (hence additive) functors from an abelian category to another one.

It can be shown that the pairing (*) can be extended canonically to a pairing

$$A^{\wedge}_{\mathrm{ab}} \times B^{\wedge}_{\mathrm{ab}} \to (\mathrm{Ab}) \tag{6}$$

commuting to small direct limits in each variable, and identifying (up to equivalence) each left hand factor to the category of functors from the other functor to (Ab) which commute with small direct limits (much in the same way as the corresponding relationship between $A^{\wedge}$ and $B^{\wedge}$, with (Ab) being replaced by (Sets)), and the accordingly the functor (5) is equally fully faithful, and extends to a fully faithful functor from $B^{\wedge}_{\mathrm{ab}}$ to $\mathbf{Homadd}(A^{\wedge}_{\mathrm{ab}}, (\mathrm{Ab}))$, inducing in fact an equivalence between $B^{\wedge}_{\mathrm{ab}}$ and the full subcategory $\mathbf{Hom}_!(A^{\wedge}_{\mathrm{ab}}, (\mathrm{Ab}))$ of $\mathbf{Hom}(A^{\wedge}_{\mathrm{ab}}, (\mathrm{Ab}))$ made up by all functors $A^{\wedge}_{\mathrm{ab}} \to (\mathrm{Ab})$ which commute to small direct limits. But for what we have in mind at present, these niceties are not too relevant yet it seems – all what matters is that an object L of Addinf(B) defines an exact functor [p. 379]

$$\widetilde{L} : A^{\wedge}_{\mathrm{ab}} \to (\mathrm{Ab}),$$

depending functorially on L, in an infinitely additive way. Thus, as noted in section 92 (but where Addinf(B) was replaced by the smaller category Add(B), which has turned out insufficient for our purposes), whenever we have a chain complex $L_\bullet$ in Addinf(B), we get a corresponding functor

$$\widetilde{L}_\bullet : A^{\wedge}_{\mathrm{ab}} \to \mathrm{Ch}_\bullet(\mathrm{Ab}) \tag{7}$$

from $A^{\wedge}_{\mathrm{ab}}$ to the category of chain complexes of (Ab), which is moreover an *exact* functor. Generalizing slightly the terminology introduced in section 93, where we restricted to chain complexes with components in Add(B) rather than in Add(B), we'll say that $\widetilde{L}_\bullet$ is an *abelianizator for A*, if the following diagram commutes up to isomorphism:

$$\begin{tikzcd}
A^{\wedge} \arrow[r] \arrow[d, "\substack{\mathrm{Wh}_A \\ (\text{abelianization})}"'] & \mathrm{Hot}_A \arrow[r] & (\mathrm{Hot}) \arrow[d, "\substack{\text{“absolute”} \\ \text{abelianization} \\ \text{functor}}"] \\
A^{\wedge}_{\mathrm{ab}} \arrow[r, "\widetilde{L}_\bullet"] & \mathrm{Ch}_\bullet(\mathrm{Ab}) \arrow[r] & (\mathrm{Hotab})
\end{tikzcd} \tag{8}$$

More accurately, an abelianizator is a pair $(\widetilde{L}_\bullet, \lambda)$, where λ is an isomorphism of functors $A^{\wedge} \to (\mathrm{Hotab})$ making the diagram commute. Here, I like to view the abelianization functor

$$(\mathrm{Hot}) \to (\mathrm{Hotab}) \overset{\mathrm{def}}{=} \mathrm{D}_\bullet(\mathrm{Ab}) \tag{9}$$

as the one described directly in section 100 via integrators of arbitrary modelizing objects in (Cat), without any reference to an auxiliary test category such as $\triangle$ or the like.

The point of (8) is that via an “abelianizator” for A, we want to be able to give a) a *simultaneous* handy expression, in terms of “computable” [p. 380] chain complexes in (Ab), of abelianization of homotopy types modelized by a variable object X in $A^\wedge$, and b) we want that the chain complex in (Ab) expressing abelianization of X, should be expressible in terms of the "tautological abelianization" $\mathrm{Wh}_A(X) = \mathbb{Z}^{(X)}$ of X itself, by a formula moreover which should make sense functorially with respect to an arbitrary abelian presheaf, i.e., an object F in $A^\wedge_{\mathrm{ab}}$.

The main fact I have in view here is that whenever the chain complex $L_\bullet$ in $\mathrm{Addinf}(B)$ is endowed with an augmentation

$$L_\bullet \to \mathbb{Z}_B \tag{10}$$

turning it into a resolution of $\mathbb{Z}_B$, i.e., into a (quasi-special) *integrator* for B, then ipso facto $L_\bullet$ is an abelianizator for A, the commutation isomorphism λ being canonically defined by the augmentation (10).

Some comments, before proceeding to a proof. Presumably, the converse of our statement holds too – namely that the natural functor we'll get from quasi-special integrators for B to abelianizators $(L_\bullet, \lambda)$ for B is an equivalence (even an isomorphism!) between the relevant categories. I don't feel like pursuing this – the more relevant fact here, whether or not a converse as contemplated holds, is that we can pin down at any rate a special class of abelianizators for A, namely those which come from (quasi-special) integrators for B, and these abelianizators are defined up to chain homotopism in $\mathrm{Addinf}(B)$. In this sense, *we get an existence and unicity statement for abelianizators in A*, as strong as we possibly could hope for. In practical terms, it would seem, *an abelianizator for A will be no more no less than just a* (*quasi-special*) *integrator for B, namely a projective resolution of $\mathbb{Z}_B$ in $B^\wedge_{\mathrm{ab}}$, whose components satisfy a mild extra assumption besides being projective*.

Here, I am struck by a slight discrepancy in terminology, as we would rather have a correspondence

$$\begin{cases} \text{integrators for } B \to \text{abelianizators for } A \\ \text{quasi-special int.s for } B \to \text{quasi-special abelian.s for } A, \end{cases}$$

and the same for "special" integrators and abelianizators. As I still feel that the general appellation of an "integrator" for *any* projective resolution of $\mathbb{Z}_B$ is adequate (without insisting that the components should be in $\mathrm{Addinf}(B)$), this kind of forces us to extend accordingly still the notion of an abelianizator for A. This does make sense, using the pairing (6) (which we had dismissed as an "irrelevant nicety for the

[p. 381] time being"!), and the corresponding equivalence

$$B^\wedge_{\mathrm{ab}} \xrightarrow{\approx} \mathbf{Hom}_!(A^\wedge_{\mathrm{ab}}, (\mathrm{Ab})), \tag{11}$$

where the index ! denotes the full subcategory of **Hom** made up with functors commuting to small direct limits. It is immediate that projective objects in $B^\wedge_{\mathrm{ab}}$ give rise to objects in $\mathbf{Hom}_!$ which are *exact* functors from $A^\wedge_{\mathrm{ab}}$ to (Ab), and I'll have to check that the converse also holds. If so, a chain complex in $B^\wedge_{\mathrm{ab}}$ with projective components can be interpreted as being just an *arbitrary exact functor commuting to small sums*

$$A^\wedge_{\mathrm{ab}} \to \mathrm{Ch}_\bullet(\mathrm{Ab}),$$

(never minding whether or not it can be described "computationally" in terms of objects in Add(B) or in Addinf(B)!) – which is all that is needed in order to complete the diagram (8), and wonder if it commutes up to isomorphism! And the most natural statement here is that this is indeed so whenever this functor, viewed as a chain complex in the abelian category

$$\mathbf{Hom}_!(A^\wedge_{\mathrm{ab}}, (\mathrm{Ab})), \tag{12}$$

is a (projective) resolution of the canonical object $\widetilde{\mathbb{Z}}_B$ of the category (12), coming from the object $\mathbb{Z}_B$ of the left-hand side of (11). Now, this functor is just the familiar "direct limit" functor

$$\widetilde{\mathbb{Z}}_B \simeq \varinjlim_B : A^\wedge_{\mathrm{ab}} \stackrel{\mathrm{def}}{=} \mathbf{Hom}(B, (\mathrm{Ab})) \to (\mathrm{Ab}), \tag{13}$$

which can be equally interpreted as

$$\widetilde{\mathbb{Z}}_B \simeq p^{\mathrm{ab}}_{A!} : A^\wedge_{\mathrm{ab}} \to (\mathrm{Ab}), \tag{14}$$

namely (abelian) "integration" with respect to the map in (Cat)

$$p_A : A \to \mathbb{A}_0.$$

Thus, ultimately, *abelianizators for A* (or what we may call "standard abelianizators", if there should turn out to be any others, and that they are worth looking at) *turn out to be no more, no less than just a projective resolution, in the category* (12) *of functors from $A^\wedge_{\mathrm{ab}}$ to* (Ab) *commuting with small direct limits, of the most interesting object in the category, namely the functor*

$$p^{\mathrm{ab}}_{A!} \simeq \varinjlim_B : A^\wedge_{\mathrm{ab}} \to (\mathrm{Ab}). \tag{15}$$

We are far indeed from the faltering reflections of section 91, about computing homology and cohomology of homotopy models described in terms of test categories deduced some way or other from cellular decompositions of spheres!

11.8. [p. 382]

103 It has been over three weeks now I haven't been working on the notes. Most part of this time was spent wandering in the Pyrenees with some friends (a kind of thing I hadn't been doing since I was a boy), and touring some other friends living the simple life around there, in the mountains. I was glad to meet them and happy to wander and breathe the fresher air of the mountains – and very happy too after two weeks to be back in the familiar surroundings of my home amidst the gentle hills covered with vineyards. . . Yesterday I resumed mathematical work – I had to spend the day doing scratchwork in order to get back into it, now I feel ready to go on with the notes. I'll have to finish in the long last with

Integrators versus cointegrators.

that abelianization story I got into unpremeditatedly – which turns out to be essentially the same thing as some systematics about (commutative) cohomology and homology, in the context of “models” in (Cat), or in a category $A^\wedge$ (with A is (Cat)). We were out for proving a statement about “integrators” for a small category B being “abelianizators” for the dual category $A = B^{\mathrm{op}}$. The proof I had in mind for this is somewhat indirect via cohomology, and follows the proof I gave myself a very long time ago (in case $A\mathbb{\Delta}$), that the usual semi-simplicial boundary operations do give the correct (topos-theoretic) cohomology invariants for any object X in $A^\wedge$ (i.e., any semisimplicial set), for any locally constant coefficients on $A_{/X}$. The idea was to replace $A_{/X}$ by the dual category $(A_{/X})^{\mathrm{op}} = {}_{X\backslash}A^{\mathrm{op}}$ (which, according to a nice result of Quillen, has a homotopy type canonically isomorphic to the one defined by $A_{/X}$), and use the canonical functor

$$f = (p_X)^{\mathrm{op}} : (A_{/X})^{\mathrm{op}} \to A^{\mathrm{op}} \stackrel{\mathrm{def}}{=} B,$$

which is a cofibration with discrete fibers, and hence gives rise, for any abelian presheaf F on the category C upstairs, to an isomorphism

$$\mathrm{R}\Gamma(C, F) \simeq \mathrm{R}\Gamma(B, f_*(F))$$

(due to $\mathrm{R}f_*(F) \xleftarrow{\sim} f_*(F)$, as f_* is exact, due to the fact that f is a cofibration with discrete fibers). We'll get Quillen's result about the isomorphism $C \simeq C^{\mathrm{op}}$ in (Hot), for any object C in (Cat), very smoothly in part VI, as a result of the asphericity story of part IV. However, I now realize that the proof of the fact about abelianizators via Quillen's result and cohomology is rather awkward, as what we're after now is typically a result on homology, not cohomology – and I was really turning it
[p. 383] upside down in order to fit it at all costs into the more familiar (to me) cohomology pot! Therefore, I'm not going to write out this proof, as “the” natural proof is going to come out by itself, once we got a good conceptual understanding of homology, cohomology and abelianization, in the context of “spaces” embodies by objects of (Cat). Thus, I feel what is mainly needed now is an overall review of the relevant notions and facts along these lines – most of which we've come in touch with before, be it only “en passant”.

Before starting, just an afterthought on terminology. It occurred to me that the name of an “integrator” (for A), for a projective resolution of the constant abelian presheaf $\mathbb{Z}_A$ in $A^\wedge_{\mathrm{ab}}$, is inaccurate – as it was meant to suggest that its main use is for allowing computation, for an arbitrary abelian presheaf (or complex of such presheaves) F on A, of $\mathrm{R}\Gamma(A, F)$, which we were thinking of by that time as the “integration” of F over A (or over the associated topos). But it has turned out that for the sake of coherence with a broader use of the notions of “integration” and “cointegration” (compare section 69), the appropriate designation of $\mathrm{R}\Gamma(A, F)$ is “*co*integration” of F over A, not integration. Therefore, the appropriate designation for a projective resolution $L^A_\bullet$ of $\mathbb{Z}_A$, allowing computational expression of cointegration, is “cointegrator” (for A) rather than “integrator”. On the other hand, in terms of the dual category

$B = A^{\mathrm{op}}$, it turns out that such $L^A_\bullet$ allows computational expression of *integration* (i.e., homology) over B, and therefore it seems adequate to call $L^A_\bullet$ also an *"integrator" for B*. Moreover, it turns out that such an integrator for B is equally an "abelianizator" for B, i.e., it allows simultaneous computation of the "abelianizations" of the homotopy types defined by arbitrary objects X in $A^\wedge$, in terms of the abelianization $\mathrm{Wh}_A(X) = \mathbb{Z}^{(X)}$ of X (cf. sections 93 and 102) – and possibly the converse holds too. Whether this is so or not, there doesn't seem at present much sense to bother about abelianizators which do not come from integrators, while the latter have the invaluable advantage (besides mere existence) of being unique up to homotopism. Thus, in practical terms, it would seem that abelianizators (for a given small category B) are no more no less than just integrators (for the same B, i.e., cointegrators for $A = B^{\mathrm{op}}$) – and I would therefore suggest to simply drop the designation "abelianizator" for the benefit of the synonym "integrator", which fits more suggestively into the pair of dual notions integrator—cointegrator. [p. 384]

104 I'll have after all to give a certain amount of functorial "general nonsense" which I've tried to bypass so far.

Overall review on abelianization (1)*: Case of pseudo-topoi.*

A) Pseudo-topoi and adjunction equivalences. In what follows, ordinary capital letters as $A, B, \ldots$ will generally denote small categories (mostly objects in (Cat)), whereas round capital letters $\mathcal{A}, \mathcal{B}, \mathcal{M}$ will denote $\mathcal{U}$-categories which may be "large", for instance $\mathcal{A} = A^\wedge$, $\mathcal{B} = B^\wedge$, etc. For two such categories $\mathcal{A}, \mathcal{B}$, we denote by

$$\mathbf{Hom}_!(\mathcal{A}, \mathcal{B}), \quad \mathbf{Hom}^!(\mathcal{A}, \mathcal{B}) \tag{1}$$

the full subcategories of the functor category $\mathbf{Hom}(\mathcal{A}, \mathcal{B})$, made up with all functors which commute with small direct or inverse limits respectively. This notation is useful mainly in case $\mathcal{A}$ and $\mathcal{B}$ are stable under small direct resp. inverse limits, in which case the same holds true for the corresponding category (1), because as a full subcategory of $\mathbf{Hom}(\mathcal{A}, \mathcal{B})$ (where direct resp. inverse limits exist and are computed componentwise) it is stable under direct resp. inverse limits. Thus, the inclusion functors

$$\mathbf{Hom}_!(\mathcal{A}, \mathcal{B}) \to \mathbf{Hom}(\mathcal{A}, \mathcal{B}), \quad \mathbf{Hom}^!(\mathcal{A}, \mathcal{B}) \to \mathbf{Hom}(\mathcal{A}, \mathcal{B}) \tag{2}$$

commute with direct resp. inverse limits, i.e., those limits in the categories (1) are computed equally componentwise.

The canonical inclusion

$$\mathcal{A} \hookrightarrow \mathbf{Hom}(\mathcal{A}^{\mathrm{op}}, (\mathrm{Sets}))$$

factors into a fully faithful inclusion functor

$$\mathcal{A} \hookrightarrow \mathbf{Hom}^!(\mathcal{A}^{\mathrm{op}}, (\mathrm{Sets})). \tag{3}$$

Let's recall the non-trivial useful result:

Proposition 1. *Assume the $\mathcal{U}$-category $\mathcal{A}$ is stable under small direct limits, and admits a small full subcategory A which is "generating for monomorphisms", i.e., any monomorphism $i : X \to Y$ in $\mathcal{A}$ such that* $\operatorname{Hom}(Z,i) : \operatorname{Hom}(Z,X) \to \operatorname{Hom}(Z,Y)$ *is bijective for any Z in C, is an isomorphism. Then the fully faithful functor (3) is an equivalence, i.e., any functor*

$$\mathcal{A}^{\mathrm{op}} \to (\mathrm{Sets})$$

that commutes with small inverse limits is representable.

For a proof, see SGA 4 I 8.12.7.

[Artin, Grothendieck, and Verdier (SGA 4.1)]

Corollary 1. *If $\mathcal{A}$ satisfies the assumptions above, then $\mathcal{A}$ is equally stable under small* inverse *limits.*

[p. 385]

For the sake of brevity, we'll say that a $\mathcal{U}$-category satisfying the assumptions of prop. 1 is a *pseudo-topos* (as these conditions are satisfied for any topos). We get at once the

Corollary 2. *Let $\mathcal{A}, \mathcal{B}$ be two pseudo-topoi. Then a functor from $\mathcal{A}$ to $\mathcal{B}^{\mathrm{op}}$ (resp. from $\mathcal{B}^{\mathrm{op}}$ to $\mathcal{A}$) has a right adjoint (resp. a left adjoint) iff it commutes to small direct limits (resp. to small inverse limits). Thus, taking right and left adjoints we get two equivalences of categories, quasi-inverse to each other*

$$\mathbf{Hom}_!(\mathcal{A}, \mathcal{B}^{\mathrm{op}}) \leftrightarrows \mathbf{Hom}^!(\mathcal{B}^{\mathrm{op}}, \mathcal{A}), \tag{4}$$

and the two members of (4) are canonically equivalent to the category

$$\mathbf{Hom}^{!!}(\mathcal{A}^{\mathrm{op}}, \mathcal{B}^{\mathrm{op}}; (\mathrm{Sets})) \tag{5}$$

of functors

$$\mathcal{A}^{\mathrm{op}} \times \mathcal{B}^{\mathrm{op}} \to (\mathrm{Sets})$$

which commute with small inverse limits with respect to either variable (the other being fixed), (5) being viewed as a full subcategory of $\mathbf{Hom}(\mathcal{A}^{\mathrm{op}} \times \mathcal{B}^{\mathrm{op}}, (\mathrm{Sets}))$.

Remarks. 1) The first-hand side of (4) is tautologically isomorphic to the category $\mathbf{Hom}^!(\mathcal{A}^{\mathrm{op}}, \mathcal{B})$ (as for any two $\mathcal{U}$-categories we have the tautological isomorphism

$$(\mathbf{Hom}_!(\mathcal{P}, \mathcal{Q}))^{\mathrm{op}} \simeq \mathbf{Hom}^!(\mathcal{P}^{\mathrm{op}}, \mathcal{Q}^{\mathrm{op}}) \quad), \tag{6}$$

thus, the equivalence (4) can be seen more symmetrically as an equivalence

$$\mathbf{Hom}^!(\mathcal{A}^{\mathrm{op}}, \mathcal{B}) \simeq \mathbf{Hom}^!(\mathcal{B}^{\mathrm{op}}, scrA), \tag{4'}$$

both categories being equivalent to (5) using the equivalence (3) for the second, and the corresponding equivalence for $\mathcal{B}$ for the first, plus the tautological isomorphism

$$\mathbf{Hom}^!(\mathcal{P}, \mathbf{Hom}^!(\mathcal{Q}, \mathcal{M})) \simeq \mathbf{Hom}^{!!}(\mathcal{P}, \mathcal{Q}; \mathcal{M}), \tag{7}$$

for any three $\mathcal{U}$-categories $\mathcal{P}, \mathcal{Q}, \mathcal{M}$.

2) When $\mathcal{A}$ is a topos, $\mathcal{B}$ any $\mathcal{U}$-category stable under small inverse limits, then we may interpret the category $\mathbf{Hom}^!(\mathcal{A}^{\mathrm{op}}, \mathcal{B})$ as the category of $\mathcal{B}$-*valued sheaves* on the topos (defined by) $\mathcal{A}$. When $\mathcal{A}$ and $\mathcal{B}$ are both topoi, then the equivalence (4) states that $\mathcal{B}$-valued sheaves on $\mathcal{A}$ can be identified with $\mathcal{A}$-valued sheaves on $\mathcal{B}$, and both may be identified with set-valued "bi-sheaves" on $\mathcal{A} \times \mathcal{B}$. In case the topoi $\mathcal{A}$, $\mathcal{B}$ are defined respectively by $\mathcal{U}$-sites A, B (not necessarily small ones), these bisheaves can be interpreted in a rather evident way as bisheaves on $A \times B$, namely functors [p. 386]

$$A^{\mathrm{op}} \times B^{\mathrm{op}} \to (\mathrm{Sets})$$

which are sheaves with respect to each variable (the other being fixed). It is easy to check that the category of all such bisheaves is again a topos, and that the latter is a 2-product of the two topoi $\mathcal{A}$, $\mathcal{B}$ in the 2-category of all topoi – it plays exactly the same geometrical role as the usual product for two topological spaces...

B) Abelianization of a pseudo-topos. Let $\mathcal{A}$ be a pseudo-topos, and let's denote by

$$\mathcal{A}_{\mathrm{ab}} \tag{8}$$

the category of abelian group-objects in $\mathcal{A}$. It is immediate that the forgetful functor

$$\mathcal{A}_{\mathrm{ab}} \to \mathcal{A} \tag{9}$$

commutes with small direct limits (and that such limits exist in $\mathcal{A}_{\mathrm{ab}}$, whereas they exist in $\mathcal{A}$ by cor. 1 above) – thus, we may expect that this functor admits a left adjoint. When so, this will be denoted by

$$\mathrm{Wh}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_{\mathrm{ab}}, \tag{10}$$

we'll write also

$$\mathrm{Wh}(X) = \mathbb{Z}^{(X)}$$

when no confusion may arise. The abelianization functor exists for instance when $\mathcal{A}$ is a topos, in this case it is well-known that $\mathcal{A}_{\mathrm{ab}}$ is not only an additive category, but an *abelian* category with small filtering direct limits which are exact, and a small generating subcategory. This in turn ensures, as well-known too, that any object of $\mathcal{A}_{\mathrm{ab}}$ can be embedded into an injective one, and from this follows (cf. SGA 4 I 7.12) that $\mathcal{A}_{\mathrm{ab}}$ admits also a small full subcategory which is *co*generating with respect to epimorphisms, in other words that $\mathcal{A}_{\mathrm{ab}}$ is not only an abelian pseudo-topos, but that the dual category $(\mathcal{A}_{\mathrm{ab}})^{\mathrm{op}}$ is a pseudo-topos too. Conversely (kind of), without assuming $\mathcal{A}$ to be a topos, if we know some way or other (but this may be hard to check directly...) that $(\mathcal{A}_{\mathrm{ab}})^{\mathrm{op}}$ is a pseudo-topos, then it follows from cor. 2 above that the abelianization functor $\mathrm{Wh}_{\mathcal{A}}$ exists, and this in turn implies that $\mathcal{A}_{\mathrm{ab}}$ is a pseudo-topos, i.e., admits a small full subcategory which is generating with respect to [Artin, Grothendieck, and Verdier (SGA 4.1)] [p. 387] monomorphisms (as we see by taking such a full subcategory A in $\mathcal{A}$, and the full subcategory in $\mathcal{A}_{\mathrm{ab}}$ generated by $\mathrm{Wh}_{\mathcal{A}}(A)$).

Let now $\mathcal{M}$ be an *additive* $\mathcal{U}$-category, which is moreover a *pseudo-cotopos*, i.e., the dual category $\mathcal{M}^{\mathrm{op}}$ is a pseudo-topos. Using twice the corollary 2 above, for the pair of pseudotopoi $(\mathcal{A}, \mathcal{M}^{\mathrm{op}})$ and $(\mathcal{A}_{\mathrm{ab}}, \mathcal{M}^{\mathrm{op}})$, we get the sequence of equivalences of categories

$$\mathbf{Hom}_!(\mathcal{A},\mathcal{M}) \approx \mathbf{Hom}^!(\mathcal{M},\mathcal{A})^{\mathrm{op}} \xleftarrow{\approx} \mathbf{Hom}^!(\mathcal{M},\mathcal{A}_{\mathrm{ab}})^{\mathrm{op}} \approx \mathbf{Hom}_!(\mathcal{A}_{\mathrm{ab}},\mathcal{M}),$$

where the second equivalence of categories comes from the fact that any functor

$$f : \mathcal{M} \to \mathcal{A}$$

from an *additive* category $\mathcal{M}$ to a category $\mathcal{A}$, which commutes with finite products, factors canonically through $\mathcal{A}_{\mathrm{ab}} \to \mathcal{A}$. We are interested now in the composite equivalence

$$\mathbf{Hom}_!(\mathcal{A},\mathcal{M}) \approx \mathbf{Hom}_!(\mathcal{A}_{\mathrm{ab}},\mathcal{M}), \tag{11}$$

defined under the only assumption that $\mathcal{M}$ is additive and the categories $\mathcal{A}$, $\mathcal{A}_{\mathrm{ab}}$ and $\mathcal{M}^{\mathrm{op}}$ are pseudotopoi (without having to assume the existence of the abelianization functor $\mathrm{Wh}_{\mathcal{A}}$). This equivalence is functorial for variable additive pseudo-cotopos $\mathcal{M}$, when we take as "maps" $\mathcal{M} \to \mathcal{M}'$ functors which commute to small direct limits (a fortiori, these are right exact and hence additive). In case $\mathcal{A}_{\mathrm{ab}}$ itself is among the eligible $\mathcal{M}$'s, i.e., is a pseudo-cotopos (not only pseudotopos), we may say that $\mathcal{A}_{\mathrm{ab}}$ *2-represents the 2-functor* $\mathcal{M} \mapsto \mathbf{Hom}_!(\mathcal{A},\mathcal{M})$ on the 2-category of all additive pseudo-cotopoi and functors between these commuting to small direct limits. As we noticed above, the assumption just made implies that $\mathrm{Wh}_{\mathcal{A}}$ exists. On the other hand, assuming merely existence of $\mathrm{Wh}_{\mathcal{A}}$ (besides $\mathcal{A}$ being a pseudo-topos), which implies that $\mathcal{A}_{\mathrm{ab}}$ is equally a pseudotopos as we say above, it is readily checked that the equivalence (11) can be described as

$$F \mapsto F \circ \mathrm{Wh}_{\mathcal{A}} : \mathbf{Hom}_!(\mathcal{A}_{\mathrm{ab}},\mathcal{M}) \xrightarrow{\approx} \mathbf{Hom}_!(\mathcal{A},\mathcal{M}). \tag{12}$$

Thus, we get the

Proposition 2. *Let $\mathcal{A}$ be a pseudotopos such that the abelianization functor* (10) *exists* (*for instance $\mathcal{A}$ a topos*)*. Then $\mathcal{A}_{\mathrm{ab}}$ is a pseudotopos and an additive category. Moreover, for any additive category $\mathcal{M}$ which is a pseudo-cotopos, the functor* (12) *is an equivalence of categories.*

[p. 388] **Corollary 1.** *Let $\mathcal{A}$ be a pseudotopos such that $\mathcal{A}_{\mathrm{ab}}$ is a pseudotopos* (*for instance, $\mathcal{A}$ is a topos*)*. Then the abelianization functor $\mathrm{Wh}_{\mathcal{A}}$ exists, and this functor is 2-universal for functors from $\mathcal{A}$ into $\mathcal{U}$-categories which are both additive and are pseudotopoi* (*maps between these being functors which commute with small direct limits*)*.*

By duality, using (6), we can restate the equivalence (12) as

$$F \mapsto F \circ \mathrm{Wh}^{\mathrm{op}} : \mathbf{Hom}^!({\mathcal{A}_{\mathrm{ab}}}^{\mathrm{op}},\mathcal{N}) \xrightarrow{\approx} \mathbf{Hom}^!(\mathcal{A}^{\mathrm{op}},\mathcal{N}), \tag{13}$$

valid provided $\mathcal{A}$ is a pseudo-topos, $\mathrm{Wh}_{\mathcal{A}}$ exists, and $\mathcal{N}$ is an additive category which is moreover a pseudotopos.

Take for instance $\mathcal{N} = (\mathrm{Ab})$, the category of abelian groups, i.e.,

$$\mathcal{N} = \mathcal{B}_{\mathrm{ab}}, \quad \text{where } \mathcal{B} = (\mathrm{Sets}),$$

then as already noticed above the left hand side of (13) is canonically equivalent with $\mathbf{Hom}^!(\mathcal{A}_{\mathrm{ab}}{}^{\mathrm{op}}, \mathcal{B}) = \mathbf{Hom}^!(\mathcal{A}_{\mathrm{ab}}{}^{\mathrm{op}}, (\mathrm{Sets}))$, as $\mathcal{A}_{\mathrm{ab}}{}^{\mathrm{op}}$ is additive; on the other hand, by prop. 1 applied to the pseudotopos $\mathcal{A}_{\mathrm{ab}}$ we get

$$\mathcal{A}_{\mathrm{ab}} \xrightarrow{\approx} \mathbf{Hom}^!(\mathcal{A}_{\mathrm{ab}}{}^{\mathrm{op}}, (\mathrm{Sets})) \quad \big(\xleftarrow{\approx} \mathbf{Hom}^!(\mathcal{A}_{\mathrm{ab}}{}^{\mathrm{op}}, (\mathrm{Ab}))\big),$$

and hence an equivalence

$$\mathcal{A}_{\mathrm{ab}} \xrightarrow{\approx} \mathbf{Hom}^!(\mathcal{A}^{\mathrm{op}}, (\mathrm{Ab})), \quad F \mapsto \big(X \mapsto \mathrm{Hom}(X, F)\big), \tag{14}$$

valid whenever $\mathcal{A}$ is a pseudotopos such that $\mathrm{Wh}_{\mathcal{A}}$ exists. When $\mathcal{A}$ is a topos, this corresponds to the familiar fact that an abelian group object in the category of sheaves (of sets) on a topos, can be interpreted equally as a sheaf on the topos with values in the category (Ab) of abelian groups.

C) Interior and exterior operations $\otimes_{\mathbb{Z}}$ and $\mathrm{Hom}_{\mathbb{Z}}$.

In the first place, I want to emphasize the basic tensor product operation

$$(F, G) \mapsto F \otimes_{\mathbb{Z}} G : \mathcal{A}_{\mathrm{ab}} \times \mathcal{A}_{\mathrm{ab}} \to \mathcal{A}_{\mathrm{ab}} \tag{15}$$

between abelian group objects of the pseudotopos $\mathcal{A}$, defined as usual argumentwise as the solution of the universal problem, expressed by the "Cartan isomorphism"

$$\mathrm{Hom}_{\mathcal{A}_{\mathrm{ab}}}(F \otimes G, H) \simeq \mathrm{Bil}_{\mathbb{Z}}(F, G; H), \tag{16}$$

where we dropped the subscript $\mathbb{Z}$ in the tensor product, and where $\mathrm{Bil}_{\mathbb{Z}}$ or simply Bil denotes the set of maps $F \times G \to H$ which are "biadditive" in the usual sense of the word. Thus, the existence of (15) just means, [p. 389]
by definition, that for any pair (F, G) of objects in $\mathcal{A}_{\mathrm{ab}}$, the functor in H

$$H \mapsto \mathrm{Bil}(F, G; H)$$

is representable. It is clear that this functor commutes with small inverse limits, hence by prop. 1 it is representable, provided we know that $\mathcal{A}_{\mathrm{ab}}$ is a pseudo-cotopos (for instance, when $\mathcal{A}$ is a topos, in which case the existence of tensor products is anyhow a familiar fact). The familiar Bourbaki construction of a tensor product amounts on the other hand to viewing $F \otimes G$ as a quotient of $\mathrm{Wh}_{\mathcal{A}}(F \times G) = \mathbb{Z}^{(F \times G)}$ by suitable "relations", i.e., as the cokernel of a map in $\mathcal{A}_{\mathrm{ab}}$

$$L_1 \to L_0 = \mathrm{Wh}_{\mathcal{A}}(F \times G),$$

where, as a matter fact, L_1 can be described as

$$L_1 = \mathrm{Wh}_{\mathcal{A}}(F \times F \times G) \times \mathrm{Wh}_{\mathcal{A}}(F \times G \times G).$$

Thus, if we know beforehand that cokernels exist in $\mathcal{A}_{\text{ab}}$ (which would indeed follow from $\mathcal{A}_{\text{ab}}$ being a pseudotopos, but may be checked more readily in terms of suitable exactness properties of $\mathcal{A}$ directly), plus the existence of course of $\text{Wh}_{\mathcal{A}}$, the tensor product functor (15) exists. (On the other hand, no use is made here of the assumption that $\mathcal{A}$ be a pseudotopos.)

Let's assume the tensor product functor (15) exists. Then it is readily checked it is associative and commutative up to canonical isomorphisms, giving rise to the usual compatibilities. If moreover $\text{Wh}_{\mathcal{A}}$ exists, we readily get the canonical isomorphism

$$(17)\quad \text{Wh}_{\mathcal{A}}(X\times Y) \overset{\text{def}}{=} \mathbb{Z}^{(X\times Y)} \xleftarrow{\sim} \text{Wh}_{\mathcal{A}}(X)\otimes \text{Wh}_{\mathcal{A}}(Y) \quad \left(\overset{\text{def}}{=} \mathbb{Z}^{(X)}\otimes\mathbb{Z}^{(Y)}\right),$$

compatible of course with the commutativity and associativity isomorphisms for the operations $\times$ and $\otimes$. If on the other hand $\mathcal{A}$ admits moreover a final object (as it does if $\mathcal{A}$ is a pseudotopos and hence stable under small direct limits), then

$$(18)\qquad \mathbb{Z}_{\mathcal{A}} \overset{\text{def}}{=} \text{Wh}_{\mathcal{A}}(e) = \mathbb{Z}^{(e)}$$

is a two-sided unit for the tensor product operation.

In what follows, we are interested in categories of the type

$$(19)\qquad \mathcal{A}^{\mathcal{M}} \overset{\text{def}}{=} \mathbf{Hom}_!(\mathcal{A},\mathcal{M}), \quad \mathcal{A}_{\mathcal{N}} \overset{\text{def}}{=} \mathbf{Hom}^!(\mathcal{A}^{\text{op}},\mathcal{N})$$

[p. 390] where now $\mathcal{A}$ is assumed to be a fixed pseudotopos, and $\mathcal{M}$ and $\mathcal{N}$ are *additive* categories, $\mathcal{M}$ being moreover a pseudo-cotopos, $\mathcal{N}$ a pseudotopos. We assume moreover that Wh_{A} exists, and hence the categories (19) can be interpreted up to equivalence, via (12) and (13), as

$$(19')\qquad \mathbf{Hom}_!(\mathcal{A}_{\text{ab}},\mathcal{M}), \quad \mathbf{Hom}^!({\mathcal{A}_{\text{ab}}}^{\text{op}},\mathcal{N}).$$

Let's remark that the dual of a category of one of the types (19) (or equivalently, (19')) is isomorphic to a category of the other type, more accurately, by (6) we get

$$(20)\qquad \mathbf{Hom}_!(\mathcal{A},\mathcal{M})^{\text{op}} \xrightarrow{\sim} \mathbf{Hom}^!(\mathcal{A}^{\text{op}},\mathcal{N}),$$

i.e., $(\mathcal{A}^{\mathcal{M}})^{\text{op}} \xrightarrow{\sim} \mathcal{A}_{\mathcal{N}}$, with $\mathcal{N} = \mathcal{M}^{\text{op}}$. In case $\mathcal{A}$ is a topos, the objects of the second category $\mathcal{A}_{\mathcal{N}}$ in (19) (or equivalently, in (19')) can be interpreted as $\mathcal{N}$-*valued sheaves* on the topos $\mathcal{A}$, whereas the object of the first, $\mathcal{A}^{\mathcal{M}}$, may be called, correspondingly, *cosheaves on $\mathcal{A}$ with values in* $\mathcal{M}$. Thus, in virtue of (20), $\mathcal{M}$-valued cosheaves on $\mathcal{A}$ can be interpreted as $scrN = \mathcal{M}^{\text{op}}$-valued sheaves on $\mathcal{A}$, the corresponding categories of cosheaves and sheaves being however *dual* to each other. In the next subsection D), when $\mathcal{A} = A^{\wedge}$, we'll interpret moreover $\mathcal{M}$-valued cosheaves on $\mathcal{A}$ (or on A, as we'll call them equivalently) as $\mathcal{M}$-valued *sheaves* on the (topos associated to the) dual category $B = A^{\text{op}}$, and in this context the difference between the categories of cosheaves and of sheaves (which for the time being appear as categories *dual* to each other) will disappear altogether, provided we allow the ground topos $\mathcal{A}$ to change (from $\mathcal{A} = A^\wedge$ to the "dual" topos $\mathcal{B} = B^\wedge$).

In terms of the expressions (19') of the category of "cosheaves" and "sheaves" we are interested in, we want now to define an external operation of the fixed category $\mathcal{A}_{\mathrm{ab}}$ on those categories, using the tensor product operation (15) in $\mathcal{A}_{\mathrm{ab}}$. What is needed visibly for this end is that for fixed F, the functor

$$(21) \qquad G \mapsto F \otimes G$$

from $\mathcal{A}_{\mathrm{ab}}$ to itself should commute with small direct limits – hence composing it with a functor L in $\mathbf{Hom}_!(\mathcal{A}_{\mathrm{ab}}, \mathcal{M})$ will yield a functor in the same category, which will be the looked-for external tensor product $F \circledast L$, i.e.,

$$(22) \qquad F \circledast L(G) = L(F \otimes G),$$

which we may write more suggestively as

$$(22') \qquad G * (F \circledast L) = (G \otimes F) * L,$$

with the notation

$$(23) \qquad H * L \overset{\mathrm{def}}{=} L(H), \quad \text{for } H \text{ in } \mathcal{A}_{\mathrm{ab}},\ L \text{ in } \mathcal{A}^{\mathcal{M}} = \mathbf{Hom}_!(\mathcal{A}, \mathcal{M}),$$

which will be convenient mainly in the context of the next subsection (when $\mathcal{A}$ is of the type $A^\wedge$). [p. 391]

The exactness property needed for the functor (21) is equivalent with the property that for any object H in $\mathcal{A}_{\mathrm{ab}}$, the functor

$$G \mapsto \mathrm{Hom}(F \otimes G, H) \simeq \mathrm{Bil}(F, G; H)$$

from ${\mathcal{A}_{\mathrm{ab}}}^{\mathrm{op}}$ to (Sets) commute with small inverse limits. As $\mathcal{A}_{\mathrm{ab}}$ is a pseudotopos (prop. 2), this is equivalent by prop. 1 with this functor being representable. By definition of Bil, we get

$$\mathrm{Bil}(F, G; H) \simeq \mathrm{Hom}_{\mathcal{A}_{\mathrm{ab}}^\wedge}(G, \mathbf{Hom}_{\mathbb{Z}}(F, H)),$$

where the object

$$(24) \qquad \mathbf{Hom}_{\mathbb{Z}}(F, H)$$

is taken in the category of presheaves $\mathcal{A}_{\mathrm{ab}}^\wedge$ (cheating a little with universes here...). To sum up, the condition we want for (21) just boils down to the representability of the abelian group objects (24) in $\mathcal{A}_{\mathrm{ab}}^\wedge$, for any two objects F, G in $\mathcal{A}_{\mathrm{ab}}$, i.e., essentially to the existence of "internal **Hom**'s" in the category $\mathcal{A}_{\mathrm{ab}}$ (endowed with the tensor product $\otimes_{\mathbb{Z}}$), satisfying the familiar Cartan isomorphism formula

$$(25) \qquad \mathrm{Hom}(F \otimes G, H) \simeq \mathrm{Hom}(G, \mathbf{Hom}(F, H)).$$

To sum up, what is needed for a nice formalism of interior and exterior tensor products and Hom's for "sheaves" and "presheaves" on the pseudotopos $\mathcal{A}$, are the following assumptions on $\mathcal{A}$:

1) Tensor products and corresponding internal **Hom**'s exist in $\mathcal{A}_{\mathrm{ab}}$, and

2) the abelianization functor $\mathrm{Wh}_{\mathcal{A}}$ (15) exists,

the latter assumption being needed in order to feel at ease with the equivalence between the categories (19) of sheaves and cosheaves, and their "abelianized" interpretations (19').

Under these assumptions, we define the exterior tensor product operation of $\mathcal{A}_{\mathrm{ab}}$ upon a category of cosheaves $\mathcal{A}^{\mathcal{M}}$

$$\mathcal{A}_{\mathrm{ab}} \times \mathcal{A}^{\mathcal{M}} \to \mathcal{A}^{\mathcal{M}}, \quad (F, L) \mapsto F \circledast L, \tag{26}$$

by formula (22) (which may be written also under the form (22')). This operation has the obvious associativity property

$$F \circledast (G \circledast L) \simeq (F \otimes G) \circledast L, \tag{27}$$

[p. 392] and moreover the unit $\mathbb{Z}_{\mathcal{A}}$ for the internal tensor product in $\mathcal{A}_{\mathrm{ab}}$ operators on $\mathcal{A}^{\mathcal{M}}$ as the identity functors, i.e.,

$$\mathbb{Z}_{\mathcal{A}} \circledast L \simeq L. \tag{28}$$

Using the tautological duality relation (20) between categories of cosheaves (with values in $\mathcal{M}$) and categories of sheaves (with values in $\mathcal{N} = \mathcal{M}^{\mathrm{op}}$), we deduce accordingly an associative and unitary operation of $\mathcal{A}_{\mathrm{ab}}$ on any category of the type $\mathcal{A}_{\mathcal{N}}$, namely $\mathcal{N}$-valued sheaves on $\mathcal{A}$. This operation is most conveniently denoted by the **Hom** symbol

$$(F, K) \mapsto \mathbf{Hom}(F, K) : {\mathcal{A}_{\mathrm{ab}}}^{\mathrm{op}} \times \mathcal{A}_{\mathcal{N}} \to \mathcal{A}_{\mathcal{N}}, \tag{29}$$

its explicit description in terms of K, viewed as a functor

$$K : {\mathcal{A}_{\mathrm{ab}}}^{\mathrm{op}} \to \mathcal{N}$$

is by

$$\mathbf{Hom}(F, L)(G) = K(G \otimes F) \tag{30}$$

for K in $\mathcal{A}_{\mathcal{N}}$, F and G in $\mathcal{A}_{\mathrm{ab}}$. This may be written more suggestively as

$$\mathrm{Hom}(G, \mathbf{Hom}(F, L)) \simeq \mathrm{Hom}(G \otimes F, K), \tag{30'}$$

similar to (22'), where we use the notation Hom (non-bold!) in analogy to (23) for denoting $K(H)$, namely

[In the typescript this parenthetical remark says "non-underlined!", as internal **Hom**'s there are underlined rather than in boldface.]

$$\mathrm{Hom}(H, K) \stackrel{\mathrm{def}}{=} K(H). \tag{31}$$

If confusion is feared, we may put a subscript $\mathbb{Z}$ in all Hom's and **Hom**'s just introduced, as well as in the internal and external tensor product operations $\otimes$, $*$, $\circledast$.

Comments. Take for instance

$$\mathcal{N} = (\mathrm{Ab}), \quad \text{hence } \mathcal{A}_{\mathcal{N}} \simeq \mathcal{A}_{\mathrm{ab}}$$

by prop. 1 (compare (14)). By this equivalence, the Hom in (31) is just the usual Hom set corresponding to the category structure of $\mathcal{A}_{\mathrm{ab}}$ (the Hom endowed moreover with the structure of abelian group, coming from the fact that $\mathcal{A}_{\mathrm{ab}}$ is an additive category), whereas formula (30') shows that the "external" **Hom** in this case is nothing but the usual internal Hom as contemplated in (24). This I hope will convince the reader of the adequacy of the notation used (in case of exterior operation of abelian sheaves on $\mathcal{N}$-valued sheaves), and of the convention (31). We would like to give a similar justification for the notations (22) (23) used in connection with operation of abelian sheaves on $\mathcal{M}$-valued emphcosheaves, by interpreting this as the internal tensor product operation in $\mathcal{A}_{\mathrm{ab}}$, for suitable choice of $\mathcal{N}$. This I am afraid cannot be done for an arbitrary $\mathcal{A}$ satisfying our assumptions, even when $\mathcal{A}$ is a topos and even when it is of the special type $A\hat{}$, as I do not know of any $\mathcal{M}$ such that [p. 393]

$$(*) \qquad \mathcal{A}^{\mathcal{M}} \simeq \mathcal{A}_{\mathrm{ab}}.$$

However, in case $\mathcal{A} = A\hat{}$, introducing the dual category $B = A^{\mathrm{op}}$ and $\mathcal{B} = B\hat{}$, we get (see D) below)

$$(**) \qquad \mathcal{A}^{\mathcal{M}} \approxeq \mathcal{B}_{\mathcal{M}} \approxeq \mathcal{B}_{\mathrm{ab}} \quad \text{if} \quad \mathcal{M} = (\mathrm{Ab}),$$

hence we do get a canonical isomorphism (*) provided $A = B$ say and hence $\mathcal{A} = \mathcal{B}$. Let's look at any rate at the simplest case, namely when A is a final object in (Cat), hence $\mathcal{A}$ can be identified with (Sets), and $\mathcal{A}_{\mathrm{ab}}$ with (Ab), the identification between the categories (Ab) and $\mathbf{Hom}_!((\mathrm{Ab}),(\mathrm{Ab}))$ being obtained (we hope!) by associating to every object L in (Ab), the functor

$$F \mapsto L \otimes F : (\mathrm{Ab}) \to (\mathrm{Ab}).$$

This being so, the external operation (22) of (Ab) on $\mathbf{Hom}_! \approxeq (\mathrm{Ab})$ can be interpreted (using this identification) as the interior tensor product operation in (Ab). On the other hand, the operation $*$ of (23) is equally interpreted as nothing but the tensor product in (Ab), which justifies the notation suggesting a tensor product. It would be nice checking corresponding compatibilities for a general object A in (Cat) satisfying $A = A^{\mathrm{op}}$, namely a direct sum of one-object categories A_i defined each in terms of a *commutative* monoid M_i – I didn't work it out myself, sorry!

It should be noted that the relationship between the two exterior Hom's in (29) and (31) is essentially the same as between the two exterior tensor-type operations (26) and (23), $\mathrm{Hom}(F,K)$ designating an object in $\mathcal{N}$ and $\mathbf{Hom}(F,K)$ an $\mathcal{N}$-valued sheaf on $\mathcal{A}$, just as $F * L$ designates an object in $\mathcal{M}$ and $F \circledast L$ an $\mathcal{M}$-valued cosheaf on $\mathcal{A}$; the graphical device of *bold-facing* the symbol Hom (used for sheaves) corresponds to the device of *circling* the symbol $*$ (used for cosheaves). [in the typescript: *underlining*] With this luxury of explanations, I hope the notations introduced here are getting through...

[p. 394] 12.8. and 13.8.

105 Let's go on with the overall review of abelianization.

Review (2): duality equivalences for "algebraic" topoi and abelian topoi.

D) Duality for topoi of the type $A^\wedge$, and tentative generalizations. The main fact, it seems, which will give rise to duality statements for topoi of the type $A^\wedge$ is the following, rather familiar one:

Proposition 3. *Let A be a small category, $\mathcal{M}$ a $\mathcal{U}$-category stable under small direct limits,*

$$\varepsilon_A : A \hookrightarrow A^\wedge \tag{32}$$

the canonical inclusion functor. Then the following functor is an equivalence of categories:

$$F \mapsto F \circ \varepsilon_A : \mathbf{Hom}_!(A^\wedge, \mathcal{M}) \to \mathbf{Hom}(A, \mathcal{M}). \tag{33}$$

As I am at a loss to give a reference for this standard fact of category theory, I'll give in guise of a proof the indication that a quasi-inverse functor for (33) is given by the familiar construction

$$i \mapsto i_! : \mathbf{Hom}(A, \mathcal{M}) \to \mathbf{Hom}_!(A^\wedge, \mathcal{M}), \tag{34}$$

where for any functor

$$i : A \to \mathcal{M},$$

the functor

$$i_! : A^\wedge \to \mathcal{M}$$

is defined by the formula

$$i_!(F) = \varinjlim_{a \text{ in } A_{/F}} i(a). \tag{35}$$

It is readily checked (and we have already used a number of times) that this functor admits a right adjoint

$$i^* : \mathcal{M} \to A^\wedge, \quad i^*(x) = (a \mapsto \mathrm{Hom}(i(a), x)), \tag{36}$$

and hence $i_!$ commutes to small direct limits, i.e., is in $\mathbf{Hom}_!(A^\wedge, \mathcal{M})$, hence (34).

Remark. If $\mathcal{M}$ is a pseudotopos, then by prop. 3 above and by corollary 2 of prop. 1 (p. 385) the functor

$$i \mapsto i^* : \mathbf{Hom}(A, \mathcal{M}) \to \mathbf{Hom}^!(\mathcal{M}, A^\wedge) \tag{37}$$

(which in any case is fully faithful) is equally an equivalence of categories.

The equivalence (33) of prop. 3 can be interpreted by saying that the [p. 395]
canonical functor (32) from A to $A^\wedge$ is 2-*universal*, for functors of A into $\mathcal{U}$-categories stable under small direct limits, taking as "maps" between such categories functors which commute to small direct limits. Thus, the $\mathcal{U}$-category $A^\wedge$ may be viewed as "the" category deduced from A by adding arbitrary direct limits (disregarding the direct limits which may perchance already exist in A...).

Taking the duals of the two members of (33), we get an equivalent statement of prop. 3:

Corollary 1. *Let $\mathcal{N}$ be a $\mathcal{U}$-category stable under small inverse limits. Then the functor*

$$F \mapsto \varepsilon_A^{\mathrm{op}} : \mathbf{Hom}^!(A^{\wedge\,\mathrm{op}}, \mathcal{N}) \to \mathbf{Hom}(A^{\mathrm{op}}, \mathcal{N}) \tag{38}$$

is an equivalence of categories.

In terms of the topos

$$\mathcal{A} = A^\wedge,$$

we may interpret the left-hand side of (33) as the category of $\mathcal{M}$-valued *cosheaves* on this topos, which by prop. 3 can be interpreted (up to equivalence) as the category of functors from A to $\mathcal{M}$. Dually, the left-hand side of (38) can be viewed as the category of $\mathcal{N}$-valued *sheaves* on the topos $\mathcal{A}$, which (via the right-hand side) can be interpreted up to equivalence as the category of functors $A^{\mathrm{op}} \to \mathcal{N}$, i.e., as the category of $\mathcal{N}$-valued *presheaves* on A. As A endowed with the coarsest ("chaotic") site structure is a generating site for the topos $\mathcal{A}$, the equivalence (38) may be viewed as a particular case of the familiar fact, according to which (up to equivalence) the category of $\mathcal{N}$-valued sheaves on a topos can be constructed in terms of $\mathcal{N}$-valued sheaves on any $\mathcal{U}$-site defining this topos. (When the site structure is chaotic, then those sheaves are just arbitrary $\mathcal{N}$-valued presheaves.)

Assume now that the $\mathcal{U}$-category $\mathcal{M}$ is stable under *both* types of small limits (direct and inverse). Then applying (33) for $(A, \mathcal{M})$ and (38) for $(B, \mathcal{M})$ where $B = A^{\mathrm{op}}$, we get an equivalence

$$\delta_A^{\mathcal{M}} : \mathbf{Hom}_!(A^\wedge, \mathcal{M}) \xrightarrow{\approx} \mathbf{Hom}^!(B^{\wedge\,\mathrm{op}}, \mathcal{M}), \tag{39}$$

i.e. (as announced in yesterday's notes, p. 390), we get:

Corollary 2. *Let A be a small category, $B = A^{\mathrm{op}}$ the dual category, $\mathcal{M}$ any $\mathcal{U}$-category stable under small direct and inverse limits. Then the category of $\mathcal{M}$-valued* cosheaves *on the topos $A^\wedge$ is equivalent to the category of* [p. 396]
$\mathcal{M}$-valued sheaves *on the topos $B^\wedge$.*

This equivalence, defined up to unique isomorphism, is deduced from the diagram of canonical equivalences

$$\begin{tikzcd}[column sep=small]
\mathbf{Hom}_!(A^\wedge, \mathcal{M}) \overset{\mathrm{def}}{=} (A^\wedge)^{\mathcal{M}} \arrow[dr, "\approx"] & & (B^\wedge)_{\mathcal{M}} \overset{\mathrm{def}}{=} \mathbf{Hom}^!(B^{\wedge\,\mathrm{op}}, \mathcal{M}) \arrow[dl, "\approx"'] \\
& \mathbf{Hom}(A, \mathcal{M}) = \mathbf{Hom}(B^{\mathrm{op}}, \mathcal{M}) &
\end{tikzcd} \quad , \tag{40}$$

and depends upon the choice of a quasi-inverse of the second vertical equivalence in (40) (which choice can be made, via the dual of (35), via the choice of small inverse limits in $\mathcal{M}$).

Remark. It is felt that the way we got the equivalence (39) via (40), the role of A and B in it is symmetric. To give a more precise statement, consider the equivalence deduced from (39) by passing to the dual categories of the two members – using the tautological isomorphisms (6) of page 385, we get an equivalence

(39') $\quad (\delta_A^{\mathcal{M}})' : \mathbf{Hom}^!((A^\wedge)^{\mathrm{op}}, \mathcal{N}) \xrightarrow{\approx} \mathbf{Hom}_!(B^\wedge, \mathcal{N}), \quad$ where $\mathcal{N} = \mathcal{M}^{\mathrm{op}}$;

this equivalence is *canonically quasi-inverse to the equivalence* $\delta_B^{\mathcal{N}}$ *in opposite direction*, associated to the pair $(B, \mathcal{N} = \mathcal{M}^{\mathrm{op}})$ instead of $(A, \mathcal{M})$.

In the rest of this subsection D), we'll elaborate on some particular cases of the equivalence (39) between cosheaves and sheaves.

Case 1º). Assume $\mathcal{M} =$ (Sets), then by prop. 1 (p. 384) the right-hand side of (39) is canonically equivalent to $B^\wedge$ itself, hence we get an equivalence

(41) $$B^\wedge \xrightarrow{\approx} \mathbf{Hom}_!(A^\wedge, (\mathrm{Sets})).$$

If we want to keep track of the symmetry aspect described in the remark above, we may consider the functor (41) as being deduced from a canonical "pairing" between the categories $A^\wedge$ and $B^\wedge$

(42) $$\delta_A : A^\wedge \times B^\wedge \to (\mathrm{Sets}),$$

which is an object in

$$\mathbf{Hom}_{!!}(A^\wedge, B^\wedge; (\mathrm{Sets})),$$

i.e., which commutes to small direct limits in each variable. (For this interpretation, compare the dual statement contained in formula (7) of page 385 – and note that (41), being an equivalence, commutes to small direct limits, i.e., is in a category $\mathbf{Hom}_!(B^\wedge, \mathbf{Hom}_!(A^\wedge, (\mathrm{Sets})))$.)
[p. 397] This pairing gives rise, in a symmetric way, to the functor (41) (which is an equivalence) *and* to a functor

(41') $$A^\wedge \xrightarrow{\approx} \mathbf{Hom}_!(B^\wedge, (\mathrm{Sets}))$$

which (it turns out) is none other (up to canonical isomorphism) than (41) with B replaced by A (and hence A replaced by B), and therefore is equally and equivalence. *Thus, the pairing* (42) *between the two topoi* $A^\wedge$ *and* $B^\wedge$ *has the remarkable property that it defines an equivalence of each of these topoi with the category of* (*set-valued*) *cosheaves on the other.* I do not know of any other example of a pair of topoi related in such a remarkable way, which we may express by saying that the two topoi are "*dual*" to each other.

We still have to give an explicit expression for the pairing (42), plus a convenient notation. I'll write

(43) $$\delta_A(F, G) \overset{\mathrm{def}}{=} F * G \quad \text{for } F \text{ in } A^\wedge,\ G \text{ in } B^\wedge,$$

and I'll use the canonical equivalence (valid for any pair of small categories A, B – not necessarily dual to each other – and any $\mathcal{U}$-category stable under direct limits), deduced by twofold application of prop. 3, plus the tautological isomorphism similar to (7) p. 385:

$$(44)\qquad \mathbf{Hom}_{!!}(A\hat{~}, B\hat{~}; \mathcal{M}) \xrightarrow{\approx} \mathbf{Hom}(A \times B, \mathcal{M}), \quad F \mapsto F \circ (\varepsilon_A \times \varepsilon_B),$$

which shows that (42) is known up to canonical isomorphism when we know its restriction to the full subcategory $A \times B = A \times A^{\mathrm{op}}$, identifying as usual an object a of A to its image in $A\hat{~}$, and similarly for B. If b is an object of A, we'll denote by b^{op} the same object viewed as an object of $A^{\mathrm{op}} = B$. With these conventions (including (43)) we get the nice formula

$$(45)\qquad a * b^{\mathrm{op}} = \mathrm{Hom}_A(b, a) \quad (= \mathrm{Hom}_B(a^{\mathrm{op}}, b^{\mathrm{op}})), \quad \text{for } a, b \text{ in } A,$$

which has the required symmetry property – which, for general objects F in $A\hat{~}$ and G in $B\hat{~}$, can be stated as a bifunctorial isomorphism

$$(46)\qquad F * G \simeq G * F,$$

where the operation $*$ in the first member refers to the pair (A, B), and in the second to the pair (B, A).

From (45) we easily deduce the more general formula for $F * G$, when either F or G is in A resp. B, namely [p. 398]

$$(47)\qquad a * G \simeq G(a), \quad F * b^{\mathrm{op}} \simeq F(b).$$

Remarks. We are mainly interested here in abelianization and (commutative) homology and cohomology, and hence in sheaves and cosheaves with values in additive (even abelian) categories, we are not going to use for the time being the relationship between $A\hat{~}$ and $B\hat{~}$ just touched upon. We could elaborate a great deal more on it, for instance introducing a canonical pairing (more accurately, a bi-sheaf) with opposite variance to (42)

$$(48)\qquad A^{\wedge\,\mathrm{op}} \times B^{\wedge\,\mathrm{op}} \to (\mathrm{Sets})$$

(or what amounts to the same, canonical functors adjoint to each other

$$A^{\wedge\,\mathrm{op}} \to B\hat{~}, \quad B^{\wedge\,\mathrm{op}} \to A\hat{~} \quad),$$

deduced (via the equivalence dual to (44)

$$\mathbf{Hom}^{!!}(A^{\wedge\,\mathrm{op}}, B^{\wedge\,\mathrm{op}}; \mathcal{N}) \xrightarrow{\approx} \mathbf{Hom}(A^{\mathrm{op}} \times B^{\mathrm{op}}, \mathcal{N}) \quad)$$

from the co-pairing

$$A^{\mathrm{op}} \times B^{\mathrm{op}} = B \times A \to (\mathrm{Sets})$$

given by

$$(b^{\mathrm{op}}, a) \mapsto \mathrm{Hom}(b, a) \quad (\text{for } a, b \text{ in } A).$$

The two pairings (42) and (48) can be given a common interpretation as Hom-sets in a suitable category

$$\mathcal{E} = \mathcal{E}(A),$$

which is the union of the two full subcategories $A^\wedge$ and $B^{\wedge\mathrm{op}}$ (which may be interpreted as deduced from A by adjoining respectively small direct and small inverse limits to it) intersecting in the common subcategory A, the Hom-sets in the two directions between an object F of $A^\wedge$ and an object G^{op} of $B^{\wedge\mathrm{op}}$ (corresponding to an object G of $B^\wedge$) being given respectively by the pairings (48) and (42). The full relationship between these pairings is most conveniently expressed, it seems, by the composition law of maps in $\mathcal{E}(A)$, and associativity for this law. The symmetry of the situation with respect to the pair (A, B) is expressed by the canonical isomorphism of categories

[unreadable footnote]

$$\mathcal{E}(A)^{\mathrm{op}} \simeq \mathcal{E}(B) \quad (\text{where } B = A^{\mathrm{op}}).$$

Case 2º). Of direct relevance for the abelianization story is the particular case of the equivalence (39), obtained by taking

$$\mathcal{M} = (\mathrm{Ab}).$$

Using formula (12) (page 387) for the pair $(A^\wedge, (\mathrm{Ab}))$ and formula (14) for $B^\wedge$ in guise of $\mathcal{A}$, we get the canonical equivalence [p. 399]

$$B^\wedge_{\mathrm{ab}} \xrightarrow{\approx} \mathbf{Hom}_!(A^\wedge_{\mathrm{ab}}, (\mathrm{Ab})), \tag{49}$$

which should be viewed as the "abelian" analogon of the equivalence (41) above (corresponding to the case $\mathcal{M} = (\mathrm{Sets})$). This equivalence again may be viewed as described (in analogy to (45)) by a canonical pairing

$$A^\wedge_{\mathrm{ab}} \times B^\wedge_{\mathrm{ab}} \to (\mathrm{Ab}), \tag{50}$$

which commutes to small direct limits in each variable, i.e., can be viewed as an object in the category of "abelian bi-cosheaves"

$$\mathbf{Hom}_{!!}(A^\wedge_{\mathrm{ab}}, B^\wedge_{\mathrm{ab}}; (\mathrm{Ab})), \tag{*}$$

and gives rise simultaneously to the equivalence (49), and to the symmetric equivalence

$$A^\wedge_{\mathrm{ab}} \xrightarrow{\approx} \mathbf{Hom}_!(B^\wedge_{\mathrm{ab}}, (\mathrm{Ab})) \tag{49'}$$

of the category of abelian sheaves on $A^\wedge$ with the category of abelian cosheaves on $B^\wedge$ (which is just (49) with A replaced by B, up to canonical isomorphism at any rate).

Using the equivalence

$$\mathbf{Hom}_{!!}(A^\wedge_{\mathrm{ab}}, B^\wedge_{\mathrm{ab}}; (\mathrm{Ab})) \xrightarrow{\approx} \mathbf{Hom}(A \times B, (\mathrm{Ab})) \tag{51}$$

(which is a particular case of the evident abelian analogon of the equivalence (44)), we see that the pairing (50) is described, up to canonical isomorphism, by its composition with

$$(a,b) \mapsto (\mathrm{Wh}_{A^\wedge}(a), \mathrm{Wh}_{B^\wedge}(b)) : A \times B \to A^\wedge_{\mathrm{ab}} \times B^\wedge_{\mathrm{ab}},$$

and the latter, as is readily checked, is given by

(52) $$\mathrm{Wh}_A(a) *_{\mathbb{Z}} \mathrm{Wh}_B(b^{\mathrm{op}}) \simeq \mathbb{Z}^{(\mathrm{Hom}(b,a))} \quad \text{for } a, b \text{ in } A,$$

where we have written Wh_A instead of $\mathrm{Wh}_{A^\wedge}$ for brevity and accordingly for B, and where the pairing (50) is denoted by the symbol $*_{\mathbb{Z}}$, in analogy with the notation $*$ in (43), the index $\mathbb{Z}$ being added in order to avoid confusion with the non-abelian case (43) (and the index being dropped when no such confusion is to be feared). The formula (52) can be written, with different notations

(52') $$\mathbb{Z}^{(a)} *_{\mathbb{Z}} \mathbb{Z}^{(b^{\mathrm{op}})} \simeq \mathbb{Z}^{(\mathrm{Hom}(b,a))} \quad \text{for } a, b \text{ in } A.$$

Comparing with the similar formula (45), this suggests the generalization

(53) $$\mathrm{Wh}_A(F) *_{\mathbb{Z}} \mathrm{Wh}_B(G) \simeq \mathbb{Z}^{(F*G)},$$

or with the exponential notation

(53') $$\mathbb{Z}^{(F)} *_{\mathbb{Z}} \mathbb{Z}^{(G)} \simeq \mathbb{Z}^{(F*G)},$$

valid for F in $A^\wedge$ and G in $B^\wedge$. As both members of (53) commute with small direct limits in each variable, the formula (53) follows from the particular case (52), in view of the equivalence of categories (51). [p. 400]

Remarks. 1) In order to appreciate the significance of the pairing (50), we may forget altogether about the non-additive categories $A^\wedge$ and $B^\wedge$, and view (50) as a remarkable "duality" relationship between two additive $\mathcal{U}$-categories, stable under small direct limits, say $\mathcal{P}$ and $\mathcal{Q}$, endowed with a "pairing"

$$\mathcal{P} \times \mathcal{Q} \to (\mathrm{Ab}) \quad \text{in } \mathbf{Hom}_{!!}(\mathcal{P}, \mathcal{Q}; (\mathrm{Ab})),$$

giving rise to two functors which are *equivalences*

$$\mathcal{Q} \xrightarrow{\approx} \mathbf{Hom}_!(\mathcal{P}, (\mathrm{Ab})), \quad \mathcal{P} \xrightarrow{\approx} \mathbf{Hom}_!(\mathcal{Q}, (\mathrm{Ab})),$$

identifying each of $\mathcal{P}$, $\mathcal{Q}$ to the category of "abelian cosheaves" on the other. In the particular case (50), $\mathcal{P} = A^\wedge_{\mathrm{ab}}$ and $\mathcal{Q} = B^\wedge_{\mathrm{ab}}$ with $B = A^{\mathrm{op}}$, each of these categories is even an "abelian topos" by which I mean an abelian category $\mathcal{P}$ stable under small filtering direct limits, with the latter being *exact*, and moreover $\mathcal{P}$ admitting a small generating subcategory. (These categories are sometimes called, somewhat misleadingly, "Grothendieck categories". Of course, an "abelian topos" is by no means a category which is a topos, besides being abelian!) There are many other examples of dual pairs of abelian topoi. One evident generalization is by taking

$$\mathcal{P} = A_k^\wedge, \quad \mathcal{Q} = B_k^\wedge, \quad \text{where again } B = A^{\mathrm{op}},$$

where k is any commutative ring, and $A_k^\wedge \overset{\text{def}}{=} A^\wedge_{(k\text{-Mod})}$ is the category of presheaves of k-modules on A, or equivalently, of objects in $A^\wedge$ endowed with a structure of a k-module – and accordingly for the notation $B_k^\wedge$. Indeed, the generalities B) and C) in yesterday's notes about abelian sheaves and cosheaves, as well as today's, could be developed replacing throughout abelian group objects and additive categories by k-module objects and k-linear categories. In case k is not supposed commutative, one still should get a duality pairing

$$A^\wedge_{k^{\mathrm{op}}} \times B_k^\wedge \to (\mathrm{Ab}),$$

where k^{op} denotes the ring opposite to k (i.e., a duality pairing between presheaves on A of right k-modules, and copresheaves on A of left k-
[p. 401] modules), given in terms of $*_{\mathbb{Z}}$ in (50) by the formula

$$M *_k N = (M *_{\mathbb{Z}} N)^\natural,$$

where in the right-hand side $P = M *_{\mathbb{Z}} N$ is viewed as a bi-k-module via the right and left k-module structures on M and N respectively and bifunctoriality of $*_{\mathbb{Z}}$, and where for any bimodule P, we write

$$P^\natural \overset{\text{def}}{=} P/\text{sub-}\mathbb{Z}\text{-module generated by elements of the type } s.x - x.s \text{ for } x \text{ in } P \text{ and } s \text{ in } k.$$

I confess I didn't do the checking that this does give rise indeed to a duality pairing as desired. When $A = B =$ final category, then the pairing above is just the pairing given by usual tensor product

$$(M, N) \mapsto M \otimes_k N$$

between right and left k-modules, which is immediately checked to be dualizing indeed. More generally, posing

$$\mathcal{P} = (k^{\mathrm{op}}\text{-Mod}), \quad Q = (k\text{-Mod}),$$

it is immediately checked that for any additive category $\mathcal{M}$ stable under small direct limits, we get a canonical equivalence

[Clearly, the "d" in k_{d} is for "dexter" (on the right), and the "s" below is for "sinister" (on the left).]

$$\mathbf{Hom}_!(\mathcal{P}, \mathcal{M}) \xrightarrow{\approx} (k\text{-}\mathcal{M}), \quad F \mapsto F(k_{\mathrm{d}}),$$

where $(k\text{-}\mathcal{M})$ denotes the category of objects L of $\mathcal{M}$ endowed with a structure of a "left k-module in $\mathcal{M}$", i.e., a ring homomorphism $k \to \mathrm{End}_{\mathcal{M}}(L)$, k_{d} denotes k viewed as a right k-module, and $F(k_{\mathrm{d}})$ is viewed as an object of $(k\text{-}\mathcal{M})$ via the operations of k on it coming from left multiplication of k upon k_{d}. A quasi-inverse equivalence is obtained by associating to an object L in $(k\text{-}\mathcal{M})$ the functor

$$M \mapsto M \otimes_k L : (k^{\mathrm{op}}\text{-Mod}) \to \mathcal{M}.$$

Dually, we get an equivalence (if $\mathcal{N}$ stable under small inv. limits)

$$\mathbf{Hom}^!(\mathcal{Q},\mathcal{N}) \xrightarrow{\approx} (k\text{-}\mathcal{N}), \quad F \mapsto F(k_s),$$

where k_s denotes k viewed as a left k-module, so that the *contravariant* functor F transforms the endomorphisms of k_s (obtained by *right* operation of k on k_s via right multiplication) into a *left* operation of k on $F(k_s)$; the quasi-inverse is given by the familiar Hom_k operation, it associates to the left k-module L in $\mathcal{M}$ the functor

$$M \mapsto \mathrm{Hom}_k(M,L) : (k\text{-Mod}) \to \mathcal{N}.$$

Comparing the two pairs of equivalences, we get the equivalence [p. 402]

$$(*) \qquad \mathbf{Hom}_!(scrP,\mathcal{M}) \approx \mathbf{Hom}^!(\mathcal{Q}^{\mathrm{op}},\mathcal{M}),$$

valid when $\mathcal{M}$ is stable under both (small) direct and inverse limits, and which should be viewed as an abelian analogon of the equivalence (39).

2) It is well-known that an abelian topos $\mathcal{P}$ is equivalent to a category (k-Mod), for a suitable ring k (not necessarily commutative) iff it admits an object L which is a) generating, and b) "*ultraprojective*", i.e., the functor $X \mapsto \mathrm{Hom}(L,X)$ commutes to small direct limits. The condition b) (for an object of an *abelian* category stable under small direct limits or equivalently, under small direct sums) is equivalent with L being: b_1) projective, and b_2) of "*finite presentation*", i.e., the functor $X \mapsto \mathrm{Hom}(L,X)$ commutes with small filtering direct limits. This observation suggests one common feature of all the examples of abelian duality pairings considered so far, namely that the abelian topoi under consideration in the pairing *have a small set of ultraprojective generators*. I don't know if a structure theory of such categories (which are the abelian analogons for topoi equivalent to topoi of the special type $A^\wedge$, with A in (Cat)) has been worked out yet. I didn't do it at any rate – but the natural thing to expect is that these abelian topoi $\mathcal{P}$ (which we may call "*algebraic*" ones, just as an ordinary topos equivalent to one of the type $A^\wedge$ may be called "algebraic", which equally means that the set of ultraprojective objects in it is generating...) are exactly those equivalent to a category of the type

$$\mathrm{Homadd}(P^{\mathrm{op}},(\mathrm{Ab})),$$

where P is any *small additive* category, and where Homadd denotes the category of additive functors from one additive category to another. Instead of assuming P small, we may as well take P merely "essentially small", i.e., equivalent to a small category, with the benefit that for a given $\mathcal{P}$, there is a canonical choice of an additive category P together with an equivalence

$$\mathcal{P} \xrightarrow{\approx} \mathrm{Homadd}(P^{\mathrm{op}},(\mathrm{Ab})),$$

namely by taking

$P =$ full subcategory of $\mathcal{P}$ made up with all ultraprojective objects in $\mathcal{P}$.

As we saw earlier, in case $\mathcal{P} = A^{\wedge}_{\mathrm{ab}}$, P is nothing but the abelian Karoubi envelope of the category A, namely the Karoubi envelope of the additive category $\mathrm{Add}(A)$ (cf. sections 93 and 99). Another choice for P in this case would be just $\mathrm{Add}(A)$ itself, whose objects are more amenable to
[p. 403] computations.

Associating to any small [additive] category P the algebraic abelian topos $\mathbf{Homadd}(P^{\mathrm{op}}, (\mathrm{Ab}))$ should be viewed of course as the abelian analogon of $A \mapsto A^{\wedge}$, associating to a small category A the corresponding algebraic topos. It merits a notation of its own, say

$$P^{\&} \overset{\mathrm{def}}{=} \mathbf{Homadd}(P^{\mathrm{op}}, (\mathrm{Ab})),$$

and as in the non-additive case, we get a canonical inclusion functor

$$\varepsilon_P : P \to P^{\&}$$

which is additive. (Its composition with the canonical functor $P^{\&} \to P^{\wedge}$ is the canonical inclusion previously denoted by ε_P too from P to $P^{\wedge}$.) Next thing we'll expect, in analogy to prop. 3, is that for any additive category $\mathcal{M}$ stable under small direct limits, the following canonical functor is an equivalence of categories:

$$(**) \qquad \mathbf{Hom}_!(P^{\&}, \mathcal{M}) \xrightarrow{\approx} \mathbf{Homadd}(P, \mathcal{M}), \quad F \mapsto F \circ \varepsilon_P.$$

The proof, via construction of a quasi-inverse functor, should be about the same as for prop. 3, which should go through once we get the abelian analogon of the well-known fact in $A^{\wedge}$, that any object F in $A^{\wedge}$ can be recovered as a direct limit in $A^{\wedge}$ of objects of A, according to $A_{/F}$ as an indexing category – which makes us expect that we get too:

$$F \xleftarrow{\sim} \varinjlim_{a \text{ in } P^{\&}_{/F}} a \quad (\text{direct limit in } P^{\&}).$$

From (**) we get as in cor. 1, passing to the dual categories, the dual equivalence

$$\mathbf{Hom}^!(P^{\&\mathrm{op}}, \mathcal{N}) \xrightarrow{\approx} \mathbf{Homadd}(P^{\mathrm{op}}, \mathcal{N}),$$

valid if $\mathcal{N}$ is an additive category stable under small inverse limits. Hence, if $\mathcal{M}$ is additive and stable under both types of limits, the equivalence

$$(***) \qquad \mathbf{Hom}_!(P^{\&}, \mathcal{M}) \xrightarrow{\approx} \mathbf{Hom}^!(Q^{\&\mathrm{op}}, \mathcal{M}), \quad \text{with } Q = P^{\mathrm{op}},$$

between abelian cosheaves on $P^{\&}$ and abelian sheaves in $Q^{\&}$, with values in the same additive category (in analogy to (39)). In the particular case $\mathcal{M} = (\mathrm{Ab})$, this then gives rise to the equivalence

$$Q^{\&} \approx \mathbf{Hom}_!(P^{\&}, (\mathrm{Ab}))$$

and to the corresponding pairing

$$P^{\&} \times Q^{\&} \to (\mathrm{Ab})$$

which is a duality, namely induces an equivalence between each abelian topos $P^{\&}$, $Q^{\&}$ and the category of abelian cosheaves on the other. [p. 404]

We may call an abelian topos "*reflexive*" if it can be inserted in a pair $(\mathcal{P}, \mathcal{Q})$ of dually paired abelian topoi – where $\mathcal{Q}$, or the "dual" of $\mathcal{P}$, is defined up to equivalence in terms of $\mathcal{P}$ as $\mathbf{Hom}_!(\mathcal{P}, (\mathrm{Ab}))$, the category of cosheaves on $\mathcal{P}$ with values in (Ab). Thus, it seems that a sufficient condition for reflexivity is "algebraicity" of $\mathcal{P}$, namely the existence of a small generating family made up with ultraprojective objects. (NB In the non-abelian case, it is well-known that a topos $\mathcal{A}$ is "algebraic", i.e., equivalent to a topos $A^\wedge$, iff it admits such a generating family – and as we saw in 1º), as a consequence of (39) for $\mathcal{M} = (\mathrm{Sets})$, such a topos is indeed "reflexive".) I wouldn't be too surprised if this sufficient condition for reflexivity turned out to be necessary too, at any rate if we want a property stronger still than reflexivity, namely validity of a duality equivalence (***) for sheaves and cosheaves with values in an arbitrary additive $\mathcal{U}$-category stable under small direct and inverse limits, satisfying (for varying $\mathcal{M}$) suitable compatibility assumptions.

3) With respect to this duality equivalence (***), I am a little unhappy still, as I do not see how to get (for a general dual pair $\mathcal{P}$, $\mathcal{Q}$ of abelian topoi) a functor i one direction or the other between the two categories

$$\mathbf{Hom}_!(\mathcal{P}, \mathcal{M}), \quad \mathbf{Hom}^!(\mathcal{Q}^{\mathrm{op}}, \mathcal{M}),$$

in terms of just the duality pairing. The same perplexity holds in the non-abelian case. This is one of the reasons that make me feel that I haven't yet a thorough understanding of the duality formalism I am developing here, except in the "algebraic" case (granting for the latter that the tentative theory just outlined for algebraic *abelian* topoi is indeed correct).

4) To finish with the comments on the (pre-homological) duality formalism for algebraic topoi and algebraic abelian topoi, I still would like to add that the category $\mathcal{E}(A)$ (union of $A^\wedge$ and $B^{\wedge\,\mathrm{op}}$, with $B = A^{\mathrm{op}}$) introduced in 1º) (cf. remark on page 398) admits also an abelian analogon. In the non-abelian case still, the simplest way to construct the category $\mathcal{E}(A)$, is via an equivalent category canonically embedded in the category $\mathbf{Hom}(B^\wedge, (\mathrm{Sets}))$ as a strictly full subcategory ("strict" referring to the fact that with any object it contains all isomorphic ones), namely the union $\overline{\mathcal{E}}(A)$ of the (strictly full) subcategories $\mathbf{Hom}_!(B^\wedge, (\mathrm{Sets}))$ (equivalent to $A^\wedge$) and the subcategory of representable functors (equivalent to $B^{\wedge\,\mathrm{op}}$). The intersection of these two categories contains of course A (embedded in $\mathbf{Hom}(B^\wedge, (\mathrm{Sets}))$ by associating to a in A the functor $G \mapsto G(a)$ from $B^\wedge = \mathbf{Hom}(A, (\mathrm{Sets}))$ to (Sets)), but in general need not be quite equivalent to A – it turns out to be the "Karoubi envelope" of A, obtained by adjoining to A formally images (or equivalently, coimages) of projectors in A. The more immediate interpretation of this intersection, is that it is equivalent to the dual category of the category of ultraprojective objects in $B^\wedge$ (and the latter can be viewed as $\mathrm{Kar}(B)$, but formation of the Karoubi envelope up to equivalence commutes to taking dual categories...). All these constructions immediately extend [p. 405]

[There seems to be another unreadable footnote here...]

to the abelian set-up, starting with a small *additive* category P, instead of A.

After this endless procession of remarks, which are really digressions for what we're after (namely abelianization and duality in the context of small categories as homotopy models), it is time to resume our main line of thought in this subsection, namely looking at interesting particular cases for the general duality relation (39).

Case 3º). This is the case when $\mathcal{M}$ is an *additive* category, stable under both types of small limits. If we assume moreover that $\mathcal{M}$ and $\mathcal{M}^{\mathrm{op}}$ are both pseudotopoi, using the equivalences (12) and (13) (p. 387), (39) may be interpreted as an equivalence

$$\mathbf{Hom}_!(A^\wedge_{\mathrm{ab}}, \mathcal{M}) \approxeq \mathbf{Hom}^!(B^\wedge_{\mathrm{ab}}, \mathcal{M}), \tag{54}$$

interpreting $\mathcal{M}$-valued cosheaves on the abelian topos $A^\wedge_{\mathrm{ab}}$ in terms of $\mathcal{M}$-valued sheaves on the dual abelian topos $B^\wedge_{\mathrm{ab}}$, as anticipated in a more general situation in the remark above (cf. formula (***) on page 403). There, however, the assumption that $\mathcal{M}$ and/or $\mathcal{M}^{\mathrm{op}}$ should be pseudotopoi didn't seem to come in at all, so we expect this condition to be irrelevant indeed. This will of course follow, if the same holds for (12) (hence by duality for (13)), namely that the canonical functor

$$\mathbf{Hom}_!(A^\wedge_{\mathrm{ab}}, \mathcal{M}) \xrightarrow{\approxeq} \mathbf{Hom}_!(A^\wedge, \mathcal{M}), \quad F \mapsto F \circ \mathrm{Wh}_A, \tag{55}$$

is an equivalence, under the only assumption that the additive category $\mathcal{M}$ is stable under small direct limits (without assuming that $\mathcal{M}$ be a pseudotopos). The line of thought of the remark 2 above suggests a
[p. 406] way for proving this, via an equivalence

$$\mathbf{Hom}_!(A^\wedge_{\mathrm{ab}}, \mathcal{M}) \xrightarrow{\approxeq} \mathbf{Homadd}(\mathrm{Add}(A), \mathcal{M}), \quad F \mapsto F \circ j_A, \tag{56}$$

where

$$j_A : \mathrm{Add}(A) \to A^\wedge_{\mathrm{ab}}$$

is the canonical inclusion functor (cf. section 97). This, and the dual equivalence (deduced from (55), taking $\mathcal{N} = \mathcal{M}^{\mathrm{op}}$)

$$\mathbf{Hom}^!({A^\wedge_{\mathrm{ab}}}^{\mathrm{op}}, \mathcal{N}) \xrightarrow{\approxeq} \mathbf{Homadd}(\mathrm{Add}(A)^{\mathrm{op}}, \mathcal{N}), \tag{57}$$

valid for any additive category stable under small inverse limits, will immediately imply an equivalence (54) by a direct argument as in the remark above, without passing through the non-abelian case (39). At any rate, (56) implies that (55) is an equivalence, as is seem by looking at the commutative diagram

$$\begin{tikzcd} & \mathbf{Hom}_!(A^\wedge, \mathcal{M}) \arrow[dr] & \\ \mathbf{Hom}_!(A^\wedge_{\mathrm{ab}}, \mathcal{M}) \arrow[ur] \arrow[dr] & & \mathbf{Hom}(A, \mathcal{M}) \\ & \mathbf{Homadd}(\mathrm{Add}(A), \mathcal{M}) \arrow[ur] & \end{tikzcd} \tag{58}$$

,

where the two right-hand arrows are equivalences, which implies that one of the two left-hand arrows is an equivalence iff the other is.

Thus, for getting (54) and (55) without extraneous assumptions on $\mathcal{M}$, we are left with proving (56). Now, writing

$$\mathcal{P} = A^{\wedge}_{\mathrm{ab}}, \quad P = \mathrm{Add}(A),$$

we do have indeed an equivalence

$$\mathcal{P} \approxeq P^{\&} \overset{\mathrm{def}}{=} \mathbf{Homadd}(P^{\mathrm{op}}, (\mathrm{Ab})),$$

as seen from (57) taking $\mathcal{N} = (\mathrm{Ab})$ (which satisfies the extra assumptions). So we may as well prove (56) in the more general case when P is any small additive category and $\mathcal{P}$ is defined as $P^{\&}$, namely prove the equivalence (**) of page 403 above, as I don't expect the particular case at hand here to be any simpler. The suggestion for a proof there there seems convincing, I guess I should check it works during some in-between scratchwork...

14.8. and 15.8 [p. 407]

106 **E) A formulaire around the basic operations $*$ and** Hom**.** I would like to dwell a little more still on the duality formalism weaving around formula (39) (p. 395), stating that for two given small categories A and B dual to each other

Review (3): A formulaire for the basic integration and cointegration operations $*$ and Hom*.*

$$B = A^{\mathrm{op}}, \quad A = B^{\mathrm{op}},$$

and any $\mathcal{U}$-category $\mathcal{M}$ stable under both types of small limits, $\mathcal{M}$-valued *cosheaves* on $A^{\wedge}$ may be interpreted as $\mathcal{M}$-valued *sheaves* on the dual topos $B^{\wedge}$. This identification preserves variance (i.e., (39) is an *equivalence* of categories, not an antiequivalence:

(59) $$(A^{\wedge})^{\mathcal{M}} = \mathbf{Hom}_!(A^{\wedge}, \mathcal{M}) \xrightarrow{\approxeq} \mathbf{Hom}^!((B^{\wedge})^{\mathrm{op}}, \mathcal{M}) = B^{\wedge}_{\mathcal{M}} \quad).$$

It should not be confused with the tautological interpretation of $\mathcal{M}$-valued cosheaves on $A^{\wedge}$ as $\mathcal{M}^{\mathrm{op}}$-valued sheaves on the *same* topos, an identification *reversing variance*, as expressed by the canonical antiequivalence between the corresponding categories – an anti-isomorphism even (reflecting its tautological nature):

(60) $$\mathbf{Hom}_!(A^{\wedge}, \mathcal{M})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{Hom}^!((A^{\wedge})^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}), \text{ i.e., } ((A^{\wedge})^{\mathcal{M}})^{\mathrm{op}} \xrightarrow{\sim} A^{\wedge}_{\mathcal{M}}.$$

In the latter formula, the basic topos $A^{\wedge}$ remains the same in both sides, it is the category of values that changes from $\mathcal{M}$ to the dual one $\mathcal{M}^{\mathrm{op}}$, whereas in formula (59) = (39), it is the opposite. In terms of the tautological formula (60) (a particular case of formula (6) p. 385), the not-so-tautological formula relating cosheaves and sheaves can be reformulated as a formula in terms of sheaves only (due to our preference for sheaves rather than cosheaves...):

(61) $$\mathbf{Hom}^!(A^{\wedge}, \mathcal{M})^{\mathrm{op}} \approxeq \mathbf{Hom}^!(B^{\wedge}, \mathcal{M}^{\mathrm{op}}), \quad \text{i.e.,} \quad (A^{\wedge}_{\mathcal{M}})^{\mathrm{op}} \approxeq B^{\wedge}_{\mathcal{M}^{\mathrm{op}}},$$

namely $\mathcal{M}$-valued sheaves on the topos $A^\wedge$ can be interpreted as sheaves on the dual topos with values in the dual category $\mathcal{M}^{\mathrm{op}}$, this interpretation *reversing variances*. In the homology and cohomology formalism which is to follow, due to habits of long standing, I prefer systematically to take as coefficients *sheaves* rather than cosheaves – hence rule out cosheaves in favor of sheaves via (60). From this point of view the relevant basic duality statement is (61) rather than (59).

On the *cosheaves* side, yesterday's diagram (58) of equivalences gives a fourfold description of cosheaves on the topos $A^\wedge$ with values in an *abelian* category $\mathcal{M}$ stable under small direct limits. We could still
[p. 408] enlarge this diagram, by including in it a fifth category equivalent to the four others, namely

$$\mathbf{Homaddinf}(\mathrm{Addinf}(A), \mathcal{M}),$$

the category of infinitely additive functors from the infinitely additive envelope of A into M (cf. section 99 p. 366 for description of the category $\mathrm{Addinf}(A)$). Rather than writing down the larger diagram here, I'll write down the dual enlarged one, for the dual topos $B^\wedge$ and for various expressions of the category of *sheaves* on this topos, with values in a category $\mathcal{M}$ stable this time under small inverse limits:

(62)
$$\begin{tikzcd}
 & \mathbf{Hom}^!((B^\wedge)^{\mathrm{op}}, \mathcal{M}) \arrow[dr, "\approx"] & \\
\mathbf{Hom}^!((B^\wedge_{\mathrm{ab}})^{\mathrm{op}}, \mathcal{M}) \arrow[ur, "\approx"] \arrow[d, "\approx"'] & & \mathbf{Hom}(B^{\mathrm{op}}, \mathcal{M}) \\
\mathbf{Hommultinf}(\mathrm{Addinf}(B)^{\mathrm{op}}, \mathcal{M}) \arrow[rr, "\approx"] & & \mathbf{Homadd}(\mathrm{Add}(B)^{\mathrm{op}}, \mathcal{M}) \arrow[u, "\approx"']
\end{tikzcd},$$

where **Hommultinf** denotes the category of "infinitely multiplicative" functors from one additive category stable under infinite products to another. Recalling for the extreme right term of (62) that $B^{\mathrm{op}} = A$, we see that this term is *identical* to the corresponding term in the diagram (even the enlarged one) (58) – hence, if $\mathcal{M}$ is stable under both types of limits, the ten categories occurring altogether in the two diagrams are mutually equivalent (as a matter of fact, there are nine only which are mutually different). It may be noted that there is still another pair of corresponding terms in the two diagrams for which the equivalence between them may be viewed as tautological, namely

$$\mathbf{Homadd}(\mathrm{Add}(A), \mathcal{M}) \approxeq \mathbf{Homadd}(\mathrm{Add}(B)^{\mathrm{op}}, \mathcal{M}),$$

due to the tautological equivalence of categories

$$\mathrm{Add}(B)^{\mathrm{op}} \approxeq \mathrm{Add}(B^{\mathrm{op}}) = \mathrm{Add}(A).$$

As emphasized in yesterday's notes, the canonical pairing (deduced from (59) by taking $\mathcal{M} = (\mathrm{Ab})$)

(63) $$A^\wedge_{\mathrm{ab}} \times B^\wedge_{\mathrm{ab}} \to (\mathrm{Ab}), \quad (F, G) \mapsto F *_{\mathbb{Z}} G,$$

deserves special attention, giving rise to an equivalence between each of the mutually dual abelian topoi $A^\wedge_{\mathrm{ab}}$, $B^\wedge_{\mathrm{ab}}$ with the category of abelian cosheaves on the other

$$(64)\qquad B^\wedge_{\mathrm{ab}} \xrightarrow{\approx} \mathbf{Hom}_!(A^\wedge_{\mathrm{ab}}, (\mathrm{Ab})), \quad A^\wedge_{\mathrm{ab}} \xrightarrow{\approx} \mathbf{Hom}_!(B^\wedge_{\mathrm{ab}}, (\mathrm{Ab}))$$

(cf. (49) and (49') page 399). It should be kept in mind that besides this duality pairing between two abelian topoi, there is important extra structure in this abelianized duality context, embodied by the *tensor product structure* on both abelian topoi $A^\wedge_{\mathrm{ab}}$ and $B^\wedge_{\mathrm{ab}}$, as contemplated in section 104 C), in a somewhat more general context. Corresponding to this extra structure on $A^\wedge_{\mathrm{ab}}$ say, we saw that this category of abelian sheaves "operates" covariantly (by an operation denoted by $\circledast_{\mathbb{Z}}$ or simply $\circledast$) on any category of $\mathcal{M}$-valued cosheaves on $A^\wedge_{\mathrm{ab}}$, and contravariantly (by an operation denoted by $\mathbf{Hom}_{\mathbb{Z}}$ or simply $\mathbf{Hom}$, if no confusion may arise) on any category of $\mathcal{N}$-valued sheaves on $A^\wedge_{\mathrm{ab}}$, where $\mathcal{M}$, $\mathcal{N}$ are additive categories, stable under small direct resp. inverse limits. The latter operation (cf. page 392) [p. 409]

$$(65)\qquad (L,K) \mapsto \mathbf{Hom}_{\mathbb{Z}}(L,K) : (A^\wedge_{\mathrm{ab}})^{\mathrm{op}} \times A^\wedge_{\mathcal{N}} \to A^\wedge_{\mathcal{N}},$$

involving sheaves, will be used in the sequel "tel quel", its definition is of a tautological character, independent of duality. As for the former operation involving cosheaves, we may view it via the duality relation (59) as an operation of $A^\wedge_{\mathrm{ab}}$ on $\mathcal{M}$-valued *sheaves* on the dual abelian topos $B^\wedge_{\mathrm{ab}}$, and this operation will be denoted by the same symbol $\circledast_{\mathbb{Z}}$: ["tel quel" = "as is"]

$$(66)\qquad (L,M') \mapsto L \circledast_{\mathbb{Z}} M' : A^\wedge_{\mathrm{ab}} \times B^\wedge_{\mathcal{M}} \to B^\wedge_{\mathcal{M}}.$$

Replacing A by B in (66), we get an operation of $B^\wedge_{\mathrm{ab}}$ upon $A^\wedge_{\mathcal{M}}$,

$$(67)\qquad (M,L') \mapsto M \circledast_{\mathbb{Z}} L' : A^\wedge_{\mathcal{M}} \times B^\wedge_{\mathrm{ab}} \to A^\wedge_{\mathcal{M}}.$$

Whenever convenient, we'll allow ourselves to write $L' \circledast M$ instead of $M \circledast L'$ (which doesn't seem to lead to any trouble), and will henceforth (unless special need should arise) drop the subscripts $\mathbb{Z}$.

Thus, for a given category $A^\wedge_{\mathcal{M}}$ of $\mathcal{M}$-valued sheaves on $A^\wedge_{\mathrm{ab}}$, there is a twofold operation on this category, namely $A^\wedge_{\mathrm{ab}}$ itself operates (the operation defined by L in $A^\wedge_{\mathrm{ab}}$ depending contravariantly on L) as well as the dual abelian topos $B^\wedge_{\mathrm{ab}}$ (the operation defined by L' in $B^\wedge_{\mathrm{ab}}$ depending covariantly on L'), $\mathcal{M}$ being any additive category stable under small inverse and direct limits (in order to ensure existence of both types of operations). I would like to dwell a little more on this twofold structure, as I don't feel to have understood it thoroughly yet. It is this second operation mainly which hasn't become really familiar yet, still less its relationship to the first, more familiar operation is understood. I'll have to play around a little more with it for being really at ease. It's worth the while, as the $\mathbf{Hom}$ and corresponding Hom operation is the key operation for expressing cohomology of $A^\wedge$ (with coefficients in K), where the $\circledast$ and corresponding $*$ operation is the key for expressing [p. 410] homology of $A^\wedge$ (with coefficients in M, where K and M are the sheaves occurring in (65) and (67) respectively).

A typical special case is the one when A is the final category, hence $B = A$ and $A^\wedge_{\mathcal{M}} \simeq \mathcal{M}$, in which case (65) and (67) are the two familiar exterior operations of (Ab) on any additive category stable under the two types of small limits

$$(L,X) \mapsto \mathbf{Hom}_{\mathbb{Z}}(L,X) : (\mathrm{Ab})^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M},$$

and

$$(X,L) \mapsto X \otimes_{\mathbb{Z}} L : (\mathrm{Ab}) \times \mathcal{M} \to \mathcal{M},$$

each one of these two operations being deducible from the other by the usual device of replacing $\mathcal{M}$ by the dual category $\mathcal{M}^{\mathrm{op}}$. When $\mathcal{M} =$ (Ab), these are just the usual internal Hom $=$ **Hom** and tensor product operations. This very particular case shows at once that we shouldn't expect in general the operations $\mathbf{Hom}_{\mathbb{Z}}(L,-)$ of $A^\wedge_{\mathrm{ab}}$ and $- * L'$ of $B^\wedge_{\mathrm{ab}}$ upon $A^\wedge_{\mathcal{M}}$ to commute up to isomorphism – we shouldn't expect, for a given L in $A^\wedge_{\mathrm{ab}}$ or a given L' in $B^\wedge_{\mathrm{ab}}$ the commutation relation to hold, except when this object is "projective of finite presentation", i.e., is a direct factor of an object of $\mathrm{Add}(A)$ resp. of $\mathrm{Add}(B) \approxeq \mathrm{Add}(A)^{\mathrm{op}}$. Another fact becoming evident by this particular case, is that whereas it is true that in the equivalence (64)

$$B^\wedge_{\mathrm{ab}} \approxeq \mathbf{Hom}_!(A^\wedge_{\mathrm{ab}}, (\mathrm{Ab}))$$

a projective object L' in $B^\wedge_{\mathrm{ab}}$ gives rise to a functor $A^\wedge_{\mathrm{ab}} \to (\mathrm{Ab})$ which is *exact* (besides commuting to small direct limits) – and even to a functor commuting to small inverse limits if L' is ultraprojective, i.e., projective and of finite presentation – the converse to this (as contemplated on page 381) does *not* hold true. Indeed, in the particular case $A = \Delta_0$, when L' is just an object in (Ab), the exactness property envisioned, i.e, exactness of the functor $M \mapsto M \otimes_{\mathbb{Z}} L'$ from (Ab) to itself, only means that L' is a *flat* $\mathbb{Z}$-module (i.e., torsion-free), which does *not* imply that it is projective (i.e., free).

[p. 411] The feeling I had earlier today, that the familiar looking operation (65) $\mathbf{Hom}(L,K)$ of abelian sheaves on $A^\wedge$ upon $\mathcal{M}$-valued ones ($\mathcal{M}$ an additive category stable under small inverse limits) was well-understood, whereas the less familiar one $M * L'$ in (67) was not, turns out to be mistaken. In computational terms, and writing F for K in (65) and M in (67), the three basic data

$$F \text{ in } A^\wedge_{\mathcal{M}}, \quad L \text{ in } A^\wedge_{\mathrm{ab}}, \quad L' \text{ in } B^\wedge_{\mathrm{ab}}$$

should surely be interpreted as just functor

$$F : A^{\mathrm{op}} \to \mathcal{M}, \quad L : A^{\mathrm{op}} \to (\mathrm{Ab}), \quad L' : B^{\mathrm{op}} = A \to (\mathrm{Ab}), \tag{68}$$

and the practical question of "computing" $\mathbf{Hom}(L,F)$ or $F \circledast L'$ thus amounts to describing, directly in terms of these data, the corresponding objects in $A^\wedge_{\mathcal{M}}$ as again a functor

$$\mathbf{Hom}(L,F) \quad \text{or} \quad F \circledast L' : A^{\mathrm{op}} \to \mathcal{M}.$$

It would seem that neither of the two can be expressed in simplistic computational terms, via the data (68). I feel I have to come to terms with this fact and get as close as I can to an explicit expression of both. The point I want to make first, is that this question of expressing $\mathbf{Hom}(L,F)$, or of expressing the operation $F \circledast L'$, is essentially the same, via replacement of $\mathcal{M}$ by $\mathcal{M}^{\mathrm{op}}$, and mere interchange of A and B. More accurately, passing from F to the corresponding functor F^{op} between the dual categories, we may view the data (68) as being functors

(68') $\quad F^{\mathrm{op}} : B^{\mathrm{op}} \to \mathcal{N}, \quad L' : B^{\mathrm{op}} \to (\mathrm{Ab}), \quad L : A^{\mathrm{op}} \to (\mathrm{Ab})$
(where $\mathcal{N} = \mathcal{M}^{\mathrm{op}}$),

i.e., a set of data like (68), with $(A, \mathcal{M})$ replaced by the dual pair $(B, \mathcal{N})$. This being understood, we have the tautological isomorphisms

(69)
$$\begin{cases} \mathbf{Hom}(L,F)^{\mathrm{op}} \simeq F^{\mathrm{op}} \circledast L \\ (F \circledast L')^{\mathrm{op}} \simeq \mathbf{Hom}(L', F^{\mathrm{op}}) \end{cases},$$

where the first members involve operations relative to the pair $(A, \mathcal{M})$, the second members operations relative to the dual pair $(B, \mathcal{N})$. This makes very clear, it seems to me, that the operations **Hom** (65) and $\circledast$ (67) may be viewed as the same type of operation, simply viewed with two different pairs of spectacles – one being $(A, \mathcal{M})$, the other the dual pair $(B, \mathcal{N})$. Thus, if we got a good understanding of one of the two operations, embodied by a comprehensive formulaire for it, by just dualizing we should get just as good a formulaire and corresponding comprehension for the dual operation.

[I'm leaving in "formulaire" (form), even though "formula" seems to work better. . .]

Now, it is clear indeed that it is the operations (65) which is closer to my experience, it makes sense however, independently of any duality statements, in the vastly more general context of topoi $\mathcal{A}$ (or even only pseudotopoi satisfying some mild extra conditions, cf. section 104 C)) instead of just $A^{\wedge}$, provided we make on $\mathcal{M}$ the mild extra assumption of being a pseudotopos, needed in this more general context in order to ensure equivalence between the category $\mathcal{A}_{\mathcal{M}}$ of $\mathcal{M}$-valued sheaves, and the category $\mathbf{Hom}^{!}(\mathcal{A}_{\mathrm{ab}}{}^{\mathrm{op}}, \mathcal{M})$ (compare (13) p. 388). What I should do then is, first to write down a basic "formulaire" for the **Hom** operation in this general and familiar context, then see how it can be used for clarifying the computational puzzle raised on the previous page, in the case of the **Hom** operation, and finally dualize the formulaire and computational insight, for getting a hold on the dual operation $\circledast$. [p. 412]

We'll need too the Hom operation (non-bold-face) of (31) p. 392

(70) $$(L,F) \mapsto \mathrm{Hom}(L,F) : \mathcal{A}_{\mathrm{ab}}{}^{\mathrm{op}} \times \mathcal{A}_{\mathcal{M}} \to \mathcal{M},$$

with values in $\mathcal{M}$, not $\mathcal{A}_{\mathcal{M}}$, where $\mathrm{Hom}(L,F)$ denotes the value on L of the functor
$$\widetilde{F} : \mathcal{A}_{\mathrm{ab}}{}^{\mathrm{op}} \to \mathcal{M}$$
defined by F, F being viewed for the time being as an object in $\mathbf{Hom}^{!}(\mathcal{A}^{\mathrm{op}}, \mathcal{M})$:

(71) $$F : \mathcal{A}^{\mathrm{op}} \to \mathcal{M}.$$

(In case $\mathcal{A} = A^\wedge$, we get the description (68) of F by taking the restriction of (71) to the subcategory A of $A^\wedge$.)

In the following formulaire, L, L' are objects in $\mathcal{A}_{\mathrm{ab}}$, F is an object in $\mathcal{A}_{\mathcal{M}}$, where $\mathcal{M}$ is an additive category stable under small inverse limits, which moreover is assumed to be a pseudotopos (i.e., admits a small set of objects generating with respect to monomorphisms, and is stable under small direct limits) in case the topos $\mathcal{A}$ is not equivalent to a category $A^\wedge$. We denote by

$$X \mapsto \mathbb{Z}^{(X)}, \quad \mathcal{A} \to \mathcal{A}_{\mathrm{ab}}$$

the abelianization functor, which in the case $\mathcal{A} = A^\wedge$ is just "componentwise abelianization", i.e.,

$$\mathbb{Z}^{(X)}(a) = \mathbb{Z}^{(X(a))} \quad \text{for } a \text{ in } A.$$

We recall that the constant sheaf $\mathbb{Z}_{\mathcal{A}}$ on $\mathcal{A}$ with value $\mathbb{Z}$ can be also described as

$$\mathbb{Z}_{\mathcal{A}} = \mathbb{Z}^{(e)}, \tag{72}$$

[p. 413] where e is the final object of $\mathcal{A}$, and that the *sections* functor on $\mathcal{A}$ is defined as

$$\Gamma_{\mathcal{A}}(F) \overset{\text{def}}{=} F(e); \tag{73}$$

in case $\mathcal{A} = A^\wedge$, this can equally be interpreted as the inverse limit functor for the functor $A^{\mathrm{op}} \to \mathcal{M}$ defined by F:

$$\Gamma_{\mathcal{A}}(F) \simeq \varprojlim_{A^{\mathrm{op}}} F(a). \tag{74}$$

We are now ready to give a basic formulaire for the operations **Hom** and Hom, and their relations to the abelianization functor and to the sections functor (i.e., to inverse limits, in case $\mathcal{A} = A^\wedge$).

$$\left\{\begin{array}{ll} \left\{\begin{array}{ll} \text{a} & \mathrm{Hom}(\mathbb{Z}^{(X)}, F) \simeq F(X) \\ \text{a')} & \mathbf{Hom}(\mathbb{Z}^{(X)}, F) \simeq (Y \mapsto F(X \times Y) : \mathcal{A}^{\mathrm{op}} \to \mathcal{M}) \end{array}\right. \\ \left\{\begin{array}{ll} \text{b} & \mathrm{Hom}(L', \mathbf{Hom}(L, F)) \simeq \mathrm{Hom}(L' \otimes L, F) \\ \text{b')} & \mathbf{Hom}(L', \mathbf{Hom}(L, F)) \simeq \mathbf{Hom}(L' \otimes L, F) \end{array}\right. \\ \left\{\begin{array}{ll} \text{c)} & \mathrm{Hom}(\mathbb{Z}_{\mathcal{A}}, F) \simeq \Gamma_{\mathcal{A}}(F) \\ \text{c')} & \mathbf{Hom}(\mathbb{Z}_{\mathcal{A}}, F) \simeq F \end{array}\right. \\ \quad\text{d)} \quad \mathrm{Hom}(L, F) \simeq \Gamma_{\mathcal{A}}\, \mathbf{Hom}(L, F) \\ \left\{\begin{array}{ll} \text{e} & \text{The functor } L \mapsto \mathrm{Hom}(L, F) : {\mathcal{A}_{\mathrm{ab}}}^{\mathrm{op}} \to \mathcal{M} \\ & \text{commutes to small inverse limits} \\ \text{e')} & \text{Similar statement as e) for} \\ & L \mapsto \mathbf{Hom}(L, F) : {\mathcal{A}_{\mathrm{ab}}}^{\mathrm{op}} \to \mathcal{A}_{\mathcal{M}}. \end{array}\right. \end{array}\right. \tag{75}$$

Comments on the formulaire (75). I have limited myself to chose canonical isomorphisms (a) to d)) and exactness properties (e) and e')) which seem to me the most relevant for what follows. Other exactness and variance properties are commutation of the functors

$$F \mapsto \operatorname{Hom}(L,F) : \mathcal{A}_{\mathcal{M}} \to \mathcal{M} \quad \text{and} \quad F \mapsto \mathbf{Hom}(L,F) : \mathcal{A}_{\mathcal{M}} \to \mathcal{A}_{\mathcal{M}}$$

to small inverse limits, and compatibility of formation of $\operatorname{Hom}(L,F)$ and $\mathbf{Hom}(L,F)$ with functors

$$u : \mathcal{M} \to \mathcal{M}'$$

commuting to small inverse limits. As for formulæ for varying topos $\mathcal{A}$, corresponding to a morphism of topoi, we'll come back upon this in a later section, in relation with the homology and cohomology invariants of maps in (Cat). Also, I am completely disregarding here compatibilities between canonical isomorphisms (surely the reader won't complain about this). All this as far as omissions are concerned.

As for the formulas included in (75), the three basic ones, including all others in a more or less formal way, are the circled ones a), e) and b). The properties a) and e) jointly can be viewed as the characterization up to canonical isomorphism, for fixed F, of the operation $\operatorname{Hom}(L,F)$, i.e., of the functor [p. 414]

$$\widetilde{F} : L \mapsto \operatorname{Hom}(L,F) : {\mathcal{A}_{\mathrm{ab}}}^{\mathrm{op}} \to \mathcal{M},$$

factoring the functor

$$F : \mathcal{A}^{\mathrm{op}} \to \mathcal{M}$$

via the abelianization functor $\mathrm{Wh}_A : X \mapsto \mathbb{Z}^{(X)}$. In terms of a), the formula b) can be viewed as essentially the definition of $\mathbf{Hom}(L,F)$ via $\operatorname{Hom}(-,F)$, more specifically we get

$$\text{(76)} \qquad \mathbf{Hom}(L,F)(X) \simeq \operatorname{Hom}(\mathbb{Z}^{(X)}, \mathbf{Hom}(L,F)) \simeq \operatorname{Hom}(\mathbb{Z}^{(X)} \otimes L, F).$$

Taking $L = \mathbb{Z}^{(X)}$ and using

$$\mathbb{Z}^{(X)} \otimes \mathbb{Z}^{(Y)} \simeq \mathbb{Z}^{(X \times Y)}$$

((17) page 389), (76) gives a'), whereas b') follows via (76) applied to both members, from associativity of the operation $\otimes$. Formula c) is the particular case of a) for $X = e$, in the same way c') follows from a'). Formula d) follows from c), b) and the relation

$$\mathbb{Z}_{\mathcal{A}} \otimes L \simeq L.$$

The exactness property e') is equivalent to the similar exactness statement for the functors

$$L \mapsto \mathbf{Hom}(L,F)(X) \simeq \operatorname{Hom}(\mathbb{Z}^{(X)} \otimes L, F),$$

for X in $\mathcal{A}$, and thus reduces to e) with L replaced by $\mathbb{Z}^{(X)} \otimes L$.

I would like now to come back to the question of "computation" of $\operatorname{Hom}(L,F)$ and $\mathbf{Hom}(L,F)$. We may for this end assume $\mathcal{A}$ to be described by a *site* A – which, in case the "topology" on A defining the site structure is the chaotic one, brings us back to the particular case $\mathcal{A} = A^\wedge$ we are mainly interested in at present. Accordingly, we'll consider the objects F in $\mathcal{A}_{\mathcal{M}}$ as being functors

$$F : A^{\mathrm{op}} \to \mathcal{M} \tag{77}$$

satisfying the standard exactness properties for sheaves (with respect to the given site structure on A). In terms of (71), this is just the composition of the functor (71) with the canonical functor

$$A \to \mathcal{A} = A^\sim, \tag{78}$$

[p. 415] associating to an object a in A the presheaf represented by it, in the most common case when this presheaf is a sheaf for any choice of a, otherwise we take the sheaf associated to it. In the first case (which we may reduce to if we prefer, by suitable choice of the site A for given topos $\mathcal{A}$) the functor (78) is fully faithful and moreover and embedding, therefore, we'll identify an object a in A with the corresponding object in $\mathcal{A}$. Thus, the description (76) of **Hom** in terms of Hom may be interpreted, from this point of view, as a formula with $X = a$ in A, i.e., as describing the sheaf $\mathbf{Hom}(L,F)$ as a functor on A^{op}. Accordingly, the question of describing the sheaf $\mathbf{Hom}(L,F)$ is reduced to the question of describing the objects $\mathrm{Hom}(L',F)$ in $\mathcal{M}$, for $L' = \mathbb{Z}^{(X)} \otimes L$. Thus, the main question here is to give a "computational" description of the object $\mathrm{Hom}(L,F)$ in $\mathcal{M}$, for L in $\mathcal{A}_{\mathrm{ab}}$ and F in $\mathcal{A}_{\mathcal{M}}$, i.e., F and L being sheaves on A

$$F : A^{\mathrm{op}} \to \mathcal{M}, \quad L : A^{\mathrm{op}} \to (\mathrm{Ab}). \tag{79}$$

The rule of the game here is to do so, using just a) in case of $X = a$ in A, and the exactness property e).

It seems most convenient here to introduce again the additive envelope $\mathrm{Add}(A)$ of the category A, which we'll assume to be small in what follows, and the canonical additive functor

$$\varepsilon_{\mathrm{ab}} : \mathrm{Add}(A) \to \mathcal{A}_{\mathrm{ab}}, \tag{80}$$

extending the functor

$$a \mapsto \mathbb{Z}^{(a)} : A \to \mathcal{A}_{\mathrm{ab}}.$$

For a given F (77), it follows from formula (75 a)) that the composition

$$\widetilde{F} \circ \varepsilon_{\mathrm{ab}}^{\mathrm{op}} : \mathrm{Add}(A)^{\mathrm{op}} \xrightarrow{\varepsilon_{\mathrm{ab}}^{\mathrm{op}}} {\mathcal{A}_{\mathrm{ab}}}^{\mathrm{op}} \xrightarrow{\widetilde{F}} \mathcal{M}$$

is just the canonical extension $\mathrm{Add}(F)$ of F to $\mathrm{Add}(A)^{\mathrm{op}}$, whose value on the general object

$$x = \bigoplus_{i \in I} \mathrm{Wh}_A(a_i) \quad (I \text{ a finite indexing set})$$

of $\mathrm{Add}(A)$ (where $\mathrm{Wh}_A(a) = \mathbb{Z}^{(a)}$ as an object in $\mathrm{Add}(A) \subset A^\wedge_{\mathrm{ab}}$) is just

$$\mathrm{Add}(F)(x) = \prod_{i \in I} F(a_i). \tag{81}$$

Now, it is easily checked that for any object L in $\mathcal{A}_{\mathrm{ab}}$, i.e., any sheaf $L : A^{\mathrm{op}} \to (\mathrm{Ab})$, we have a canonical isomorphism in $\mathcal{A}_{\mathrm{ab}}$

$$L \xleftarrow{\sim} \varinjlim_{(x,u) \text{ in } \mathrm{Add}(A)_{/L}} \varepsilon_{\mathrm{ab}}(x) \tag{82}$$

(compare with the similar isomorphism on page 403). Using (75 e), we deduce from this the expression [p. 416]

$$\mathrm{Hom}(L,F) = \widetilde{F}(L) \simeq \varinjlim_{(x,u) \text{ in } \mathrm{Add}(A)_{/L}} \mathrm{Add}(F)(x), \tag{83}$$

which in an evident way is functorial in L for variable L.

This is about the best which can be done in general, it seems to me, by way of "computational" expression of $\mathrm{Hom}(L,F)$ in terms of F and L given as in (79). Of course, the symbol $\mathrm{Add}(A)_{/L}$ is relative to the canonical functor $\varepsilon_{\mathrm{ab}}$ (80), which is a full embedding in case the site structure on A is the chaotic one, i.e., $\mathcal{A} = A^{\wedge}$. In computational terms, this category is rather explicit, an object of the category is just a pair

$$(x,u) = \big((a_i)_{i\in I}, (u_i)_{i\in I}\big)$$

where I is a finite indexing set, $(a_i)_{i\in I}$ a family of objects of A, and for i in I, u_i is an element of $L(a_i)$ – I'll leave to the reader the description of maps between such objects. The value of $\mathrm{Add}(F)(x)$ is given by (81) above.

Remark. The expression (83) of $\mathrm{Hom}(L,F) = \widetilde{F}(L)$ makes sense, provided only the additive category $\mathcal{M}$ is stable under small inverse limits, without having to assume that $\mathcal{M}$ be a pseudotopos. This makes us suspect that the functor

$$\mathcal{P} \mapsto \mathcal{P} \circ \mathrm{Wh} : \mathbf{Hom}^!(\mathcal{A}_{\mathrm{ab}}^{\mathrm{op}}, \mathcal{M}) \to \mathcal{A}_{\mathcal{M}}$$

is an equivalence ((13) p. 388) without this extra assumption, provided $\mathcal{A}$ is an actual topos (not only a pseudotopos as in loc. cit.). Indeed, we get a reasonable candidate for a quasi-inverse functor

$$F \mapsto \widetilde{F} : \mathcal{A}_{\mathcal{M}} \to \mathbf{Hom}^!(\mathcal{A}_{\mathrm{ab}}^{\mathrm{op}}, \mathcal{M}).$$

The only point still to check, with (83) defining $\widetilde{F}$ for given F in $\mathcal{A}_{\mathcal{M}}$, is that we get a functorial isomorphism

$$\widetilde{F}(\mathbb{Z}^{(a)}) \simeq F(a)$$

for a in A. In case $\mathcal{A} = A^{\wedge}$, this follows from the fact that (80) is fully faithful, hence $\mathrm{Add}(A)_{/L}$ for $L = \mathbb{Z}^{(a)}$ admits a final object – hence the limit (83) is the value of $\mathrm{Add}(F)$ on the latter, namely $F(a)$.

I feel the little program on the **Hom** and Hom operations, as contemplated on page 412, is by now completed; all we've got to do still is to dualize to get corresponding results for $*$ and $\circledast$. It's just a matter [p. 417] of essentially copying the formulaire (75), which I'll do for the sake of getting more familiar with the more unusual operations $*$ and $\circledast$. Now of course, we'll have to restrict to the case $\mathcal{A} = A^\wedge$, and use the interpretation (68) of the data F, L' as functors on A^{op} and on A with values in $\mathcal{M}$ and (Ab) respectively, where now $\mathcal{M}$ is an additive category stables under small *direct* limits. By duality, the "sections" or "inverse limits" functor $\varprojlim_{A^{\mathrm{op}}}$ (or "cointegration") is replaced by the direct limit functor $\varinjlim_{A^{\mathrm{op}}}$ (or "integration"). With this in mind, we get the following transcription of (75):

$$
(84)\qquad
\begin{cases}
\begin{cases}
\text{a)} & F * \mathbb{Z}^{(b^{\mathrm{op}})} \simeq F(b) \quad \text{for any } b \text{ in } A \text{, hence } b^{\mathrm{op}} \text{ in } B\\
\text{a')} & F \circledast \mathbb{Z}^{(b^{\mathrm{op}})} \simeq (a \mapsto F(a \vee b) \simeq \varinjlim_{(x,\alpha) \text{ in } {}_{a\vee b\backslash}A} F(x))
\end{cases}\\
\begin{cases}
\text{b)} & (F * L') * L'' \simeq F * (L' \otimes L'')\\
\text{b')} & (F \circledast L') * L'' \simeq F \circledast (L' \otimes L'')
\end{cases}\\
\begin{cases}
\text{c)} & F * \mathbb{Z}_{B^\wedge} \simeq \varinjlim_{A^{\mathrm{op}}} F\\
\text{c')} & F \circledast \mathbb{Z}_{B^\wedge} \simeq F
\end{cases}\\
\quad\text{d)} \quad F * L' \simeq \varinjlim_{A^{\mathrm{op}}} F \circledast L'\\
\begin{cases}
\text{e)} & \text{The functor } L' \mapsto F * L' : B^\wedge_{\mathrm{ab}} \to \mathcal{M}\\
& \text{commutes to small direct limits}\\
\text{e')} & \text{Similar statement as e) for}\\
& L' \mapsto F \circledast L' : B^\wedge_{\mathrm{ab}} \to A^\wedge_{\mathcal{M}}.
\end{cases}
\end{cases}
$$

Comments. This formulaire doesn't look wholly symmetric to (75), due to the fact that we gave (75) in a somewhat more general context than topoi of the type $A^\wedge$ only. This accounts for the letter X or Y in (75) (designating there an arbitrary object of $A^\wedge$) being replaced by a small letter a or b (designating objects in A), which allows the dualization to be done. A slight trouble then occurs when A is not stable under binary products $a \times b$, these products are only in $A^\wedge$ not in A, which accounts for the slightly more complicated formula a') of (84) in comparison to (75 a')), whose more explicit form, in the present context of data as in (68), would be

$$
(85)\qquad \mathbf{Hom}(\mathbb{Z}^{(a)}, F) \simeq \big(b \mapsto F(a \times b) \simeq \varprojlim_{(x,\alpha) \text{ in } A_{/a\times b}} F(x)\big).
$$

Accordingly, the symbol $a \vee b$ ("sum") in (84 a')) denotes the element $(a^{\mathrm{op}} \times b^{\mathrm{op}})^{\mathrm{op}}$ of $(B^\wedge)^{\mathrm{op}}$ and can be identified with the *sum* of a and b in
[p. 418] the category A whenever the sum exists in A. Accordingly, the category ${}_{a\vee b\backslash}A$, dual to $B_{/a^{\mathrm{op}}\times b^{\mathrm{op}}}$, can be described as

$$
(86)\qquad {}_{a\vee b\backslash}A = \text{category of all triples } (x,u,v) \text{, with } x \text{ in } A \text{ and } u : a \to x,\ v : b \to x \text{ maps in } A,
$$

the maps in this category from (x,u,v) to (x',u',v') being just maps $x \to x'$ "compatible" with the pairs $\alpha = (u,v)$ and $\alpha' = (u',v')$ in the obvious way.

Remarks. 1) An interesting particular case (although admittedly a little strange looking in our modelizing context!) is the one when A is an additive category, hence stable under both binary sum and product operation, the two operations being canonically isomorphic, and written as $a \oplus b$. In this case, comparison of (85) and (86) shows that for a given object a in A, hence a^{op} in B, the operation $\mathrm{Hom}(\mathbb{Z}^{(a)}, -)$ on $A^{\wedge}_{\mathcal{M}}$ is canonically isomorphic to the operation $- \circledast \mathbb{Z}^{(a^{\mathrm{op}})}$. This immediately extends to a canonical isomorphism

$$(87) \qquad \mathbf{Hom}(L, F) \simeq F \circledast \check{L} \quad \text{for } L \text{ in } \mathrm{Add}(A) \subset A^{\wedge}_{\mathrm{ab}},\ F \text{ in } A^{\wedge}_{\mathcal{M}},$$

where we have denoted by

$$(88) \qquad L \mapsto \check{L} : \mathrm{Add}(A)^{\mathrm{op}} \xrightarrow{\approx} \mathrm{Add}(A^{\mathrm{op}}) = \mathrm{Add}(B)$$

the canonical antiequivalence between $\mathrm{Add}(A)$ and $\mathrm{Add}(B)$. In case A is the final category, namely an additive category reduced to the zero object, and if we take moreover $\mathcal{M} = (\mathrm{Ab})$, (87) is the familiar formula of linear algebra, valid when L is a free $\mathbb{Z}$-module of finite type. It should be noted that A being stable under binary products, it follows that $\mathrm{Add}(A)$ is stable under tensor products, and similarly for $\mathrm{Add}(B)$, and that the equivalence (88) is compatible with tensor products. The relation (87) is about the only relationship I could think of between the two types of operations upon a given category $A^{\wedge}_{\mathcal{M}}$.

2) There are still two other, more trivial operations on a category $A^{\wedge}$, of a similar nature to the two operations Hom and $*$ considered so far. The more familiar one is componentwise tensor product

$$(89) \qquad (L, F) \mapsto L \otimes F : A^{\wedge}_{\mathrm{ab}} \times A^{\wedge}_{\mathcal{M}} \to A^{\wedge}_{\mathcal{M}},$$

defined by

$$L \otimes F\,(a) = L(a) \otimes F(a),$$

where the second member denotes external tensor product of the abelian group $L(a)$ with the object $F(a)$ of $\mathcal{M}$ (defined when $\mathcal{M}$ is additive and stable under small direct limits). The other, deduced from (89) by duality [p. 419]

$$(90) \qquad (L', F) \mapsto \mathbf{Hom}(L', F) : B^{\wedge}_{\mathrm{ab}} \times A^{\wedge}_{\mathcal{N}} \to A^{\wedge}_{\mathcal{N}}$$

is defined when the additive category $\mathcal{N}$ is stable under small inverse limits, and can be equally described as taking external Hom's componentwise

$$\mathbf{Hom}(L', F)(a) = \mathrm{Hom}(L'(a), F(a)).$$

These operations make sense too when $A^{\wedge}$ is replaced by an arbitrary topos $\mathcal{A}$, $B^{\wedge}_{\mathrm{ab}}$ being replaced by the category of abelian cosheaves on $\mathcal{A}$. It doesn't seem worthwhile here to dwell on them, as they don't seem to be so relevant for the homology and cohomology formalism we want to develop in the next sections. I like to point out, though, that in the cohomology formalism of ringed topoi the tensor product operation

(89) and the derived operation $\overset{\mathrm{L}}{\otimes}$ on the relevant derived categories $\mathrm{D}_\bullet$ play an important role, and it is likely therefore that in a more extensive development of the homology and cohomology formalism within the context of topoi $A^\wedge$ and maps in (Cat), the same will hold for the dual operation (90) too.

The reader who may feel confused by the manifold use of the symbol **Hom** should notice that there is no possibility of confusion reasonably between (90) and (65) (p. 409), as the argument L' in (90) is in $B^\wedge_{\mathrm{ab}}$, whereas the argument L in (65) is in an altogether different category $A^\wedge_{\mathrm{ab}}$. In the case when A is the final category say, hence $A = B$, and a confusion might arise, the two operations turn out to be actually the same (up to canonical isomorphism). A similar remark applies to the fear of confusion between the kindred operations $\otimes$ and $\circledast$. I daresay I devoted a considerable amount of attention on terminology and notation around the abelianization story – and it does seem that a pretty coherent formalism is emerging indeed.

17.8.

107 *Review* (4)*: Case of general ground ring k.*

F) Extension of ground ring from $\mathbb{Z}$ to k (k-linearization). I would like still to make a quick review of the main facts and formulas of the last two sections, replacing throughout the ground ring $\mathbb{Z}$ by an arbitrary *commutative* ring k, and additive categories $\mathcal{M}$ and additive functors between these, by k-additive categories and k-additive functors. This will allow us to check that the conceptual and notational set-up we got so far extends smoothly to k-linearization.

[p. 420] Let's recall that a *k-additive category* $\mathcal{M}$ is an additive category endowed with the extra structure given by a homomorphism of commutative rings

$$k \to \mathrm{End}(\mathrm{id}_{\mathcal{M}}), \tag{91}$$

where the second member denotes the (commutative) ring of all endomorphisms of the identity functor of $\mathcal{M}$ to itself. Defining accordingly the notion of *k-additive functor* between two k-additive categories $\mathcal{M}$, $\mathcal{M}'$, we'll denote by

$$\mathbf{Hom}_k(\mathcal{M},\mathcal{M}') \subset \mathbf{Hom}(\mathcal{M},\mathcal{M}') \tag{92}$$

the full subcategory of $\mathbf{Hom}(\mathcal{M},\mathcal{M}')$ made up with such functors. Thus, we get a canonical fully faithful inclusion

$$\mathbf{Hom}_k(\mathcal{M},\mathcal{M}') \hookrightarrow \mathbf{Hom}_{\mathbb{Z}}(\mathcal{M},\mathcal{M}') \overset{\mathrm{def}}{=} \mathrm{Homadd}(\mathcal{M},\mathcal{M}'). \tag{93}$$

Wed defined accordingly the categories $\mathbf{Hom}_{k!}$, $\mathbf{Hom}^!_k$ as full subcategories of (92), and the category

$$\mathbf{Hom}_k(\mathcal{P},\mathcal{Q};\mathcal{M}) \subset \mathbf{Biadd}(\mathcal{P},\mathcal{Q};\mathcal{M})$$

the full subcategory of $\mathbf{Hom}(\mathcal{P}\times\mathcal{Q},\mathcal{M})$ made up with *k-bilinear* functors, namely functors k-additive in each argument (in case $k = \mathbb{Z}$, this is the

[Actually, it was previously denoted by just $\mathbf{Hom}(\mathcal{P},\mathcal{Q};\mathcal{M})$. . .]

category denoted previously by **Biadd**), and similarly for the notations $\mathbf{Hom}_{k!!}$ and $\mathbf{Hom}^{!!}_{k}$.

It should be noted that for a given additive category $\mathcal{M}$, there is a "best" choice for endowed it with a k-linear structure, in such a way that any k'-linear structure just corresponds to "ground ring restriction" with respect to suitable (well-defined) ring homomorphism

$$k' \to k;$$

we just take the "tautological" linear structure with

$$k = \mathrm{End}(\mathrm{id}_{\mathcal{M}}),$$

and (91) the identity.

If A is any small category, we'll denote by

(94) $$A^\wedge_k = A^\wedge_{k\text{-Mod}} \simeq \mathbf{Hom}(A^{\mathrm{op}}, (k\text{-Mod}))$$

the category of objects in $A^\wedge$ endowed with a structure of k-module, i.e., the category of presheaves on A with values in the category (k-Mod) of k-modules (in the given basic universe $\mathcal{U}$). This is of course a k-additive category, which for $k = \mathbb{Z}$ reduces to the category of additive presheaves [p. 421]
on A:

$$A^\wedge_{\mathbb{Z}} \overset{\text{def}}{=} A^\wedge_{\mathrm{ab}}.$$

We have, for a homomorphism of commutative rings

$$k \to k',$$

a corresponding functor between additive topoi

(95) $$A^\wedge_k \to A^\wedge_{k'}, \quad F \mapsto F \otimes_k k' = (a \mapsto F(a) \otimes_k k'),$$

by which we may interpret if we wish, in a rather evident way the k'-linear topos $A^\wedge_{k'}$ as deduced from the k-linear one $A^\wedge_k$ by "ground ring extension" $k \to k'$, namely as the solution of a 2-universal problem with respect to categories $\mathbf{Hom}_{k!}(A^\wedge_k, \mathcal{M})$, where $\mathcal{M}$ is a k'-additive category stable under small direct limits. The k-abelianization functor

(96) $$A^\wedge \to A^\wedge_k, \quad X \mapsto k^{(X)} \quad \big(\simeq (a \mapsto k^{(X(a))}) \big)$$

or $\mathrm{Wh}_{A^\wedge, k}$, is defined as the composition

$$A^\wedge \to A^\wedge_{\mathbb{Z}} = A^\wedge_{\mathrm{ab}} \to A^\wedge_k,$$

where the first functor is the familiar abelianization $X \mapsto \mathbb{Z}^{(X)}$, and the second is ground ring extension for $\mathbb{Z} \to k$. If $\mathcal{M}$ is any k-additive category stable under small direct limits, (96) gives rise to a functor which is an equivalence of categories $F \mapsto (X \mapsto F(k^{(X)}))$

(97) $$\mathbf{Hom}_{k!}(A^\wedge_k, \mathcal{M}) \xrightarrow{\approx} \mathbf{Hom}_{!}(A^\wedge, \mathcal{M}) \quad (\xrightarrow{\approx} \mathbf{Hom}(A, \mathcal{M})),$$

where the second equivalence is the familiar one of prop. 3 (p. 394), independent of any abelian assumptions. Dually, we get an equivalence

$$\mathbf{Hom}^!_k((A^\wedge_k)^{\mathrm{op}}, \mathcal{M}) \xrightarrow{\approx} \mathbf{Hom}((A^\wedge)^{\mathrm{op}}, \mathcal{M}) \quad (\xrightarrow{\approx} \mathbf{Hom}(A^{\mathrm{op}}, \mathcal{M})), \tag{98}$$

where $\mathcal{M}$ is any k-additive category stable under small inverse limits. From (97) (98) and replacing in (98) A by the dual category $B = A^{\mathrm{op}}$, and assuming the k-additive category $\mathcal{M}$ is stable under both types of small limits, we get the duality equivalence

$$\mathbf{Hom}_{k!}(A^\wedge_k, \mathcal{M}) \approx \mathbf{Hom}^!_k(B^\wedge_k, \mathcal{M}) \quad (\approx \mathbf{Hom}(A, \mathcal{M})). \tag{99}$$

This may be viewed as giving two alternative descriptions, by the two members of (99), of the category

$$B^\wedge_{\mathcal{M}} = \mathbf{Hom}(B^{\mathrm{op}} = A, \mathcal{M})$$

of $\mathcal{M}$-valued presheaves on B (defined without any use of the k-additive [p. 422] structure of $\mathcal{M}$). The left-hand side interpretation (99), via $\mathcal{M}$-valued k-additive cosheaves on the k-additive topos $A^\wedge_k$, gives rise to the operations $*_k$ and $\circledast_k$ of $A^\wedge_k$ upon $B^\wedge_{\mathcal{M}}$ (operations previously denoted by $*$ and $\circledast$ when $k = \mathbb{Z}$ and no confusion would arise from dropping subscripts), and similarly the interpretation by right-hand side of (99), via $\mathcal{M}$-valued k-additive sheaves on the k-additive topos $B^\wedge_k$, gives rise to the operations Hom_k and $\mathbf{Hom}_k$ of $B^\wedge_k$ upon $B^\wedge_{\mathcal{M}}$. Replacing in this comment A by B, hence $B^\wedge$ by $A^\wedge$, namely in terms of operations upon the category of $\mathcal{M}$-valued sheaves on the topos $A^\wedge$ (or $\mathcal{M}$-valued presheaves on A), we get the mutually dual pair of operations

$$\begin{aligned} (F, L') &\mapsto F *_k L' : A^\wedge_{\mathcal{M}} \times B^\wedge_k \to \mathcal{M}, \\ (F, L') &\mapsto F \circledast_k L' : A^\wedge_{\mathcal{M}} \times B^\wedge_k \to A^\wedge_{\mathcal{M}} \end{aligned} \tag{100}$$

and

$$\begin{aligned} (L, F) &\mapsto \mathrm{Hom}_k(L, F) : A^\wedge_k \times A^\wedge_{\mathcal{M}} \to \mathcal{M}, \\ (L, F) &\mapsto \mathbf{Hom}_k(L, F) : A^\wedge_k \times A^\wedge_{\mathcal{M}} \to A^\wedge_{\mathcal{M}}. \end{aligned} \tag{101}$$

The operations (100) are ruled by formulaire (84) (with subscripts k added), whereas the operations (101) are ruled by formulaire (75) with subscripts (see moreover for the latter comments on page 417, and formula (85) for (75 a')); they are valid provided the additive category is stable under small direct resp. inverse limits. Moreover, we get a "computational" expression of $\mathrm{Hom}_k(L, F)$ by a formula extending (83) which we'll still have to write down, and correspondingly for $F *_k L'$ (by a dual formula, which we forgot to include in the previous section). To do so, we have to introduce still

$$\mathrm{Add}_k(A) \subset A^\wedge_k, \tag{102}$$

the k-additive envelope of A, which may be described (beside by the familiar 2-universal property in the context of k-additive categories and functors from A into these) as the full subcategory of $A^\wedge_k$ generated by finite sums of objects of the type $k^{(a)}$ with a in A – i.e., the general object of $\mathrm{Add}_k(A)$ may be written

$$\bigoplus_{i\in I} k^{(a_i)},$$

where $(a_i)_{i\in I}$ is any finite family of objects of A. When the finiteness condition on I is dropped, we get a larger full subcategory

$$\mathrm{Addinf}_k(A) \subset A^\wedge_k, \tag{103}$$

which may also be interpreted as "the" solution of the 2-universal problem of sending A into categories which are k-additive and moreover infinitely additive, i.e., stable under small direct sums. Enlarging the subcategories (102) and (103) of $A^\wedge_k$ by adjoining all objects of $A^\wedge_k$ isomorphic to direct factors of objects in the considered subcategory, we get to (strictly) full subcategories of $A^\wedge_k$ containing the latter, which may be interpreted as being just the subcategory $\mathrm{Proj}(A^\wedge_k)$ of *projective* objects of $A^\wedge_k$ when starting with (103), and as the subcategory $\mathrm{UlProj}(A^\wedge_k)$ of *ultraprojective* objects, namely objects projective and of finite presentation, when starting with (102). These may be equally interpreted as the abstract Karoubi envelops of the categories (103) and (102), deduced from these formally by adjoining images (=coimages) of projectors (or equivalently, as 2-universal solutions of the 2-universal problem of sending the given category (103) or (102) into "karoubian categories", namely categories stable under images (=coimages) of projectors, with maps between these being functors commuting to those images or coimages of projectors): [p. 423]

$$\mathrm{Proj}(A^\wedge_k) \approxeq \mathrm{Kar}(\mathrm{Addinf}_k(A)), \quad \mathrm{UlProj}(A^\wedge_k) \approxeq \mathrm{Kar}(\mathrm{Add}_k(A)). \tag{104}$$

Accordingly, these two categories may be equally described, directly in terms of A, as the solutions of the two 2-universal problems, obtained from mapping A into k-additive karoubian categories, which in the first case (corresponding to $\mathrm{Proj}(A^\wedge_k)$) are moreover assume infinitely additive.

To sum up the situation, we get in $A^\wedge_k$ a diagram of four remarkable full subcategories (102), (103), (104), which may be interpreted (as well as $A^\wedge_k$ itself) as the solutions of five corresponding "k-additive" 2-universal problems, in terms of sending A into k-additive categories satisfying suitable extra exactness assumptions (namely being karoubian for the two categories in (104), being infinitely additive for the two categories $\mathrm{Addinf}_k(A)$ and its Karoubi envelope $\mathrm{Proj}(A^\wedge_k)$, and being stable for small direct limits in case of $A^\wedge_k$). Including equally the non-additive categories A and $A^\wedge$ and the functors $A \to \mathrm{Add}_k(A)$, $A^\wedge_k \to A^\wedge$, we get a seven term diagram of canonical functors between categories of presheaves upon

A:

$$
(105)\qquad \left\{
\begin{array}{ccccc}
A & & & & \\
\downarrow & & & & \\
\mathrm{Add}_k(A) & \hookrightarrow & \mathrm{UlProj}(A^\wedge_k) \approxeq \mathrm{KarAdd}_k(A) & & \\
\downarrow & & \downarrow & & \\
\mathrm{Addinf}_k(A) & \hookrightarrow & \mathrm{Proj}(A^\wedge_k) \approxeq \mathrm{KarAddinf}_k(A) & \hookrightarrow & A^\wedge_k \\
 & & & & \downarrow \\
 & & & & A^\wedge \quad ,
\end{array}
\right.
$$

where the five categories in the two intermediate lines are k-additive as well as all functors between them in the diagram, which are moreover fully faithful. [p. 424] For any k-additive category $\mathcal{M}$ stable under small direct limits, taking cosheaves on $A^\wedge$ with values in $\mathcal{M}$, and their restrictions to the six other categories in the diagram (105), we get a transposed seven term diagram as follows, part of which reduces to the four term diagram (58) (p. 406) in case $k = \mathbb{Z}$:

$$
(106)\qquad \left\{
\begin{array}{ccccc}
\mathbf{Hom}_!(A^\wedge, \mathcal{M}) & & & & \\
\approx\downarrow & & & & \\
\mathbf{Hom}_{k!}(A^\wedge_k, \mathcal{M}) & \xrightarrow{\approx} & \mathbf{Homaddinfkar}_k(\mathrm{Proj}(A^\wedge_k), \mathcal{M}) & \xrightarrow{\approx} & \mathbf{Homaddinf}_k(\mathrm{Addinf}_k(A), \mathcal{M}) \\
 & & \downarrow\approx & & \downarrow\approx \\
 & & \mathbf{Homaddkar}_k(\mathrm{UlProj}(A^\wedge_k), \mathcal{M}) & \xrightarrow{\approx} & \mathbf{Homadd}_k(\mathrm{Add}_k(A), \mathcal{M}) \\
 & & & & \downarrow\approx \\
 & & & & \mathbf{Hom}(A, \mathcal{M}) \qquad ,
\end{array}
\right.
$$

where the meaning of the symbols used (such as index k, suffixes "add" or "addinf" and "kar") for qualifying **Hom** and denoting various full subcategories of **Hom** categories, is clear from the explanations given previously. Replacing A by B and $\mathbf{Hom}_!(A^\wedge, \mathcal{M})$ by $\mathbf{Hom}^!(B^\wedge, \mathcal{M})$, we get a diagram "dual" to (106) (containing the five-term diagram (62) (p. 408) in case $k = \mathbb{Z}$), which we'll not write out here, valid for any k-additive category $\mathcal{M}$ stable under small inverse limits. When $\mathcal{M}$ is a k-additive category stable under both types of small limits, then the last term of the diagram (106) is equal to the last term of the dual one, hence a system of fourteen mutually equivalent categories (compare p. 408, when we considered ten among them only!), expressing as many ways for interpreting the notion of an $\mathcal{M}$-valued copresheaf on A, i.e., an object of $\mathbf{Hom}(A, \mathcal{M})$ (which is one among the fourteen…).

Let's comment a little on the significance of the various five k-additive categories appearing in (105). The largest one $A^\wedge_k$ is there precisely as the all-encompassing category of k-additive presheaves, where to carry through all kinds of k-linear constructions between presheaves on A. The significance of the (second largest) subcategory $\mathrm{Proj}(A^\wedge_k)$, made up with all projective objects of $A^\wedge_k$, comes mainly from homological algebra and emphasis upon replacing objects of $A^\wedge_k$ by projective resolutions; these are chain complexes in $\mathrm{Proj}(A^\wedge_k)$, which may be viewed as being defined (by any given object in $A_k^\wedge$) "up to chain homotopy". More sweepingly still, we get from general principles the canonical equivalence of categories

$$(*)\qquad \mathrm{D}^-(A_k^\wedge) \xleftarrow{\approx} \mathrm{K}^-(\mathrm{Proj}(A_k^\wedge)),$$

where D^- designates the "derived category bounded from above" of a given *abelian* category (defined in terms of differential operators with degree $+1$, and quasi-isomorphisms between complexes with degrees bounded from above), whereas K^- designates localization of the category of differential complexes with degrees bounded from above of a given *additive* category, localization being taken with respect to homotopisms. [p. 425]

As any object of $\mathrm{Proj}(A_k^\wedge)$ is a direct factor of an object in $\mathrm{Addinf}_k(A)$, and hence, any object in $A_k^\wedge$ is isomorphic to a quotient of an object in $\mathrm{Addinf}_k(A)$, it follows again from general principles that the categories in (*) are equally equivalent to $\mathrm{K}^-(\mathrm{Addinf}_k(A))$, hence

$$(107)\qquad \mathrm{D}^-(A_k^\wedge) \xleftarrow{\approx} \mathrm{K}^-(\mathrm{Proj}(A_k^\wedge)) \xleftarrow{\approx} \mathrm{K}^-(\mathrm{Addinf}_k(A)).$$

The advantage of $\mathrm{Addinf}_k(A)$ over $\mathrm{Proj}(A_k^\wedge)$ is that its objects, and maps between objects, are more readily described in computational terms, just working with small direct sums of objects of the type $k^{(a)}$ (with a in A), and corresponding matrices, with entries in free k-modules $k^{(\mathrm{Hom}(a,b))}$. Thus, if we call *cointegrator* (with coefficients in k) *for A* any projective resolution of the constant presheaf k_A with value k, and denote such object by L_k^A, we may view L_k^A as an object determined up to unique isomorphism, either in $\mathrm{K}^-(\mathrm{Proj}(A_k^\wedge))$, or in $\mathrm{K}^-(\mathrm{Addinf}_k(A))$ – and it is the latter interpretation which looks the most convenient. Objects in the first category, namely complexes with degrees bounded from above and projective components, which happen to be in the first (i.e., components are in $\mathrm{Addinf}_k(A)$, i.e., are direct sums of objects of the type $k^{(a)}$) may be called "*quasi-special*" (extending the terminology previously used for cointegrators and integrators, in case $k = \mathbb{Z}$). We'll call them *special* if the components are even in $\mathrm{Add}_k(A)$. The category $\mathrm{Add}_k(A)$ and its Karoubi envelope $\mathrm{UlProj}(A_k^\wedge)$ may be viewed both as embodying *finiteness conditions*, and similarly for the two corresponding K^- categories, which are of course equivalent:

$$(108)\qquad \mathrm{K}^-(\mathrm{UlProj}(A_k^\wedge)) \xleftarrow{\approx} \mathrm{K}^-(\mathrm{Add}_k(A)),$$

and presumably the canonical functor from (108) to (107) is fully faithful, under suitable coherence conditions at any rate…

[p. 426]

22.8.

108

Review (5)*: Homology and cohomology* (*absolute case*).

Since last Monday, namely for about one week, I have been mainly taken by a rather dense sequence of encounters and events, the center of which has been the unexpected news of my granddaughter Ella's death at the age of nine, by a so-called health accident. I resumed some mathematical pondering last night. Today, I got a short letter from Ronnie Brown, mainly with the announcement of the loss of his son Gabriel, twenty years old, which occurred about the same time by a climbing accident. It is a good thing that Ronnie felt like telling me in a few words about this, while we have never yet seen each other and our letters so far have been restricted to mathematics, with maybe sometimes some personal comments about his or my own involvement in mathematics. It is through these, surely, that a mutual sympathy has come into being, not merely motivated by a common interest in mathematics – and this sympathy I feel has been the main force giving life to our correspondence while mathematically speaking more than once it has been rather a "dialogue de sourds". (This is due mainly to my illiteracy homotopy in theory, and to my reluctance to get really involved in any "technical" matters, until I am really forced to by what I am just doing.)

["dialogue of the deaf"]

I want now to go on with the overall review on "abelianization" and its relation to the homology and cohomology formalism for small categories, serving as models for homotopy types.

G) Homology and cohomology (absolute case). My aim is to give a perfectly dual treatment of cohomology and homology, which is one main reason why I have to take as coefficients for both, not merely usual abelian presheaves on a given small category A, or sheaves of k-modules for a given ring k, but more generally sheaves with values in any abelian category $\mathcal{M}$, stable under small direct or inverse limits (according as to whether we are interesting in taking homology, or cohomology invariants). It will then turn out that homology of A for $\mathcal{M}$-valued presheaves (or complexes of such) is "the same" as cohomology of the dual category B, with coefficients in the corresponding $\mathcal{M}^{\mathrm{op}}$-valued ones.

As I am a lot more familiar with cohomology, it is by this I'll begin again. Here, as in the case of an arbitrary topos $\mathcal{X}$, the cohomology invariants $\mathrm{H}^i(\mathcal{X},F)$ with values in an abelian sheaf F may be viewed as being just the invariants $\mathrm{Ext}^i(\mathbb{Z}_{\mathcal{X}},F)$ in the category of all abelian sheaves, where $\mathbb{Z}_{\mathcal{X}}$ is the constant sheaf on $\mathcal{X}$ with value $\mathbb{Z}$. The similar fact holds when F is any sheaf of modules over a sheaf of rings $\mathcal{O}_{\mathcal{X}}$ on $\mathcal{X}$, with $\mathbb{Z}_{\mathcal{X}}$ being replaced by $\mathcal{O}_{\mathcal{X}}$ in the interpretation above:

[p. 427]

$$\mathrm{H}^i(\mathcal{X},F) \simeq \mathrm{Ext}^i_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}},F),$$

which is often quite useful in the cohomology formalism. We are going to restrict here to the case of a constant sheaf of rings, defined by a fixed commutative ring k, which will allow us to play around as announced with the duality relation between A and $B = A^{\mathrm{op}}$, provided moreover we take more general coefficients still, namely (pre)sheaves with value in a given k-additive category $\mathcal{M}$ stable under the relevant limits. (Presumably, the case of a locally constant commutative sheaf of rings could be dealt with too, but we'll not dive into this here!) Another important (and by now familiar?) conceptual point is that, rather than the Ext^i's which give only partial information, we are interested in the object they come from (as the "cohomology objects"), namely the objects $\mathrm{RHom}_{\mathcal{O}_{\mathcal{X}}}(L, F)$ in a suitable derived category. In the present case when $\mathcal{X}$ is the topos associated to the small category A, hence the category of $\mathcal{O}_{\mathcal{X}}$-modules has sufficiently many projective (namely direct sums of sheaves of the type $\mathcal{O}_{\mathcal{X}}^{(a)}$ with a in A), the $\mathrm{RHom}_{\mathcal{O}_{\mathcal{X}}}(L, F)$ may be computed, taking a projective resolution $L_\bullet$ of L, by the formula

$$\mathrm{RHom}_{\mathcal{O}_{\mathcal{X}}}(L, F) \simeq \mathrm{Hom}^\bullet_{\mathcal{O}_{\mathcal{X}}}(L_\bullet, F)$$

(an isomorphism in $\mathrm{D}^+(k\text{-Mod})$ say), and similarly when replacing L and F by arguments $L_\bullet$ and $F^\bullet$ in D^- and D^+ of the category of $\mathcal{O}_{\mathcal{X}}$-modules. As a result, we get a pairing, computable here using projective resolutions of the argument $L_\bullet$:

$$(*) \quad (L_\bullet, F^\bullet) \mapsto \mathrm{RHom}_{\mathcal{O}_{\mathcal{X}}}(L_\bullet, F^\bullet) : \mathrm{D}^-(\mathcal{O}_{\mathcal{X}}) \times \mathrm{D}^+(\mathcal{O}_{\mathcal{X}}) \to \mathrm{D}^+(k\text{-Mod}),$$

where k is a commutative ring and $\mathcal{O}_{\mathcal{X}}$ is endowed with a structure of k-algebra. Using $\mathbf{Hom}_{\mathcal{O}_{\mathcal{X}}}$ and its total derived functor, we get likewise

$$(**) \quad (L_\bullet, F^\bullet) \mapsto \mathrm{R}\mathbf{Hom}_{\mathcal{O}_{\mathcal{X}}}(L_\bullet, F^\bullet) : \mathrm{D}^-(\mathcal{O}_{\mathcal{X}}) \times \mathrm{D}^+(\mathcal{O}_{\mathcal{X}}) \to \mathrm{D}^+(\mathcal{O}_{\mathcal{X}}),$$

with

$$\mathrm{R}\mathbf{Hom}_{\mathcal{O}_{\mathcal{X}}}(L_\bullet, F^\bullet) \simeq \mathbf{Hom}^{\bullet\bullet}_{\mathcal{X}}(\mathcal{L}_\bullet, F^\bullet),$$

where $\mathcal{L}_\bullet$ is a projective resolution of $L_\bullet$, and $\mathbf{Hom}^{\bullet\bullet}$ stands for the simple complex associated to the double complex obtained by taking [p. 428]
Hom's componentwise (and we have the similar formula of course for the Hom's and RHom's non-bold-faced).

Taking $L_\bullet = \mathcal{O}_{\mathcal{X}}$ (or any resolution of $\mathcal{O}_{\mathcal{X}}$), the RHom invariant (*) reduces to the total derived functor $\mathrm{R}\Gamma$ of the sections functor

$$\text{(i)} \qquad \mathrm{RHom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, F^\bullet) \simeq \mathrm{R}\Gamma_{\mathcal{X}}(F^\bullet)$$

(whereas $\mathrm{R}\mathbf{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, F^\bullet) \simeq F^\bullet$ of course), which in turn allows to give the following familiar expression of RHom in terms of R**Hom**:

$$\text{(ii)} \qquad \mathrm{RHom}_{\mathcal{O}_{\mathcal{X}}}(L_\bullet, F^\bullet) \simeq \mathrm{R}\Gamma_{\mathcal{X}}(\mathrm{R}\mathbf{Hom}_{\mathcal{O}_{\mathcal{X}}}(L_\bullet, F^\bullet)),$$

coming from the similar isomorphism $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}} \simeq \Gamma_{\mathcal{X}}\, \mathbf{Hom}_{\mathcal{O}_{\mathcal{X}}}$. All this is standard cohomology formalism, valid on an arbitrary ringed topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, except for the possibility of computing RHom and R**Hom** by taking *projective resolutions of the first argument* (rather than injective ones of the second), which is special to the case when $\mathcal{X} = A^\wedge$, to which we'll now restrict.

Let now A be a fixed small category, k a fixed commutative ring, $\mathcal{M}$ a k-additive category, stable under small inverse limits. We want to define a total derived functor of the functor

(109) $$(L,F) \mapsto \mathrm{Hom}_k(L,F) : A^\wedge_k \times A^\wedge_{\mathcal{M}} \to \mathcal{M},$$

which should be a functor

(110) $$(L_\bullet, F^\bullet) \mapsto \mathrm{RHom}_k(L_\bullet, F^\bullet) : \mathrm{D}^-(A^\wedge_k) \times \mathrm{D}^+(A^\wedge_{\mathcal{M}}) \to \mathrm{D}^+(\mathcal{M}),$$

and similarly

(111) $$(L_\bullet, F^\bullet) \mapsto \mathrm{R}\mathbf{Hom}_k(L_\bullet, F^\bullet) : \mathrm{D}^-(A^\wedge_k) \times \mathrm{D}^+(A^\wedge_{\mathcal{M}}) \to \mathrm{D}^+(A^\wedge_{\mathcal{M}}).$$

For this, in order for $\mathrm{D}^+(A^\wedge_{\mathcal{M}})$ to be defined, we better assume $\mathcal{M}$ to be an *abelian* category, hence $A^\wedge_{\mathcal{M}}$ is abelian too. Of course, we'll write

(112) $$\begin{aligned} \mathrm{Ext}^i_k(L_\bullet, F^\bullet) &= \mathrm{H}^i(\mathrm{RHom}_k(L_\bullet, F^\bullet)), \\ \mathbf{Ext}^i_k(L_\bullet, F^\bullet) &= \mathrm{H}^i(\mathrm{R}\mathbf{Hom}_k(L_\bullet, F^\bullet)), \end{aligned}$$

these global and local Ext^i may be viewed as "external" Ext^i's, as contrarily to the familiar case, the components of the two arguments $L_\bullet$ and $K^\bullet$ are not in the same category – just as the Hom_k in (109) and the corresponding

(109') $$\mathbf{Hom}_k : A^\wedge_k \times A^\wedge_{\mathcal{M}} \to A^\wedge_{\mathcal{M}}$$

has arguments in the two different categories $A^\wedge_k$ and $A^\wedge_{\mathcal{M}}$.

[p. 429] As we don't know about the existence of enough injective in $A^\wedge_{\mathcal{M}}$, the only way for defining the pairings (110), (111) is now by using projective resolutions of the first argument, writing

(113) $$\begin{aligned} \mathrm{RHom}_k(L_\bullet, F^\bullet) &= \mathrm{Hom}^{\bullet\bullet}_k(\mathcal{L}_\bullet, F^\bullet), \\ \mathrm{R}\mathbf{Hom}_k(L_\bullet, F^\bullet) &= \mathbf{Hom}^{\bullet\bullet}_k(\mathcal{L}_\bullet, F^\bullet) \end{aligned}$$

where $\mathcal{L}_\bullet$ is a projective resolution of $L_\bullet$ in $A^\wedge_k$. As the latter is defined up to chain homotopy, it follows that for fixed $L_\bullet$ and $F^\bullet$, the second members of (113) are defined up to chain homotopy, i.e., they may be viewed as objects in $\mathrm{K}^+(\mathcal{M})$ and $\mathrm{K}^+(A^\wedge_{\mathcal{M}})$ respectively. They are defined as such, even without assuming $\mathcal{M}$ to be abelian and hence $\mathrm{D}^+(\mathcal{M})$ and $\mathrm{D}^+(A^\wedge_{\mathcal{M}})$ to be defined. When we make this assumption, in order to check that the formulæ (113) do define pairings as in (110) and (111), we still have to check that for a quasi-isomorphism

$$F^\bullet \to (F')\bullet$$

in $A^\wedge_{\mathcal{M}}$, the corresponding maps between RHom_k and $\mathrm{R}\mathbf{Hom}_k$ are quasi-isomorphisms too. This will follow immediately, provided we check that for fixed *projective* L in $A^\wedge_k$ and variable F in $A^\wedge_{\mathcal{M}}$, the functors

$$F \mapsto \mathrm{Hom}_k(L,F) \quad \text{and} \quad F \mapsto \mathbf{Hom}_k(L,F)$$

from $A^{\wedge}_{\mathcal{M}}$ to $\mathcal{M}$ and $A^{\wedge}_{\mathcal{M}}$ respectively are exact. Now, this is clear for Hom_k when L is of the type $k^{(a)}$, hence $\mathrm{Hom}(L,F) \simeq F(a)$, hence it follows when L is a small direct sum of objects $k^{(a_i)}$, hence

$$\mathrm{Hom}_k(L,F) = \prod_i F(a_i),$$

provided we make on $\mathcal{M}$ the mild extra assumption that *a small direct product of epimorphisms is again an epimorphism*. As any projective object of $A^{\wedge}_k$ is a direct factor of a small direct sum as above, the exactness result we want then follows, hence the looked-for pairing (110). The same argument will hold for $\mathbf{Hom}_k$, provided we check exactness of the functors

$$F \mapsto \mathbf{Hom}_k(k^{(a)}, F) = (b \mapsto F(a \times b)).$$

Here it seems we get into trouble when A is not stable under binary products – in this case there is little chance that the functor above be exact, even when restricting to the case $\mathcal{M} = (\mathrm{Ab})_k \stackrel{\mathrm{def}}{=} (k\text{-Mod})$, hence $A^{\wedge}_{\mathcal{M}} = A^{\wedge}_k$ and L, F have values in the same category (namely presheaves of k-modules). This may seem strange, as we know (and recalled above) that in this standard case there is no problem for defining a pairing (111) $\mathrm{R}\mathbf{Hom}_k$. The point here is that, whereas a reasonable $\mathrm{R}\mathbf{Hom}$ can be defined indeed, it *cannot* be computed in terms of a projective resolution of the first argument as in (113); or equivalently, that for projective L it is not necessarily true that [p. 430]

$$\mathbf{Ext}^i_k(L,F) = 0 \quad \text{for } i > 0;$$

this in turn relates to the observation that, contrarily to what happens for the notion of injective sheaves of modules (on an arbitrary topos), it is not true that the property for a sheaf of modules to be projective is stable under localization (even for a constant sheaf of rings k on a topos $A^{\wedge}$). Indeed, the localization of $k^{(a)}$ with respect to $A_{/b}$ (with a and b in A) is $k^{(a')}$ where a' is $a \times b$ viewed as an object in $A_{/b}$, and for any sheaf of k-modules F on $A_{/b}$ we have

$$\mathrm{Ext}^i_{A_{/a}}(k^{(a')}, F) = \mathrm{H}^i(A_{/a'=a\times b}, F),$$

which need not be zero for $i > 0$. If it was, this would imply that $a \times b$ is k-acyclic (rather, that its connected components are), a rather strong property indeed when $a \times b$ is not in A...

Thus, when A is not stable under binary products, it doesn't seem that there exists a pairing (111) as I expected, except (possibly) when there are enough injectives in $A^{\wedge}_{\mathcal{M}}$ – a case I do not wish to examine for the time being, as I am mainly interested now in a formalism using projective resolutions instead of injective ones. Anyhow, for the purpose of subsuming the cohomology functor $\mathrm{R}\Gamma_A$ under the RHom_k formalism, by formula

$$\mathrm{R}\Gamma_A(F^\bullet) = \mathrm{RHom}_k(k_A, F^\bullet) \simeq \mathrm{Hom}^{\bullet\bullet}_k(L^A_\bullet, F^\bullet), \tag{114}$$

where

$$L_\bullet^A \to k_A \tag{115}$$

is a projective resolution of k_A, it is the pairing RHom_k and not $\mathrm{R}\mathbf{Hom}_k$ which is the relevant one. Let's recall that a projective resolution (115) is called a *cointegrator* (for the category A, with coefficients in k), as by formula (114) it allows indeed to express "cointegration" of any $\mathcal{M}$-valued presheaf or complex of such presheaves (with degrees bounded from below).

Thus, for the time being we just got the pairing RHom_k in (110), and the corresponding functor

$$F^\bullet \mapsto \mathrm{R}\Gamma_A(F^\bullet) : \mathrm{D}^+(A^\wedge_{\mathcal{M}}) \to \mathrm{D}^+(\mathcal{M}), \tag{116}$$

[p. 431] and not the pairing $\mathrm{R}\mathbf{Hom}_k$ in (111), and hence no formula (ii) (p. 428) relating the two – which makes me feel a little silly! I'll have to come back upon this later. At present, let's dualize what we got, assuming now that $\mathcal{N}$ is a k-additive abelian category stable under small *direct* limits, and such that *a small direct sum of monomorphisms in $\mathcal{N}$ is again a monomorphism*. We then get a pairing

$$(F_\bullet, L'_\bullet) \mapsto F_\bullet \overset{\mathrm{L}}{*}_k L'_\bullet : \mathrm{D}^-(A^\wedge_{\mathcal{N}}) \times \mathrm{D}^-(B^\wedge_k) \to \mathrm{D}^-(\mathcal{N}), \tag{117}$$

defined by the formula (dual to (112))

$$F_\bullet \overset{\mathrm{L}}{*}_k L'_\bullet \simeq F_\bullet *_k \mathcal{L}'_\bullet, \tag{118}$$

where in the second member $\mathcal{L}'_\bullet$ is a projective resolution of $L_\bullet$ in $B^\wedge_k$ ($B = A^{\mathrm{op}}$ being of course the dual category of A), and the $*_k$ denotes the simple complex associated to the double complex obtained by applying $*_k$ componentwise. Using the composite equivalence

$$F_\bullet \mapsto (F_\bullet)^{\mathrm{op}} : (\mathrm{D}^-(A^\wedge_{\mathcal{N}}))^{\mathrm{op}} \approxeq \mathrm{D}^+((A^\wedge_{\mathcal{N}})^{\mathrm{op}}) \approxeq \mathrm{D}^+(B^\wedge_{\mathcal{M}}), \quad \text{with } \mathcal{M} = \mathcal{N}^{\mathrm{op}}, \tag{119}$$

we get the tautological duality isomorphism

$$\big(F_\bullet \overset{\mathrm{L}}{*}_k L'_\bullet\big)^{\mathrm{op}} \simeq \mathrm{RHom}_k(L'_\bullet, (F_\bullet)^{\mathrm{op}}), \tag{120}$$

where the expression $\overset{\mathrm{L}}{*}_k$ in the first member is relative to the pair $(A, \mathcal{N})$, whereas the expression RHom_k in the second is relative to the dual pair $(B, \mathcal{M})$. Symmetrically, we get

$$\big(\mathrm{RHom}_k(L_\bullet, F^\bullet)\big)^{\mathrm{op}} \simeq L_\bullet \overset{\mathrm{L}}{*}_k (F_\bullet)^{\mathrm{op}}, \tag{120'}$$

which is essentially the inverse isomorphism of (120), but for the pair $(B, \mathcal{M})$ instead of $(A, \mathcal{N})$.

We still should dualize the functor $\mathrm{R}\Gamma_A$ (116) (defined by (114)), which we do, recalling that Γ_A is just the inverse limit functor $\varprojlim_{A^{\mathrm{op}}}$, which is dual to the direct limit functor $\varinjlim_{A^{\mathrm{op}}}$, thus, "integration" of $\mathcal{N}$-valued presheaves is just (at least morally) the total left derived functor of the latter, and may be denoted by

$$\mathrm{L}\varinjlim_{A^{\mathrm{op}}}, \tag{121}$$

while using for $\mathrm{R}\Gamma_A$ the equivalent notation, dual to (121)

$$\mathrm{R}\varprojlim_{A^{\mathrm{op}}} = \mathrm{R}\Gamma_A. \tag{122}$$

I am not wholly happy, though, with the purely algebraic flavor of these notations, not really suggestive of the manifold geometric intuitions surrounding the familiar homology and cohomology notations $\mathrm{H}_\bullet$ and $\mathrm{H}^\bullet$. This flavor is at least partially preserved, it seems to me, in the notation $\mathrm{R}\Gamma_A$ (because of the geometric intuition tied with the sections functor), whereas there is not yet a familiar geometric notion of a "cosections functor". As we would like to have the duality symmetry reflected as perfectly as possible in the notation, I am going to use the notations [p. 432]

$$\begin{cases} \mathrm{RH}^\bullet(A, F^\bullet) = \mathrm{R}\varprojlim_{A^{\mathrm{op}}}(F^\bullet)\ (= \mathrm{R}\Gamma_A(F^\bullet)) : \mathrm{D}^+(A^\wedge_{\mathcal{M}}) \to \mathcal{M} \\ \mathrm{LH}_\bullet(A, F_\bullet) = \mathrm{L}\varinjlim_{A^{\mathrm{op}}}(F_\bullet) : \mathrm{D}^-(A^\wedge_{\mathcal{N}}) \to \mathcal{N}. \end{cases} \tag{123}$$

With these notations, the duality isomorphisms (120,120') take the form (as announced):

$$\begin{cases} (\mathrm{RH}^\bullet(A, F^\bullet))^{\mathrm{op}} \simeq \mathrm{LH}_\bullet(B, (F^\bullet)^{\mathrm{op}}) \\ (\mathrm{LH}_\bullet(A, F_\bullet))^{\mathrm{op}} \simeq \mathrm{RH}^\bullet(B, (F_\bullet)^{\mathrm{op}}) \end{cases}, \tag{124}$$

where the first members are defined in terms of cohomology resp. homology invariants with respect to the pair $(A, \mathcal{M})$ resp. $(A, \mathcal{N})$, whereas the second members denote homology resp. cohomology invariants with respect to the dual pairs $(B, \mathcal{N})$ resp. $(B, \mathcal{M})$.

Remarks. This perfect symmetry, or rather essential identity, between "homology" or "integration" and "cohomology" or "cointegration", is obtained here at the price of working with presheaves with values in rather general abelian categories, subjected to some simple exactness properties. It should be remembered moreover that for the time being, $\mathrm{RH}^\bullet$ has not been defined as the total right derived functor of the sections of inverse limits functor, therefore the notations (121) and (122) are somewhat misleading. To feel really at ease, we should still work out conditions that ensure that $A^\wedge_{\mathcal{M}}$ has enough injectives and that RHom_k can be defined also using such resolutions – in which case we'll expect too to have a satisfactory formalism for the $\mathrm{R}\mathbf{Hom}_k$ functor.

[p. 433] 23.8.

109 I still did a little scratchwork last night, about the question of existence of enough injectives or projectives in a category

Review (6)*: A further step in linearization: coalgebra structures $P \to P \otimes_k P$ in* (Cat)*.*

$$A^{\wedge}_{\mathcal{N}} = \mathbf{Hom}(A^{\mathrm{op}}, \mathcal{N}),$$

where $\mathcal{N}$ is a k-additive abelian category. Introducing the small k-additive category

$$P = \mathrm{Add}_k(A),$$

and remembering the canonical equivalence

$$A^{\wedge}_{\mathcal{N}} = \mathbf{Hom}(A^{\mathrm{op}}, \mathcal{N}) \approxeq \mathbf{Hom}_k(P^{\mathrm{op}}, \mathcal{N}),$$

the question just stated may be viewed as a particular case of the same question for a category of the type

$$P^{\&}_{\mathcal{N}} \overset{\mathrm{def}}{=} \mathbf{Hom}_k(P^{\mathrm{op}}, \mathcal{N}), \tag{125}$$

where now P is *any* small k-additive category. (Compare with the reflections on pages 403, 404.) It is immediate that in $P^{\&}_{\mathcal{N}}$ exist all types of (direct or inverse) limits which exist in $\mathcal{N}$, and they are computed "componentwise" for each argument a in P – from this follows that if $\mathcal{N}$ is abelian, so is $P^{\&}_{\mathcal{N}}$.

Proposition 4. *Assume the k-additive category $\mathcal{N}$ is stable under small direct limits, and is abelian, and that any object of $\mathcal{N}$ is isomorphic to the quotient of a projective object. Then the same holds for $P^{\&}_{\mathcal{N}}$. Assume moreover that any projective object x of $\mathcal{N}$ is k-flat, i.e., the functor*

$$U \mapsto U \otimes_k x : \mathrm{Ab}_k \to \mathcal{N}$$

is exact, i.e., transform monomorphisms into monomorphisms. Then for any projective object F in $P^{\&}_{\mathcal{N}}$, the functor

$$L' \mapsto F *_k L' : Q^{\&} \to \mathcal{N} \quad (\textit{where } Q = P^{\mathrm{op}})$$

is exact, i.e. (*as it is known to commute to small direct limits*)*, it transforms monomorphisms into monomorphisms.*

Comments. Here, the operation $*_k$ (similar to a tensor product) is defined as in the non-additive set-up (with $P^{\&}_{\mathcal{N}}$, $Q^{\&}$ being replaced by $A^{\wedge}_{\mathcal{N}}$, $B^{\wedge}_k$) reviewed in section 107, and follows from the canonical equivalence of categories

$$P^{\&}_{\mathcal{N}} \xleftarrow{\approx} \mathbf{Hom}_{k!}(Q^{\&}, \mathcal{N}) \tag{*}$$

[p. 434] (this is formula (**) of page 403 with $P, \mathcal{N}$ replaced by $Q, \mathcal{M}$). It should be noted that the assumptions made in prop. 4 are the weakest possible for the conclusions to hold (for any k-additive small category P), as these conclusions, in case $P =$ final category, just reduce to the assumptions.

Here is the outline of a proof of prop. 4. Using only stability of $\mathcal{N}$ under small direct limits (besides k-additivity) we define a canonical k-biadditive pairing

(126) $$P^{\&} \times \mathcal{N} \to P^{\&}_{\mathcal{N}}, \quad (L,x) \mapsto L \otimes x \overset{\text{def}}{=} (a \mapsto L(a) \otimes_k x)$$

(NB I recall that $P^{\&}$ is defined as

$$P^{\&} = \mathbf{Hom}_{\mathbb{Z}}(P^{\mathrm{op}}, \mathrm{Ab}) \overset{\approx}{\longleftarrow} \mathbf{Hom}_k(P^{\mathrm{op}}, \mathrm{Ab}_k),$$

here we interpret an object of $P^{\&}$ is a k-additive functor

$$L : P^{\mathrm{op}} \to \mathrm{Ab}_k \ (\overset{\text{def}}{=} (k\text{-Mod})) \quad .)$$

The relevant fact here for objects of $P^{\&}_{\mathcal{N}}$ of the type $L \otimes_k x$ is

(127) $$\begin{aligned}\mathrm{Hom}_{P^{\&}_{\mathcal{N}}}(L \otimes_k x, F) &\simeq \mathrm{Hom}_{\mathcal{N}}(x, \mathrm{Hom}_k(L,F)) \\ &\simeq \mathrm{Hom}_{P^{\&}}(L, \mathrm{Hom}(x,F)),\end{aligned}$$

where in the second term,

$$\mathrm{Hom}_k(L,F) \in \mathrm{Ob}\,\mathcal{N}$$

is defined in a way dual to $F *_k L'$ (cf. comments above), using the equivalence (dual to (*) above)

(*') $$P^{\&}_{\mathcal{N}} \simeq \mathbf{Hom}^!_k((P^{\&})^{\mathrm{op}}, \mathcal{N}),$$

which is defined only, however, when $\mathcal{N}$ is stable under small inverse limits (hence the first isomorphism in (127) makes sense only under this extra assumption); on the other hand, in the third term in (127)

$$\mathrm{Hom}(x,F) \overset{\text{def}}{=} (a \mapsto \mathrm{Hom}_{\mathcal{N}}(x, F(a)) \quad \text{in } P^{\&},$$

and the isomorphism between the first and third term in (127) makes sense and is defined without any extra assumption on $\mathcal{N}$.

We leave to the reader to check (127) (where one is readily reduced to the case when L is an object a in P, using the commutation of the three functors obtained $(P^{\&})^{\mathrm{op}} \to \mathrm{Ab}_k$ with small inverse limits). It follows, when $\mathcal{N}$ is abelian:

(128) $$L \text{ projective in } P^{\&},\ x \text{ projective in } \mathcal{N} \Rightarrow L \otimes_k x \text{ proj. in } P^{\&}_{\mathcal{N}}.$$

Assume now that any object of $\mathcal{N}$ is quotient of a projective one, and let F be any object in $P^{\&}_{\mathcal{N}}$. Formula (127) for $L = a$ in P reduces to the down-to-earth formula [p. 435]

(127') $$\mathrm{Hom}_{P^{\&}_{\mathcal{N}}}(a \otimes_k x, F) \simeq \mathrm{Hom}(x, F(a)).$$

Now, let for any a in P

$$x_a \to F(a)$$

be an epimorphism in $\mathcal{N}$, with x_a projective. From (127') we get a map

$$a \otimes_k x_a \to F$$

in $P^{\&}_{\mathcal{N}}$, hence a map

$$\mathcal{F} = \bigoplus_{a \text{ in } P} a \otimes_k x_a \to F,$$

it is easily seen that this is epimorphic (because the maps $x_a \to F(a)$ are), and $\mathcal{F}$ is projective as a direct sum of projective objects. This proves the first statement in prop. 4. For the second statement, we'll use the formula

$$(L \otimes_k) *_k L' \simeq (L *_k L') *_k x \tag{129}$$

for L in $P^{\&}$, L' in $Q^{\&}$, and x in $\mathcal{N}$ – for the proof, we may reduce to the case when L is in P, L' in $Q = P^{\mathrm{op}}$, say $L = a$ and $L' = b^{\mathrm{op}}$, in which case both members identify with $\mathrm{Hom}_P(b, a) \otimes_k x$. To prove that for F projective, $L' \mapsto F *_k L'$ takes monomorphisms into monomorphisms, using that F is a direct factor of objects of the type $a \otimes_k x$ with a in P and x in $\mathcal{N}$, we are reduced to the case $F = a \otimes_k x$, in which case by (129) the functor reduces to

$$L' \mapsto L'(a) \otimes_k x,$$

which is again exact by the assumption that any projective object in $\mathcal{N}$ (and hence x) is k-flat.

Remark. It is not automatic that a projective object in a k-additive abelian category be k-flat – take for instance $k = \mathbb{Z}$ and $\mathcal{N} = \mathrm{Ab}_{\mathbb{F}_p}$, where $\mathbb{F}_p$ is a finite prime field, then all objects in $\mathcal{N}$ are projective, whereas only the zero objects are $\mathbb{Z}$-flat.

We leave to the reader to write down the dual statement of prop. 4, concerning injectives in a category $P^{\&}_{\mathcal{M}}$, where now $\mathcal{M}$ is a k-additive abelian category stable under small inverse limits, and possessing sufficiently many injectives (hence the same holds in $P^{\&}_{\mathcal{M}}$), and assuming eventually that these injectives x are "k-coflat", i.e.,

$$U \mapsto \mathrm{Hom}_k(U, x) : \mathrm{Ab}_k^{\mathrm{op}} \to \mathcal{M}$$

[p. 436] is exact (i.e., transforms monomorphisms in Ab_k into epimorphisms in $\mathcal{M}$), which implies that for F injective in $P^{\&}$, the functor

$$L \mapsto \mathrm{Hom}_k(L, F) : P^{\&} \to \mathcal{M}$$

is exact.

To sum up, we get the

Corollary 1. *Let P be any small k-additive category, and let $\mathcal{M}$ be a k-additive category which satisfies the following assumptions:*

a) *$\mathcal{M}$ is abelian, and stable under small inverse limits,*

b) $\mathcal{M}$ *has "sufficiently many injectives",*

c) *injective objects of* $\mathcal{M}$ *are* k*-coflat,*

d) *any product (with small indexing family) of epimorphisms in* $\mathcal{M}$ *is an epimorphism.*

Consider the k*-biadditive pairing*

(130) $$(L,F) \mapsto \mathrm{Hom}_k(L,F) : P^{\&} \times P^{\&}_{\mathcal{M}} \to \mathcal{M}.$$

This pairing admits a total right derived functor

(131) $$(L_\bullet,F^\bullet) \mapsto \mathrm{RHom}_k(L_\bullet,F^\bullet) : \mathrm{D}^-(P^{\&}) \times \mathrm{D}^+(P^{\&}_{\mathcal{M}}) \to \mathrm{D}^+(\mathcal{M}),$$

which can be computed using either projective resolutions of $L_\bullet$*, or injective resolutions of* $F^\bullet$*, or both simultaneously.*

We have a dual statement, concerning the pairing

(130') $$(F,L') \mapsto F *_k L' : P^{\&}_{\mathcal{N}} \times Q^{\&} \to \mathcal{N}, \quad \text{with } Q = P^{\mathrm{op}},$$

giving rise to a total left derived functor

(131') $$(F_\bullet,L'_\bullet) \mapsto F_\bullet \overset{\mathrm{L}}{*}_k L'_\bullet : \mathrm{D}^-(P^{\&}_{\mathcal{N}}) \times \mathrm{D}^-(Q^{\&}) \to \mathrm{D}^-(\mathcal{N}),$$

using projective resolutions of either $F_\bullet$, or $L'_\bullet$, or both. Here, $\mathcal{N}$ is a k-additive category satisfying the properties dual to a) to d) above, i.e., such that $\mathcal{M} = \mathcal{N}^{\mathrm{op}}$ satisfies the properties stated in the corollary. We have the evident duality relations between the two kinds of operations RHom_k and $\overset{\mathrm{L}}{*}_k$, embodied by the formulæ (120) and (120') of page 431, where the categories $A^\wedge_k$, $A^\wedge_{\mathcal{M}}$, $A^\wedge_{\mathcal{N}}$, etc. are replaced by $P^{\&}$, $P^{\&}_{\mathcal{M}}$, $P^{\&}_{\mathcal{N}}$, etc. (where the etc.'s refer to replacement of A by $B = A^{\mathrm{op}}$ and of P by $Q = P^{\mathrm{op}}$).

Next question is to extend the RHom_k formalism to a $\mathrm{R}\mathbf{Hom}_k$ formalism (and similarly from $\overset{\mathrm{L}}{*}_k$ to $\overset{\mathrm{L}}{\circledast}_k$), as envisioned yesterday. To do so, in the wholly k-additive set-up we are now working in, we still need (in case of $\mathrm{R}\mathbf{Hom}_k$ an ("interior") tensor product structure on $P^{\&}$ (and dually for $\overset{\mathrm{L}}{\circledast}$, requiring a tensor product structure on $Q^{\&}$), so as to give rise to a k-biadditive pairing [p. 437]

(132) $$(L,F) \mapsto \mathbf{Hom}_k(L,F) : P^{\&} \times P^{\&}_{\mathcal{M}} \to P^{\&}_{\mathcal{M}},$$

(and dually,

(132') $$(F,L') \mapsto F \circledast_k L' : P^{\&}_{\mathcal{N}} \times Q^{\&} \to P^{\&}_{\mathcal{N}}),$$

for which we want to take the total right derived functor $\mathrm{R}\mathbf{Hom}_k$ (or dually, the total left derived functor $\overset{\mathrm{L}}{\circledast}_k$). As we say yesterday, though, taking projective resolutions of L will not do (except in very special cases, such as $P = \mathrm{Add}_k(A)$ with A in (Cat) stable under binary products), because for L projective in $P^{\&}$, the functor

$$F \mapsto \mathbf{Hom}_k(L,F) : P^{\&}_{\mathcal{M}} \to P^{\&}_{\mathcal{M}}$$

has little chance to be exact. Taking injective resolutions of $F_\bullet$, however, we expected *would* do in "reasonable" cases – here the question is whether for F injective, the functor

$$L \mapsto \mathbf{Hom}_k(L,F) \stackrel{\text{def}}{=} (a \mapsto \mathrm{Hom}_k(L \otimes_k a, F)) : (P^{\&})^{\mathrm{op}} \to \mathcal{M}$$

is exact (where $L \otimes_k a$ denotes the given tensor product within $P^{\&}$), i.e., whether the functors (for F in $P^{\&}_{\mathcal{M}}$, a in P)

$$(*) \qquad L \mapsto \mathrm{Hom}_k(L \otimes_k a, F) : (P^{\&})^{\mathrm{op}} \to \mathcal{M}$$

transform monomorphisms of $P^{\&}$ into epimorphisms of $\mathcal{M}$. When $\mathcal{M}$ satisfies the conditions dual to a)b)c) in the corollary, hence $L \mapsto \mathrm{Hom}_k(L,F)$ is exact when F in $P^{\&}_{\mathcal{M}}$ is injective, exactness of (*) above will follow from exactness of

$$L \mapsto L \otimes_k a : P^{\&} \to P^{\&}.$$

Now, this latter exactness holds in the case we are interesting in mainly, when $P = \mathrm{Add}_k(A)$ and hence $P^{\&} \approxeq \mathbf{Hom}(A^{\mathrm{op}}, \mathrm{Ab}_k)$ and when tensor product for presheaves on A is defined as usual, componentwise – then the objects of $\mathrm{Add}_k(A) = P$ correspond to k-flat presheaves, hence tensor product by these is exact. Thus:

Corollary 2. *Assume $\mathcal{M}$ satisfies the conditions of corollary 1 above, let A be any small category and $P = \mathrm{Add}_k(A)$. Then the pairing*

$$\mathbf{Hom}_k : A^{\wedge}_k \times A^{\wedge}_{\mathcal{M}} \to A^{\wedge}_{\mathcal{M}}$$

[p. 438] (*cf.* (109') *page 428*) *admits a total right derived functor* $\mathrm{R}\mathbf{Hom}_k$ (111)*, which may be computed using injective resolutions of the second argument $F^\bullet$ in $\mathrm{R}\mathbf{Hom}_k(L_\bullet, F^\bullet)$ (but not, in general, by using projective resolutions of $L_\bullet$), i.e.,*

$$(133) \qquad \mathrm{R}\mathbf{Hom}_k(L_\bullet, F^\bullet) \simeq \mathbf{Hom}_k^{\bullet\bullet}(L_\bullet, \mathcal{F}^\bullet),$$

where $\mathcal{F}^\bullet$ is an injective resolution of $F^\bullet$ (i.e., a complex in $A^{\wedge}_{\mathcal{M}}$ with degrees bounded from below and injective components, endowed with a quasi-isomorphism $F^\bullet \simeq \mathcal{F}^\bullet$).

Applying now $\mathrm{R}\Gamma_A$ to both members of (133), and using the similar isomorphism for RHom_k (valid by cor. 2, we get the familiar formula

$$(134) \qquad \mathrm{RHom}_k(L_\bullet, F^\bullet) \simeq \mathrm{RH}^\bullet(\mathrm{R}\mathbf{Hom}_k(L_\bullet, F^\bullet)),$$

(where in accordance with (124), we wrote $\mathrm{RH}^\bullet$ instead of $\mathrm{R}\Gamma_A$), where however $F^\bullet$ is now a complex of presheaves with values in a k-additive category $\mathcal{M}$ (satisfying the conditions dual to a) to d) in cor. 1), not just a presheaf of k-modules.

Replacing $\mathcal{M}$ by a category $\mathcal{N}$ satisfying the assumptions of cor. 1, we get likewise a total left derived functor

$$(135) \qquad (F_\bullet, L'_\bullet) \mapsto F \overset{\mathrm{L}}{\otimes}_k L'_\bullet : \mathrm{D}^-(A^{\wedge}_{\mathcal{N}}) \times \mathrm{D}^-(B^{\wedge}_k) \to \mathrm{D}^-(A^{\wedge}_{\mathcal{N}}),$$

which can be defined using projective resolutions of the first argument $F_\bullet$ (but not by using projective resolutions of $L'_\bullet$), giving rise to the isomorphism (dual to (134))

$$F_\bullet \overset{\mathrm{L}}{*}_k L'_\bullet \simeq \mathrm{LH}_\bullet(F_\bullet \overset{\mathrm{L}}{\circledast}_k L'_\bullet). \tag{134'}$$

The duality relationship between $\mathrm{R}\mathbf{Hom}_k$ and $\overset{\mathrm{L}}{\circledast}_k$ can be expressed by two obvious formulæ, similar to (120) and (120') for RHom_k and $\overset{\mathrm{L}}{*}_k$, which we leave to the reader.

Remarks. 1) If we want to consider "multiplicative structure" in the purely "k-linear" set-up, where the data is a small k-additive category P, rather than a small category A (giving rise to $P = \mathrm{Add}_k(A)$), and corresponding cohomology and homology operations $\mathrm{R}\mathbf{Hom}_k$ and $\overset{\mathrm{L}}{\circledast}_k$, not only RHom_k and $\overset{\mathrm{L}}{*}_k$, the natural thing to do, it seems, is to introduce a "*diagonal map*"

$$P \to P \otimes_k P, \tag{*}$$

where the tensor product in the second member can be defined in a rather evident way (as solution of the obvious 2-universal problem in terms of k-biadditive functors on $P \times P$), which will give rise in the "usual" way to a tensor product operation in both $P^{\&}$ and $Q^{\&}$ (where [p. 439] $Q = P^{\mathrm{op}}$). We'll come back upon this later, I expect. This structure (*) will be the k-linear analogon of the usual diagonal map

$$A \to A \times A \tag{**}$$

for a small category A, giving rise by k-linearization to

$$\mathrm{Add}_k(A) \to \mathrm{Add}_k(A \times A) \approxeq \mathrm{Add}_k(A) \otimes_k \mathrm{Add}_k(A),$$

namely a structure of type (*). It just occurred to me, through the reflections of these last days, that the "coalgebra structure" (*) may well turn out (taking $k = \mathbb{Z}$) to be the more sophisticated structure than a usual coalgebra structure (cf. p. 339 (27)), needed in order to grasp "in linear terms" the notion of a homotopy type, possibly under restrictions such as 1-connectedness, as pondered about in section 94. This looks at any rate a more "natural" object than the De Rham complex with divided powers, referred to in loc. cit., and is more evidently adapted to our point of view of using small categories as models for homotopy types. The greater sophistication, in comparison to De Rham type complexes, lies in this, that here the objects serving as models (whether small categories, or small additive categories endowed with a diagonal map) are objects in a 2-*category*, whereas De Rham complexes and the like are just objects in ordinary categories, without any question of taking "maps between maps". This feature implies, "as usual" (or in duality rather to familiar situations with tensor product functors. . .) that the (anti)commutative and associative *axioms* familiar from linear algebra (in the case of usual k-algebras or k-coalgebras), should be replaced by commutativity and associativity *data*, namely given isomorphisms (not identities) between two natural functors

$$P \to P \otimes_k P, \quad P \to P \otimes_k P \otimes_k P$$

deduced from (*). The axioms now will be more sophisticated, they will express "coherence conditions" on these data – one place maybe where this is developed somewhat, in the context of diagonal maps (*), might be Saavedra's thesis. (The more familiar case, when starting with a tensor product operation on a category, together with associativity and/or commutativity data, has been done with care by various mathematicians, including Mac Lane, Bénabou, Mme Sinh Hoang Xuan, and presumably it should be enough to "reverse arrows" in order to get "the" natural set of coherence axioms for a diagonal map (*)). The "intriguing feature" with the would-be De Rham models for homotopy types (cf. p. 341, 342), namely that the latter make sense over any commutative ground ring k, not only $\mathbb{Z}$, with corresponding notion of ring extension $k \to k'$, carries over to structures of the type (*). Indeed, for any k-additive category P, it is easy to define a k'-additive category

[Saavedra Rivano (1972); see also in particular Deligne (1990, 2002) and Deligne and Milne (1982)]

[p. 440]

$$P \otimes_k k', \quad \text{for given homomorphism } k \to k',$$

for instance as the solution of the obvious 2-universal problem corresponding to mapping P k-additively into k'-additive categories, or more evidently by taking the same objects as for P, but with

$$\mathrm{Hom}_{P'}(a,b) = \mathrm{Hom}_P(a,b) \otimes_k k'.$$

Thus, any "coalgebra structure in (Cat)" (*) over the ground ring k, gives rise to a similar structure over ground ring k'.

Of course, among the relevant axioms for the diagonal functor (*), is the existence of unit objects in $P^{\&}$ and $Q^{\&}$, which may be viewed equally as k-additive functors (defined up to unique isomorphism)

$$P^{\mathrm{op}} \to \mathrm{Ab}_k, \quad P \to \mathrm{Ab}_k,$$

playing the role I would think of "augmentation" and "coaugmentation" in the more familiar set-up of ordinary coalgebras. Denoting these objects by k_P and k_Q respectively (in analogy to the constant presheaves k_A and k_B on A and B), $\mathrm{RHom}(k_P, -)$ now allows expression of cohomology or cointegration, and $- \overset{\mathrm{L}}{*}_k (k_Q)$ allows expression of homology or integration (for complexes in $P^{\&}_{\mathcal{M}}$ say). "Constant coefficients" on P, i.e., in $P^{\&}$ may now be defined, as objects in $P^{\&}$ of the type

$$U \otimes_k k_P,$$

where U is in Ab_k, i.e., is any k-module, and hence we get homology and cohomology invariants with coefficients in any such U (or complexes of such), and surely too cup and cap products… Also, quasi-isomorphisms of structure (*) (with units) can now be defined in an evident way, hence a derived category which merits to be understood, when $k = \mathbb{Z}$, in terms of the homotopy category (Hot). I wouldn't expect of course that for any small category A, the abelianization $\mathrm{Add}(A)$ together with its diagonal map allows to recover the homotopy type, unless A is 1-connected. As was the case visibly for De Rham complexes, if we hope to recover general homotopy types (not only 1-connected ones), we should work with slightly more sophisticated structures still, involving a group (or better still, a groupoid) and an operation of it on a structure of type (*) (embodying a universal covering...). [p. 441]

Here I am getting, though, into thin air again, and I don't expect I'll ponder much more in this direction and see what comes out. The striking fact, however, here, is that quite unexpectedly, we get further hold and food for this thin-air intuition (which came up first in relation to De Rham structures with divided powers), that there may be a reasonable (and essentially just one such) notion of a "homotopy type over the ground ring k" for any commutative ring k, reducing for $k = \mathbb{Z}$ to usual homotopy types, and giving rise to base change functors

$$\mathrm{Hot}(k) \to \mathrm{Hot}(k')$$

for any ring homomorphism $k \to k'$. And I wonder whether this might not come out in some very simplistic way, in the general spirit of our "modelizing story", without having to work out in full a description of homotopy types by such sophisticated models as De Rham complexes with divided powers, or coalgebra structures in (Cat), and looking up maybe the relations between these. (How by all means hope to recover a De Rham structure from a stupid structure (*)???)

2) The condition d) in corollary 1 is needed in order to ensure that a derived functor $\mathrm{RHom}_k(L_\bullet, F^\bullet)$ may be defined using *projective* resolutions of $L_\bullet$, whereas conditions b), c) ensure that a functor RHom_k may be defined using *injective* resolutions of $F^\bullet$. It is a well-known standard fact of homological algebra that in case both methods work (namely here, when all four assumptions are satisfied) that the two methods yield the same result, which may equally be described by resolving simultaneously the two arguments. (NB condition a) is needed anyhow for Hom_k to be defined and for $P^{\&}_{\mathcal{M}}$ being an abelian category, which allows to define $\mathrm{D}^+(P^{\&}_{\mathcal{M}})$.) Our preference goes to the first method, which in case $P = \mathrm{Add}_k(A)$ and $L = k_A$, conduces to computation of cohomology $\mathrm{R}\Gamma_A(F^\bullet)$ in terms of a "cointegrator" $L^A_\bullet$ on A. However, when it comes to introducing the variant $\mathrm{R}\mathbf{Hom}_k$, this method breaks down, as we saw, it is the other one which works. We thus get a satisfactory formalism of RHom_k and $\mathrm{R}\mathbf{Hom}_k$ (including formula (134) relating them via $\mathrm{RH}^\bullet$) using only assumptions a)b)c).

3) If we want to extend the $\mathrm{R}\mathbf{Hom}_k$ formalism to the set-up when the data A is replaced by a k-additive category P endowed with a diagonal map as in remark 1), the proof on page 437 shows that what is needed is exactness of the functor $L \mapsto L \otimes a$ from $P^{\&}$ to $P^{\&}$, for any a in P – which is a "flatness" condition on a. It is easily checked that his condition is satisfied provided $\mathrm{Hom}_P(b, a)$ is a flat k-module, for any b [p. 442]

in P (more generally, an object M in $P^{\&}$ is flat for the tensor product structure in $P^{\&}$, provided $M(b)$ is flat for any b in P). Thus, there will be a satisfactory $\mathrm{R}\mathbf{Hom}_k$ theory provided the k-modules $\mathrm{Hom}_P(b,a)$, for a,b in P, are k-flat. It is immediately checked that this also means that any projective object L in $P^{\&}$ is "k-flat" (with respect to external tensor product $U \mapsto U \otimes_k L : \mathrm{Ab}_k \to P^{\&}$, as in prop. 4), or equivalently still, that this holds when L is any object a in P. In case $P = \mathrm{Add}_k(A)$, when any object of P is a finite sum of objects $k^{(x)}$ with x in A, the modules $\mathrm{Hom}(b,a)$ for b,a in L are finite sums of modules of the type $\mathrm{Hom}(k^{(y)},k^{(x)}) = k^{(\mathrm{Hom}(y,x))}$, and hence are projective, not only flat. It would seem that in the general case of P endowed with a diagonal map, the "natural" assumption to make in order to have everything come out just as nicely as when P comes from an A, is that the k-modules $\mathrm{Hom}_P(b,a)$ (for a,b in P) should be projective, not only flat. Flatness however seems to be all that is needed in order to ensure that when L in $P^{\&}$ is projective (hence $L(a)$ is flat for any a in P) and F in $P^{\&}_{\mathcal{M}}$ is injective, then the objects $\mathrm{Hom}_k(L,F)$ and $\mathbf{Hom}_k(L,F)$ in $\mathcal{M}$ and $P^{\&}_{\mathcal{M}}$ respectively are injective. This implies that for a k-additive functor

$$u : \mathcal{M} \to \mathcal{M}'$$

between categories $\mathcal{M}$, $\mathcal{M}'$ satisfying the assumptions of cor. 1, and u commuting moreover to small inverse limits (and hence to formation of $\mathrm{Hom}_k(L,F)$), we get a canonical isomorphism

$$\mathrm{R}u(\mathrm{RHom}_k(L_\bullet,F^\bullet)) \simeq \mathrm{RHom}_k(L_\bullet,\mathrm{R}u^P(F^\bullet)), \tag{136}$$

where

$$u^P : P^{\&}_{\mathcal{M}} \to P^{\&}_{\mathcal{M}'}$$

denotes the extension of u, and $\mathrm{R}u$, $\mathrm{R}u^P$ are the right derived functors. When u is exact, we may replace $\mathrm{R}u$, $\mathrm{R}u^P$ by u, u^P (applied componentwise to complexes, without any need to take an injective resolution first). There is a formula as (136) with RHom_k replaced by $\mathrm{R}\mathbf{Hom}_k$, which I skip, as well as the dual formulas.

Part VI

Schematization

24.8. [p. 443]

110 I pondered some more about homotopy types over a ground ring k, just enough to become familiar again with the idea, and more or less convinced that that there should exist such a thing, which should amount, kind of, to putting a "continuous" structures (namely the very rich structure of a scheme) upon something usually visualized as something "discrete" – namely a homotopy type. The basic analogy here is free $\mathbb{Z}$-modules M of finite type – a typical case of a "*discrete*" structure. It gives rise, though, to a *vector bundle* $W(M)$ over the absolute base $S_0 = \mathrm{Spec}(\mathbb{Z})$, whose $\mathbb{Z}$-module of sections is M, and the functor

More wishful thinking on "schematization" of homotopy types.

$$M \mapsto W(M)$$

from free $\mathbb{Z}$-modules of finite type to vector bundles over S_0 is fully faithful. When M is an arbitrary $\mathbb{Z}$-module, i.e., an abelian group, $W(M)$ still makes sense, namely as a functor

$$k \mapsto M \otimes_{\mathbb{Z}} k$$

on the category of all commutative $\mathbb{Z}$-algebra (i.e., just commutative rings); it is no longer representable by a scheme over S_0 (except precisely when M is free of finite type), but it is very close still, intuitively and technically too, to a usual vector bundle (the "vector" structure coming from the k-module structure on $W(M)(k) = M \otimes_{\mathbb{Z}} k$). Again, the functor $M \mapsto W(M)$ from Ab to the category of "generalized vector bundles" over S_0 is fully faithful. Working with semisimplicial $\mathbb{Z}$-modules (say) rather than just $\mathbb{Z}$-modules, and more specifically with those corresponding to $K(\pi, n)$ types, and using Postnikov "dévissage" of a general homotopy type, one may hope to "represent" this type, in a more or less canonical way in terms of the successive semisimplicial Postnikov fibrations, by a semi-simplicial object in the topos (say) of all functors from $\mathbb{Z}$-algebras to sets which are "sheaves" for a suitable site structure on the dual category (namely the category of affine schemes over $\mathbb{Z}$) – the so-called "flat" topology seems OK. (NB To eliminate logical difficulties, we may have to restrict somewhat the rings k used as arguments, for instance take

["dévissage" = "decomposition"]

them to be of finite type over $\mathbb{Z}$ – never mind such technicalities!). This approach may possibly work, when restricting to 1-connected homotopy types, or at any rate to the case when the fundamental groups of the connected components are abelian. If we wish a "schematization" of arbitrary homotopy types, we may think of going about it by keeping the fundamental group of groupoid "discrete", and "schematize" the
[p. 444] Postnikov truncation involving only the higher homotopy groups (π_i with $i \geq 2$). This suggests that for any integer $n \geq 1$, there may be a "schematization above level n" for a given homotopy type, leaving the Cartan-Serre truncation of level n discrete, and "schematizing" the corresponding total Postnikov fiber (involving homotopy groups π_i for $i \geq n$). Maybe such a schematization can be constructed equally for level 0, even without assuming the fundamental groupoid to be abelian, only nilpotent – but then we may have to change from ground ring $\mathbb{Z}$ to the considerably coarser one $\mathbb{Q}$ (compare comments at the end of section 94). In any case, the key step in this approach would consist in checking that, after schematization has been carried through successfully up to a certain level in the successive elementary Postnikov fibrations, the next elementary fibration (described by a cohomology class in $\mathrm{H}^{n+2}(X_n, \pi_{n+1})$) comes from a "schematic" one, and that the latter is essentially unique; in other words, that the canonical map from the "schematic" H^{n+2} (with "quasi-coherent" coefficients) to the usual "discrete" one is bijective. Maybe this hope is wholly unrealistic though. One fact which calls for some skepticism about this approach, comes in when looking at the case of an "abelian" homotopy type, described by a semisimplicial abelian group $X_\bullet$, in which case we expect that base change $\mathbb{Z} \to k$ should be just the usual base change

$$X_\bullet \mapsto X_\bullet \otimes_{\mathbb{Z}} k$$

(if $X_\bullet$ has torsion-free, i.e., flat components, at any rate). But when $X_\bullet$ has homology torsion, the universal coefficients formula shows us that the homology (= homotopy) groups of $X_\bullet \otimes_{\mathbb{Z}} k$ are *not* just the groups $\mathrm{H}_i(X_\bullet) \otimes_{\mathbb{Z}} k = \pi_i(X_\bullet) \otimes_{\mathbb{Z}} k$, as we implicitly were assuming it seems in the approach sketched above, when schematizing the homotopy groups π_i one by one via $W(\pi_i)$. Thus, maybe Postnikov dévissage isn't a possible approach towards schematization of homotopy types, and one will have to work out rather a comprehensive yoga of reconstructing a homotopy type from one kind or other of "abelianization" or "linearization" of homotopy types, endowed with suitable extra structure embodying "multiplicative" features of the homology and cohomology structure. At any rate, I did not hit upon any "simplistic way" to define homotopy types over any ground ring, and I have some doubts there is any, in terms of the general non-sense we did so far.

[p. 445] Besides this, I spent hours to try and put some order into the mess of all Hom and tensor product type operations between categories $A^\wedge_k$, $A^\wedge_{\mathcal{M}}$, $B^\wedge_k$, $B^\wedge_{\mathcal{M}}$ (or their &-style generalizations), and the duality and Cartan-type isomorphisms between these. There are a few more still than the fair bunch met with in these notes so far – I finally renounced to get really through and work out a wholly satisfactory set of notations, taking into account all symmetries in the situation. I realized this might well take days of work, while at present there is no real need yet for it. I sometimes find it difficult to find a proper balance in these notes between the need of working on reasonably firm ground, and working out suggestive terminology and notations for what is coming up, and on the other hand my resolve not to get caught again by the "Eléments de Géométrie Algébrique" style of work, when it was understood that everything had to be worked out in complete detail and in greatest generality, for the benefit of generations of "usagers" (besides my own till the end of my life!..). As a matter of fact, this whole "abelianization story", going on now for well over a hundred pages and nothing really startling coming out – just things I feel I should have known for ages, has been won (so to say) over an inner reluctance against these "digressions" in the main line of thought, the reluctance of one who is in a hurry to get through. I know well this old reluctance, feeling silly whenever working out "trivial details" with utmost care; as I know too that through this work only would come to a thorough understanding of what is going on, and new intuitions or relationships would flash up sometimes and open up unexpected landscapes and provide fresh impetus. The same has happened innumerable times too within the last seven years, when "meditating" on personal matters – constantly "the-one-in-a-hurry" has turned out to be just the servant of the inner resistances against renewal, against a fresh, innocent look upon things familiar, and consistently ignores as "irrelevant". It doesn't seem the-one-in-a-hurry gets at all discouraged for not getting his way many times – he seems to be just as stubborn as the one who likes to take his time and look up things thoroughly!

[I'm leaving in "usagers" for color, though "users" would surely do...]

25.8. [p. 446]

Complexes of "unipotent bundles" as models, and "schematic" linearization.

111 Still about "schematization" of homotopy types! Here is a tentative approach, without any explicit use of Postnikov fibrations nor abelianization, although both are involved implicitly. If n is any natural integer, I'll denote by

$$\mathrm{Hot}_n$$

the full subcategory of the pointed homotopy category $\mathrm{Hot}^\bullet$, made up with n-connected homotopy types, with the extra assumption for $n = 0$ that the fundamental group be abelian. For any (commutative) ring k, I want to define a category

$$\text{(a)} \qquad \mathrm{Hot}_n(k),$$

depending covariantly on k, in such a way that we have an equivalence

$$\text{(b)} \qquad \mathrm{Hot}_n(\mathbb{Z}) \xrightarrow{\approx} \mathrm{Hot}_n,$$

which should come from a canonical functor

$$\text{(c)} \qquad \mathrm{Hot}_n(k) \to \mathrm{Hot}_n$$

defined for any k, and which should be viewed as ground-ring restriction from k to $\mathbb{Z}$ – more generally, for any ring homomorphism

$$k' \to k$$

we expect a restriction functor, beside the ring extension functor

$$\text{(d)} \qquad \mathrm{Hot}_n(k') \to \mathrm{Hot}_n(k) \quad \text{and} \quad \mathrm{Hot}_n(k) \to \mathrm{Hot}_n(k').$$

Among other important features to expect, is that for any object X in $\mathrm{Hot}_n(k)$, the homotopy groups $\pi_i(X)$ (defined via (c)) should be naturally endowed with structures of k-modules, and the ring extension and restriction functors (d) should be compatible with these.

Here is an idea for getting such a theory. For given k, we first define an auxiliary category

$$\text{(e)} \qquad U(k),$$

whose objects may be called "unipotent bundles over k". These "bundles" will not be quite schemes over k, they will be defined as functors

$$\text{(f)} \qquad \mathrm{Alg}_{/k} \to (\mathrm{Sets})$$

where $\mathrm{Alg}_{/k}$ is the category of (commutative) k-algebras (in the basic universe $\mathcal{U}$). The opposite category may be identified with the category

$$\mathrm{Aff}_{/k}$$

[p. 447] of affine schemes over k, thus, we'll be working in the category of functors (or presheaves over $\mathrm{Aff}_{/k}$)

$$\text{(f')} \qquad (\mathrm{Aff}_{/k})^{\mathrm{op}} \to (\mathrm{Sets})$$

more specifically, $U(k)$ will be a full subcategory of this category of functors. We'll endow $\mathrm{Aff}_{/k}$ with one of the standard site structures, the most convenient one here is the fpqc topology (faithfully flat quasi compact topology), and work in the category of *sheaves* in the latter. In terms of the interpretation (f) as covariant functors on $\mathrm{Alg}_{/k}$, this just means that we are restricting to functors X which 1) commute to finite products and 2) are "compatible with faithfully flat descent", i.e., for any map

["fidèlement plat et quasi-compact"]

$$k' \to k''$$

in $\mathrm{Alg}_{/k}$ such that k'' becomes a faithfully flat algebra over k', the following diagram in (Sets)

$$X(k') \to X(k'') \rightrightarrows X(k'' \otimes_{k'} k'')$$

is exact. Thus, $U(k)$ will be defined as a full subcategory of the category of such functors, or "sheaves".

One way for defining $U(k)$ is to present it as the union of a sequence of subcategories $U_m(k)$ (m a natural integer). We'll take $U_0(k)$ to be just reduced to the final functors (i.e., $X(k')$ is a one-point set for any k' in $\mathrm{Alg}_{/k}$), and define inductively $U_{m+1}(k)$ in terms of $U_m(k)$ as follows. For any k-module M, let $W(M)$ be the corresponding "vector bundle", defined by

(g) $$W(M)(k') = M \otimes_k k',$$

then an object X is in $U_{m+1}(k)$ iff there exists an object X_m in $U_m(k)$, and a k-module $M = M_{m+1}$, in such a way that $X = X_{m+1}$ should be isomorphic to a "torsor" over X_m, with group $W(M)$. I am not too sure, here, whether we should view the objects of $U_m(k)$ as endowed with the extra structure consisting in giving the modules $M_1, \dots, M_m$ used in the inductive construction, and moreover the successive fibrations – if so, then of course the categories $U_m(k)$, and their union $U(k)$, will no longer be interpreted as a mere subcategory of the category of sheaves [p. 448] (of sets) just described. Possibly, both approaches are of interest and will yield non-equivalent notions of schematization. On the other hand, although definitely X is not representable by a usual scheme over k unless the k-modules M_i are projective of finite type (in which case X will be an affine scheme, and even isomorphic, at least locally over $\mathrm{Spec}(k)$, to standard affine space E_k^d for suitable d), it is felt that X, as far as cohomology properties go, should be very close to being an affine scheme, and that presumably its cohomology groups H^i with coefficients in "quasi-coherent" sheaves such as $W(M)$ should vanish for $i > 0$; consequently, presumably the torsors used for the inductive construction of X are trivial, which means that X is in fact isomorphic to the product of all $W(M_i)$'s. In the case when we disregard the successive fibration structure, this means that the objects of $U(k)$ are just sheaves of sets which are isomorphic to some $W(M)$ (where morally M is the direct sum of the modules M_i which have been used in our inductive definition). This gives then a rather trivial description of the objects of $U(k)$ (and all $U_m(k)$'s are already equal to $U(k)$, for $m \geq 1$!), it should be remembered, however, that maps in $U(k)$ from a $W(M)$ to a $W(M')$ are a lot more general than just k-linear maps $M \to M'$ (they may be viewed as "polynomial maps from M to M'").

The category $U(k)$ will be endowed with the *sections functor*

(h) $$X \mapsto X(k) : U(k) \to (\mathrm{Sets}).$$

Now, let A be any test category, for instance $A = \triangle$, and consider the functor

(h') $$\mathbf{Hom}(A^{\mathrm{op}}, U(k)) \to \mathbf{Hom}(A^{\mathrm{op}}, (\mathrm{Sets}))$$

induced by (h). The second member modelizes homotopy types, which therefore allows us to define homotopy invariants for objects in the first member, and hence to define the property of n-connectedness, and (if $n = 0$) of 0-connectedness with abelian fundamental group. As a matter of fact, we would like to define a subcategory $\mathcal{M}_n(k)$ of the first member, so that is should become clear that for an object X_* in it, its homotopy

groups are endowed with structures of k-modules, so that for $n = 0$ the abelian restriction on π_1 is superfluous, because automatic. To be specific, we better restrict maybe to $A = \triangle$, more familiar to us, so that X_* may be viewed as a "complex" ($n \mapsto X_n$) with components in $U(k)$.
[p. 449] The kind of restriction I am thinking of for defining the model category $\mathcal{M}_n(k)$ is:

(i) $$X_i = e \quad \text{for} \quad i \leq n,$$

where e is the final object of $U(k)$, and also maybe, if we adopt the more refined version of $U(k)$ as a strictly increasing union of subcategories $U_m(k)$,

(i') $$X_i \text{ is in } U_{i-n}, \text{ for any } i \geq n.$$

One may have to play around some more to get "the correct" description of the model category, which I tentatively propose to define simply as a suitable full subcategory

(j) $$\mathcal{M}_n(k) \subset \mathbf{Hom}(A^{\mathrm{op}}, U(k)).$$

The functor (h') allows to define a notion of weak equivalence in $\mathcal{M}_n(k)$, hence a localized category $\mathrm{Hot}_n(k)$, and a functor (c) from this category to Hot_n. The ring extension and restriction functors (d) are equally defined in an evident way, via corresponding functors on the model categories (with the task, however, to check that these are compatible with weak equivalences). The key point here is to check that for $k = \mathbb{Z}$, the functor (c) (namely (b)) is indeed an equivalence of categories. Thus, the main task seems to cut out carefully a description of a model category $\mathcal{M}_n(k)$, in terms of semisimplicial objects say, in a category such as $U(k)$, in such a way as to give rise to an equivalence of categories (b).

One point which is still somewhat misty in this (admittedly overall misty!) picture, is how to get, for an object X in $\mathrm{Hot}_n(k)$, the promised operation of k on the homotopy groups $\pi_i(X)$. I was thinking about this when suggesting the conditions (i) and (i') above on k-models for homotopy types – but I really doubt these are enough. On the other hand, it seems hard to imagine there be a good notion of homotopy types over k, without the homotopy groups to be k-modules over k, not just abelian groups. Even more still, there should be moreover a "linearization functor"

(k) $$\mathrm{Hot}_n(k) \to \mathrm{D}_\bullet(\mathrm{Ab}_k),$$

with values in the derived category of the category of chain complexes in $\mathrm{Ab}_k = k\text{-Mod}$ of k-modules, presumably coming by localization from a functor

(k') $$\mathcal{M}_n(k) \to \mathrm{Ch}_\bullet(\mathrm{Ab}_k),$$

[footnote "only for $k = \mathbb{Z}$"] and giving rise to a commutative diagram

[p. 450]

(l)
$$\begin{tikzcd} \mathrm{Hot}_n(k) \arrow[r] \arrow[d] & \mathrm{D}_\bullet(\mathrm{Ab}_k) \arrow[d] \\ \mathrm{Hot}_n \arrow[r] & \mathrm{D}_\bullet(\mathrm{Ab}) \end{tikzcd} ,$$

where the second horizontal arrow is the usual abelianization functor for homotopy types, and the second vertical one comes from the ring restriction functor $\mathrm{Ab}_k \to \mathrm{Ab} = \mathrm{Ab}_{\mathbb{Z}}$. In (k'), $\mathrm{Ch}_\bullet$ denotes the category of chain complexes, and it looks rather mysterious again how to get such a functor (k'). We may of course think of the trivial abelianization

$$X_\bullet \mapsto k^{(X_\bullet)} \overset{\text{def}}{=} (n \mapsto k^{(X_n)}),$$

where for an object X in $U(k)$, or more generally any sheaf on $\mathrm{Aff}_{/k}$, $k^{(X)}$ defines a trivial k-linearization of this sheaf, in the sense of the topos of all such sheaves. Anyhow, $k^{(X)}$ is a sheaf, not a k-module, so we should still take sections to get what we want – but this functor looks not only prohibitively large and inaccessible, but just silly! A much better choice for k-linearizing objects of $U(k)$ specifically seems the following. Disregarding the fibration structures, such an object X is isomorphic to an object $W(M)$, M some k-module. We look for a k-linearization

(m) $$X \; (= W(M)) \to W(L(X)),$$

where $L(X)$ is a suitable k-module. Now, among all maps

$$X \to W(N)$$

of X into sheaves *of the type* $W(N)$, there is a universal one, which in terms of M can be described as

$$N = \Gamma_k(M),$$

where Γ_k denotes the "algebra with divided powers generated by M", the canonical map $M \to \Gamma_k(M)$ or rather

(n) $$W(M) \to W(\Gamma_k(M)), \quad x \mapsto \exp(x) = \sum_{i \geq 0} x^{(i)}$$

being *the* "*universal* polynomial map" of M with values in a module N (or rather, of $W(M)$ into $W(N)$). Here, $x^{(i)}$ denotes the i'th divided power of x, which is an element of $\Gamma_k^i(M)$. It just occurs to me that this expression of $\exp(x)$, the universal map, is infinite, thus, it doesn't take its values in $W(N)$ actually, but in a suitable completion of it – this doesn't seem too serious a drawback, though! The point I wish to make here, is that for given X in $U(k)$, defining [p. 451]

(o) $$L_k(X) = \Gamma_k^{\wedge}(M) \overset{\text{def}}{=} \prod_{i \geq 0} \Gamma_k^i(M), \quad (k\text{-linearization of } X),$$

where M is any k-module endowed with an isomorphism

$$u : X \simeq W(M),$$

the k-module $L_k(X)$ *does not depend, up to unique isomorphism, on the choice of a pair* (M, u), because for two k-modules M, M', any morphism of *sheaves of sets*

$$v : W(M) \to W(M')$$

induces a homomorphism of k-modules

$$\Gamma_k^\wedge(v) : \Gamma_k^\wedge(M) \to \Gamma_k^\wedge(M')$$

(compatible not with multiplications, but with diagonal maps…), which will be an isomorphism if v is.

Thus, we do have, it seems to me, a good candidate for k-linearization. To check it is suitable indeed, the main point seems to check that the corresponding diagram (l) commutes up to canonical isomorphism, the crucial case of course being $k = \mathbb{Z}$. This now looks like a rather down-to-earth question, which seems to me a pretty good test, whether the intuition of schematization of homotopy types is a sound one. Let's rephrase it here. For this, let's first restate the description of the category $U(k)$ (coarse version) in the more down-to-earth terms of linear algebra. Objects may be viewed as just k-modules M, whereas (non-additive) "maps" from M to M' (defined previously as maps $W(M) \to W(M')$ of sheaves of sets) are described as just continuous k-linear maps

(p) $$f : \Gamma_k^\wedge(M) \to \Gamma_k^\wedge(M'),$$

which are moreover compatible with the natural augmentations to k, and with the natural diagonal maps:

$$\varepsilon : \Gamma_k^\wedge(M) \to k, \quad \Delta : \Gamma_k^\wedge(M) \to \Gamma_k^\wedge(M) \hat{\otimes}_k \Gamma_k^\wedge(M) \, (\simeq \Gamma_k^\wedge(M \times M)),$$

(the latter deduced from the usual linear diagonal map $M \to M \times M$). When M is looked at as being embedded in $\Gamma_k^\wedge(M)$ by the exponential map (n), it is identified (if I remember it right) to the set of elements ξ in $\Gamma_k^\wedge(M)$ satisfying the relations

(q) $$\varepsilon(\xi) = 1, \quad \Delta(\xi) = \xi \otimes \xi,$$

where ε is the augmentation and Δ the diagonal map, hence (p) induces
[p. 452] a map (in general not additive)

(p') $$\Gamma(f) : M \to M'$$

(corresponding to the action of f, viewed as a map $W(M) \to W(M')$, on *sections* of $W(M)$) – and likewise after any ring extension $k \to k'$, defining a map

$$\Gamma(f)_{k'} : M \otimes_k k' \to M' \otimes_k k'$$

from $W(M)$ to $W(M')$ – which is the description of the map of sheaves $W(M) \to W(M')$ associated to a map (p). We have thus a description, in terms of linear algebra, of a category $U(k)$, and of a "sections" functor

(r) $$\Gamma : U(k) \to (\text{Sets}), \quad X \mapsto X(k),$$

which is essentially the functor (h) above, viewed in a different light. Now to our

Question 1. The category $U(k)$ and the functor (r) being defined as above in terms of linear algebra over the ground ring k, let $X_* = (n \mapsto X_n)$ be a semisimplicial object in $U(k)$, consider the corresponding semi-simplicial set $X_*(k) = (n \mapsto X_n(k))$, and the semi-simplicial k-module $L(X_*) = (n \mapsto L(X_n))$, where, for an object M of $U(k)$, $L(M)$ is defined as

(s) $$L(M) = \Gamma_k^{\wedge}(M),$$

which depends functorially on M in $U(k)$. Then in the derived category of (Ab), is there a canonical isomorphism between $L(X_*)$ (with ground ring restricted from k to $\mathbb{Z}$) and the abelianization $\mathbb{Z}^{(X_*(k))}$ of $X_*(k)$?

[...unreadable...] $k = \mathbb{Z}$, and components X_n projective.

We may have to throw in some extra assumption on X_*, at any rate

(t) $$X_0 = e \quad \text{(final object of } U(k)),$$

giving rise to $L(X_0) \simeq k$. Also, we may have to restrict to $k = \mathbb{Z}$, or otherwise correct the obvious drawback that the two chain complexes don't have isomorphic H_0 (one is k I guess, a k-module in any case, the other is $\mathbb{Z}$), by truncating accordingly the two chain complexes ("killing" their H_0). There is a natural candidate for a map

(u) $$\mathbb{Z}^{(X_*(k))} \to L(X_*),$$

by using the functorial map

(u') $$\mathbb{Z}^{(M)} \to L(M) = \Gamma_k^{\wedge}(M),$$

deduced from the inclusion (n)

$$M \hookrightarrow L(M),$$

and we may still specify the question above, by asking *whether* (u) *induces an isomorphism for homology groups* in dimension $i > 0$.

I am not too sure whether all this isn't just complete nonsense – it is worth getting it clear whether it is or not, at any rate! There is one case of special interest, the "simplest" one in a way, namely when the simplicial maps between the X_n's, each represented by a k-module M_n, are in fact k-linear, in other words, when X_* comes from a semisimplicial k-module M_* – more specifically still, when this is a $K(\pi, n)$ type, say the nicest semisimplicial model of this, using the Kan-Dold-Puppe functor for the chain complex of k-modules, having π in degree n and zero elsewhere. Then the left-hand side of (u) gives rise to the Eilenberg-Mac Lane homology groups [p. 453]

(v) $$\mathrm{H}_i(\pi, n; \mathbb{Z}),$$

which I guess should be k-modules a priori, because of the operations of k upon π, and the question then arises whether these can be computed using the right-hand complex $L(X_*)$. Maybe such a thing is even a familiar fact for people in the know? If it turned out to be false even for $k = \mathbb{Z}$, my faith in schematization of homotopy types would be seriously shaken I confess...

* * *

After a little break for dinner, just one more afterthought. Working with the completions $\Gamma_k^\wedge$ may seem a little forbidding, and all the more so if it used for computing *homology* invariants, not cohomology (the latter more likely to involve infinite products in the corresponding (cochain) complexes...). On the other hand, as was explicitly stated from the beginning, the natural context here seems to be *pointed* homotopy types, and hence "pointed" algebraic paradigms for these – an aspect we lost sight of, when looking for a suitable description of some category $U(k)$ of "unipotent bundles" over k. It would seem that in the "question" above, we should therefore insist that X_* should be a semi-simplicial object of the category $U(k)^\bullet$ of "pointed" objects of $U(k)$, namely objects X endowed with a section over the final object e (i.e., with an element in $X(k)$). This will be automatic at any rate in terms of the condition (t), $X_0 = e$. The point I wish to make is that the category of pointed objects of $U(k)$ admits a somewhat simpler description (by choosing the marked point as the "origin" for parametrization of the given object X of $U(k)$ by a k-module M), by model-objects which are still arbitrary k-modules M, but the "maps" now being *k-linear continuous maps*

$$f : \Gamma_k(M) \to \Gamma_k(M') \tag{p'}$$

[p. 454] between the k-modules Γ_k, without having to pass to completions, satisfying compatibility with augmentations and diagonal maps, and the extra condition (expressing that $\Gamma(f)(0) = 0$ in exponential notation):

$$f(1) = 1,$$

i.e., f reduced to component $\Gamma_k^0(M) \simeq k$ of degree zero is just the identity of k with $k \simeq \Gamma_k^0(M')$. Accordingly, we have a less awkward k-linearization functor than L in (s), namely "pointed linearization" L_{pt}:

$$M \mapsto L_{\mathrm{pt}}(M) \overset{\mathrm{def}}{=} \Gamma_k(M) : U(k)^\bullet \to \mathrm{Ab}_k, \tag{s'}$$

which seems to me the better candidate for describing linearization. Thus, we better rephrase now the "question" above in terms of (s') rather than (s). One trouble however is that the comparison map (u) takes values in $L(X_*)$, not $L_{\mathrm{pt}}(X_*)$, therefore, we may still have to use the "prohibitive" $L(X_*)$ as an intermediary for comparing the complexes $\mathbb{Z}^{(X_*(k))}$ and $L_{\mathrm{pt}}(X_*)$. It may be noted now that, while the first chain complex embodies Eilenberg-Mac Lane homology (v) (in the special case considered above), the second one $L_{\mathrm{pt}}(X_*)$ (in that same case) describes the value of the total derived functor of the familiar Γ_k functor, on the "argument" π placed in degree n, and the statement that the two are "the same" does sound like some standard Dold-Puppe type result which everybody is supposed to know from the cradle – sorry!

* * *

After another break (visit, tentative nap), still another afterthought. The final shape we arrived at for the "question" above, when working in the "pointed" category $U(k)^\bullet$ of "pointed unipotent bundles over k", was whether for any semisimplicial object X_* in $U(k)^\bullet$ satisfying (t) above, i.e., $X_0 = e$ (final object of $U(k)^\bullet$), the two canonical maps of chain complexes (in fact, semisimplicial abelian groups)

(w) $$\mathbb{Z}^{(X_*(k))} \to L(X_*) \hookleftarrow L_{\mathrm{pt}}(X_*)$$

are quasi-isomorphisms (for H_i with $i > 0$). I am not too sure yet if some extra conditions on X_* are not required for this to be reasonable – I want to review two that came into my mind.

As the maps (w) are functorial for varying X_*, it would follow from a positive answer that whenever [p. 455]

(x) $$X_* \to X'_*$$

is a map of semisimplicial objects in $U(k)^\bullet$ satisfying condition (t), and such that the corresponding map

(x') $$X_*(k) \to X'_*(k)$$

is a weak equivalence, and hence the map between the $\mathbb{Z}$-abelianizations is a weak equivalence too, i.e., a quasi-isomorphism, that the same holds for the corresponding map

(y) $$L_{\mathrm{pt}}(X_*) \to L_{\mathrm{pt}}(X'_*)$$

for the "schematic" k-linearizations. Now, this is far from being an evident fact by itself, except of course in the case when the map (x) above is a homotopism. Take for instance the case when we start with a map of chain complexes in Ab_k

(x") $$M_\bullet \to M'_\bullet,$$

hence a map between the associated semisimplicial k-modules

(x"') $$M_* \to M'_*,$$

which may be viewed as giving rise to a (componentwise linear) map between the associated semisimplicial objects X_*, X'_* in $U(k)^\bullet$ via the canonical functor

$$\mathrm{Ab}_k \to U(k)^\bullet;$$

the corresponding map (y) is then just the componentwise extension of (x"') to the enveloping algebras with divided powers

(y') $$\Gamma_k(M_*) \to \Gamma_k(M'_*).$$

The map (x') can now be identified with (x"'), hence it is a weak equivalence iff (x"') is, i.e., iff (x"') is a quasi-isomorphism. If we assume moreover the components of $M_\bullet$, $M'_\bullet$ to be projective objects in Ab_k, then from the assumption that (x") is a quasi-isomorphism it does follow that it is a chain homotopism, hence by Kan-Dold-Puppe the map (x"')

is a semisimplicial homotopism, and hence the same holds for (y'), and (y') therefore is indeed a quasi-equivalence. But without the assumption that components are projective, it is surely false that the mere fact that (x") is a quasi-isomorphism, implies that (y') is – otherwise this would mean that in order to compute the left derived functor of the
[p. 456] non-additive functor

$$\Gamma_k : \mathrm{Ab}_k \to \mathrm{Ab}_k,$$

it is enough, for getting its value on a chain complex $M'_\bullet$ say, to replace $M'_\bullet$ by M'_* and apply the functor Γ_k componentwise, without first having to take a projective resolution $M_\bullet$ of $M'_\bullet$ – something rather absurd indeed! Thus, the statement made on page 454, that when taking for $M'_\bullet$ the k-module π placed in degree n and zero in all other degrees, the corresponding $L_{\mathrm{pt}}(X'_*) = L_k(M'_*)$ embodies the value of the left derived functor $\mathrm{L}\Gamma_k$ on $M'_\bullet$, is visibly incorrect if we don't assume moreover that π is projective (flat, presumably, would be enough...). Otherwise, we should first replace π by a projective (or at any rate flat) resolution, which we shift by n to get $M_\bullet$, and then take $\Gamma_k(M_*)$ to get the correct value of $\mathrm{L}\Gamma_k(M'_\bullet)$.

This convinces me that in the question as to whether the maps in (w) are quasi-isomorphisms (the more crucial one of course being the first of the two), *we should assume moreover that the components of X_* are described in terms of* projective *k-modules M_n*, or at any rate k-modules that are flat. Accordingly, we should make the same restriction on the semisimplicial schematized model X_*, in order for the description we gave of "k-linearization" as $L_{\mathrm{pt}}(X_*)$ (or $L(X_*)$, never mind which) to be topologically meaningful. Very probably, in the whole schematization set-up, namely in the very definition we gave of $U(k)$ and $U(k)^\bullet$, we should stick to the same restriction. If I insisted first (with some inner reluctance, I admit) on taking k-modules M unrestricted, this was because I was thinking of M, more specifically of the M_i's occurring in the inductive "dévissage" of an object of $U_m(k)$ (when thinking of the more refined version of $U(k)$), as essentially the homotopy groups of the homotopy type we want to modelize, or rather, as the components of the corresponding semisimplicial k-modules (denoted M'_* some minutes ago). I was still thinking of course, be it implicitly, in terms of Postnikov dévissage, despite yesterday's remark that to use such dévissage literally may cause trouble (p. 444). Thus, the feeling which gets into the fore now is that *we should kind of forget Postnikov, and work with semisimplicial "schematic" models built up with k-modules which are* projective, or at any rate flat (namely torsion free, if $k = \mathbb{Z}$).

[p. 457] It may be remarked that if M is any k-module, then the property that M be projective, or flat, can be described in terms of the isomorphism class of the corresponding object X in $U(k)^\bullet$, or equivalently, of the functor $W(M)$ on $\mathrm{Alg}_{/k}$, with values in the category of pointed sets. Indeed, the isomorphism class of the k-module $\Gamma_k(M)$ depends only on the class of X, and it is easily seen that M is projective, resp. flat, iff $\Gamma_k(M)$ is. (The "only if" is standard knowledge of commutative algebra, the "if" comes from the fact that M is a direct factor of $\Gamma_k(M)$, hence projective resp. flat if the latter is.) Using this, one even checks that it is enough to know the isomorphism class of X in $U(k)$ – because two objects of $U(k)^\bullet$ are isomorphic in $U(k)^\bullet$ iff they are in $U(k)$.

* * *

The second afterthought is that in the question on page 452, we should definitely assume $k = \mathbb{Z}$. Already when asserting hastily (p. 453) that en Eilenberg-Mac Lane homology group $\mathrm{H}_i(\pi, n; \mathbb{Z})$ should automatically inherit a structure of a k-module, whenever π had one, I was feeling uncomfortable, because after all the dependence of this group upon variable π is not at all additive, thus the operation of k upon this group stemming from its operations on π was no too likely to come out additive! To take one example, take $n = 1$, i.e., we just take ordinary group homology for π, and assume that k is free of finite type over $\mathbb{Z}$, and $\pi = M$ free of finite type over k (for instance $M = k$), hence free of finite type over $\mathbb{Z}$ too. Then it is well-known that

$$\mathrm{H}_*(M; \mathbb{Z}) \simeq \bigwedge_{\mathbb{Z}} M,$$

the exterior algebra of M over $\mathbb{Z}$, which surely is not endowed with a structure of a k-module in any natural way! This, if there was any such structure (natural or not) on the highest non-zero term (corresponding to the rank d of M over $\mathbb{Z}$), it would follow that we get a ring homomorphism from k to $\mathbb{Z} \simeq \mathrm{End}_{\mathbb{Z}}(\mathrm{H}_d \simeq \mathbb{Z})$, and we may choose k in such a way that there is no such homomorphism, for instance $k = \mathbb{Z}[T]/(T^2 + 1)$. In this case, there cannot be *any* isomorphism between the H_d's of the two members of (u)! Another point, which I hit upon first, is that when X_* is defined in terms of a semisimplicial object M_* of Ab_k, then the functorial dependence of the first term in (w) with respect to varying M_*, [p. 458] is that to a direct sum corresponds the componentwise tensor product *over* $\mathbb{Z}$, whereas for the last term of (w) we have to take tensor products *over* k (in the middle term, *completed* tensor products over k) – the two variances are clearly at odds with each other.

Thus, when working with would-be "k-homotopy types" as defined here via semisimplicial objects in $U(k)^\bullet$, *we should altogether drop the idea that the homology groups of the corresponding semisimplicial set* $X_*(k)$ *are* k*-modules*. I wouldn't really look at them as being "the" homology groups of the k-homotopy type X_*, these should be rather given via k-linearization L_{pt} and they are k-modules, indeed, they come from a canonical object of $\mathrm{D}_\bullet(\mathrm{Ab}_k)$, namely $L_{\mathrm{pt}}(X_*)$. In the example just looked at, presumably we should get the exterior algebra of M *over* k (not $\mathbb{Z}$!). This makes me suspect even that, except in the case $k = \mathbb{Z}$, this semisimplicial set $X_*(k)$ doesn't make much sense, namely its homology invariants (and presumably its homotopy groups too) are not really relevant for the k-homotopy type X_*, *which has invariants of its own which are completely different. Thus, I am not at all convinced any more that the homotopy groups* $\pi_i(X_*)$ *carry* k*-module structures* (as expected at the beginning, p. 446) – but to clear our mind on that matter, we should take off from the simplistic example when X_* comes from a semisimplicial k-module M_*, in which case of course we have by Whitehead's isomorphism

$$\pi_i(X_*(k)) \simeq \mathrm{H}_i(M_\bullet)\ (\simeq \pi_i(M_*)),$$

where $M_\bullet$ is the chain complex associated to M_*. Also, it is now becoming obvious whether weak equivalences for objects X_* should be defined (as we did) via $X_*(k)$, as suggested by the "simplistic" example above. If we define homology of X_* by the formula

(z) $$\mathrm{LH}_\bullet(X_*) \overset{\text{def}}{=} L_{\mathrm{pt}}(X_*) \quad \text{viewed as an object of } \mathrm{D}_\bullet(\mathrm{Ab}_k),$$

and accordingly, the homology modules

(z') $$\mathrm{H}_i(X_*) \overset{\text{def}}{=} \pi_i(L_{\mathrm{pt}}(X_*)) = \mathrm{H}_i(\mathrm{LH}_\bullet(X_*)),$$

maybe the better idea for defining weak equivalences $X_* \to X'_*$ is by demanding that *they should be transformed into quasi-isomorphisms by the total homology functor* $\mathrm{LH}_\bullet$, or equivalently, induce isomorphisms for the homology modules (z'). If the answer to our crucial question
[p. 459] is affirmative (with the corrections made, including $k = \mathbb{Z}$), then in case $k = \mathbb{Z}$, the new definition just given for weak equivalences is equivalent to the old one in terms of $X_*(\mathbb{Z})$ (taking $\mathbb{Z}$-valued points), provided at any rate we admit or rather assume that $X_*(\mathbb{Z})$ and $X'_*(\mathbb{Z})$ are simply connected, which will be automatic if we work in the category of (schematized) models $\mathcal{M}_1(k)$, the condition (t) above ($X_0 = e$) being replaced by (i) with $n = 1$, i.e., by

(α) $$X_0 = X_1 = e,$$

to be on the safe side! Under this extra assumption at any rate, I feel definitely more confident with the new definition of weak equivalence, via the homology invariants (z'), rather than the old one. At any rate, the question of the two definitions being equivalent or not should be cleared up, namely:

Question 2. Let k be any ring, define the category $U(k)^\bullet$ of "pointed unipotent bundles over k" in terms of projective k-modules, with maps defined as in (p') page 453. (This is equally the correct set-up for question 1 on page 452, besides the extra condition $k = \mathbb{Z}$, as we saw before.) Let

$$u : X_* \to X'_*$$

be a map of semisimplicial objects in $U(k)$, satisfying both the extra assumptions (α) above. Then is it true that the corresponding map

(β) $$X_*(k) \to X'_*(k)$$

of semisimplicial sets is a weak equivalence, iff the map

$$L_{\mathrm{pt}}(X_*) \to L_{\mathrm{pt}}(X'_*)$$

is, i.e., iff u induces an isomorphism for the homology invariants H_i defined in (z') above (via the abelianization functor $L_{\mathrm{pt}} = \Gamma_k$).

If instead of (α) we only assume $X_0 = X'_0 = e$, it seems that we may have to throw in some other extra condition on X_* and X'_*, in order for the definition of weak equivalence in terms of the mere homology invariants to be reasonable – a condition which, at the very least, and in case $k = \mathbb{Z}$ say, should ensure that the homotopy types defined by the two terms in (β) should have *abelian fundamental groups* (which doesn't look at first sight to be automatic). At any rate, the conditions (α) above, which should be viewed as a 1-connectedness assumption, are natural enough, and it is natural too to try first to push through a theory of schematization of homotopy types, under this assumption.

26.8. [p. 460]

112 I am continuing the "wishful thinking" about schematization of homotopy types – a welcome break in the "overall review" on linearization (in the context of the modelizer (Cat)), which had been getting a little fastidious lately!

Postnikov dévissage and Kan condition for schematic complexes.

I'll admit, as one firm hold in all the wishfulness, that in the "Eilenberg-Mac Lane case" of p. 453, when moreover $k = \mathbb{Z}$ and the components X_i of the semisimplicial unipotent bundle X_* are projective, the two maps (w) of page 454 are indeed quasi-isomorphisms. From this should follow the similar statement, when X_* comes from a chain complex of k-modules $M_\bullet$ with projective coefficients, by reducing to the case when only a finite number of components of $M_\bullet$ are non-zero (by suitable passage to the limit), and then by induction on the number of these components, using the fact that the three terms in (w) depend on X_* in a "multiplicative" way, namely direct sums being transformed into tensor products. (NB Under the assumptions of projectivity made, we may as well express the quasi-isomorphisms (w) we start with as being semisimplicial *homotopisms*, and remark that componentwise tensor-product of such homotopisms is again one.) From this, using the relevant spectral sequences in homology, should follow that the maps (w) are still quasi-isomorphisms, whenever X_* can be "unscrewed" ("dévissé") as a finite successive fibering with fibers of the type M_* as above. Another passage to the limit will yield the same result for an infinite dévissage, provided the fibers $M(i)_*$ ($i = 1, 2, \ldots$) are "way-out", i.e., for given n, only a finite number of components $M(i)_n$ are non-zero (it amounts to the same to demand that the sequence $M(i)_\bullet$ of corresponding chain complexes be "way out"). This will give already, it seems, a fair number of cases when (w) are quasi-isomorphisms. (Admittedly, working this out will involve a fair amount of work, especially for getting the relevant properties of $L_\bullet$ and L_{pt}, which should mimic very closely the known ones for usual linearization, including spectral sequences or, more neatly, transitivity isomorphisms in the relevant derived categories...) The main point here is that those special types of X_*'s (we may call them *Kan-Postnikov complexes* in $U(k)$) are enough in order to modelize, via the corresponding semisimplicial sets $X_*(\mathbb{Z})$, arbitrary pointed homotopy types with abelian π_1. This is seen of course using Postnikov dévissage of a given homotopy type, and replacing every homotopy group π_i by a [p. 461] shifted projective resolution (a two-step resolution will do here) $M(i)_\bullet$, as indicated on page 456, and using the corresponding semisimplicial $\mathbb{Z}$-modules $M(i)_*$ as fibers, in the successive fiberings. To see that what is being done on the "discrete" level, working with semisimplicial *sets*, can be "followed" in an essentially unique way on the "schematic" level, we hit now of course upon the key difficulty, pointed out on page 444, about the Postnikov cohomology group $\mathrm{H}^{n+2}(X(n+1), \pi_{n+1})$ being isomorphic to the corresponding "schematic" group. Now, that this is so indeed should follow from the homology isomorphisms (w) I hope, just dualizing the result to cohomology. All this seems to sound kind of reasonable, it seems, even that for a given homotopy type in Hot_0, we should be able to squeeze out this way a unique *isomorphism type*, at any rate, of semisimplicial unipotent bundles – but to see whether it does work, or if there is some major blunder which turns the whole into nonsense, will come out only from careful, down-to-earth work, which I am not prepared to dive into.

It occurred to me that the "Kan-Postnikov" complexes in $U(k)$ have some special features among all possible complexes with $X_0 = e$, and also that some extra feature *are* needed, if we want the maps in (w) to be quasi-isomorphisms. I want to dwell upon this a little. First of all, the condition $X_0 = e$ is indeed essential, as we see by taking a constant complex with value X_0, then the homology of the three chain complexes (w) reduces to degree zero, and the H_0's are respectively

$$\mathbb{Z}^{(X_0(\mathbb{Z}))} = \mathbb{Z}^{(M_0)}, \quad L_0(X_0) = \Gamma_b Z^\wedge(M_0), \quad L_{\mathrm{pt}}(X_0) = \Gamma_{\mathbb{Z}}(M_0),$$

where M_0 is the $\mathbb{Z}$-module giving rise to X_0 – and none of the two maps is an isomorphism, unless $M_0 = 0$.

Take now the next simplest case, when X_* comes from a monoid object G in $U(k)$ in the usual way; then what we are after, in dual terms of cohomology rather than homology (taking the dual complexes of those in (w)), amounts essentially to asking whether the usual discrete cohomology of the discrete monoid $G(\mathbb{Z})$ can be computed, using *polynomial cochains* rather than arbitrary ones. Now, this we did admit as "well-known" in the most evident case of all, when G is being represented as an object of $U(\mathbb{Z})$ by a projective (hence free) $\mathbb{Z}$-module M, the multiplication law is just usual addition. It still looks reasonable enough when the monoid G is a *group*, with M of finite type say. In this
[p. 462] case, the Borel theory of algebraic affine groups over a field (here, the field of fractions $\mathbb{Q}$ of $\mathbb{Z}$) tells us that $G_{\mathbb{Q}}$ is a *nilpotent* algebraic group, and that therefore it admits a composition series with factors isomorphic to the additive group $\mathbb{G}_{\mathrm{a}\mathbb{Q}}$; presumably, the same dévissage then can be obtained over the base $\mathbb{Z}$, and using induction on the length of the composition series, and the Hochschild-Serre type of relations (traditionally expressed by a spectral sequence) between group cohomology of a group, quotient group and corresponding subgroup, we should get the wished for quasi-isomorphisms (w).

Take now, however, the simplest case of a monoid which isn't a group, namely the multiplicative law on the affine line, given by the polynomial law

$$W(\mathbb{Z}) \times W(\mathbb{Z}) \to W(\mathbb{Z}) : (x, y) \mapsto xy.$$

the corresponding "discrete" monoid is just $\mathbb{Z}^{(\times)}$, namely the integers with multiplication, its H^1 with coefficients in $\mathbb{Z}$ is just the group of all homomorphisms

$$(*) \qquad \mathbb{Z}^{(\times)} \to \mathbb{Z}^{(+)} = \mathbb{Z},$$

and denoting by $\mathbb{P}$ the set of all primes and using the prime decomposition of integers, we find that

$$\mathrm{Hom}(\mathbb{Z}^{(\times)}, \mathbb{Z}^{(+)}) \simeq \mathbb{Z}^{\mathbb{P}},$$

i.e., a family $(n_p)_{p\in\mathbb{P}}$ of integers being associated the homomorphism

$$\pm \prod_{p\in\mathbb{P}} p^{\alpha_p} \mapsto \sum_{p\in\mathbb{P}} \alpha_p n_p.$$

On the other hand, the schematic H^1 consists of all homomorphisms (*) that can be expressed by a polynomial, hence induce a homomorphism of algebraic group schemes $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{a}}$, and it is well-known (and immediately checked) that there is only the zero homomorphism!

Thus, it turns out that the assumptions made yesterday on X_*, in order for the "linearization theorem" (!) to hold, namely the maps (w) to be quasi-isomorphisms, are definitely not strong enough yet! One may think of throwing in the extra condition $X_1 = e$, so as to rule out monoids altogether (and even groups, too bad!), but I don't think this helps at all (didn't try though to make a counterexample). On the other hand, just restricting to Kan-Postnikov complexes seems rather awkward, we definitely don't want to drag along Postnikov fibrations as a compulsory ingredient of the complexes we work with. The idea which comes up here is just to "drop Postnikov and keep Kan" – namely *introduce a Kan type condition on semisimplicial complexes in $U(k)$*. If we mimic formally the usual "discrete" Kan condition, we get that (for [p. 463]
given pair of integers k, n with $0 \leq k \leq n$) a certain map from X_n, to a certain finite projective limit defined in terms of the boundary maps $X_{n-1} \to X_{n-2}$, should be epimorphic. Now, clearly $U(k)$ is by no means stable under fiber products, except under very special assumptions (including differential transversality conditions, at any rate), and on the other hand one feels that the notion of "epimorphism" one will have to work with in $U(k)$ will have to be a lot more exacting than the map $X(k) \to Y(k)$ on sections being surjective, or the usual categorical meaning within $U(k)$, which looks kind of silly here. Even the most exacting surjectivity condition on $X \to Y$, namely that it admit a section doesn't quite satisfy me – what I really want is that X should be a trivial bundle over Y, more specifically that X is isomorphic to a product $Y \times Z$, in such a way that the given map $X \to Y$ identifies with the projection $Y \times Z \to Y$. Maybe this is too exacting a condition, however, and hard to check in computational terms sometimes (?), maybe we should be content with demanding only that $X \to Y$ has a section, and

moreover is "smooth", i.e., has everywhere a surjective tangent map (which may be expressed on the corresponding sheaf on $\mathrm{Aff}_{/k}$ by the familiar condition of "formal smoothness", namely possibility of lifting sections over arbitrary infinitesimal neighborhoods…). We'll have to choose at any rate some such strong "surjectivity" notion in $U(k)$, which we'll call "*submersions*" say. Thus, I feel a "Kan complex" in $U(k)$ should have boundary maps which are submersions. What we should do, is to pin down some simple "Kan condition" on a complex X_*, in terms of "submersions", in such a way as to ensure, at any rate

a) that for any pair (n,k) with $0 \le k \le n$, the object $X_*(n,k)$ of "horns of type (n,k) of X_*", expressed by the suitable finite inverse limits (in terms of boundary maps $X_{n-1} \to X_{n-2}$) is representable in $U(k)$, and

b) the canonical map $X_n \to X_*(n,k)$ is a submersion,

and such of course that all Kan-Postnikov complexes should satisfy this Kan condition, at the very least.

The first non-trivial case, in view of $X_0 = e$, is $n = 2$, in which case $X_*(2,k)$ is trivially representable by $X_1 \times X_1$, and the condition we get is that the three natural maps coming from boundary maps

$$X_2 \rightrightarrows X_1 \times X_1 \qquad (**)$$

[p. 464] should be submersions for a "*Kan complex*". In case X_* is defined by a monoid object G as above, this clearly implies that G is a group – which rules out the counterexample above!

Of course, the very first thing we'll expect from a "good notion" of Kan complexes in $U(k)$, is that for $k = \mathbb{Z}$, it should make the linearization theorem work, namely that maps in (w) p. 454 are quasi-isomorphisms. The next thing, very close to this one but for arbitrary ground ring k now, is that a map

$$X_* \to X'_* \quad (\text{with } X_0 = X'_0 = X_1 = X'_1 = e)$$

of Kan complexes in $U(k)$ is a homotopism iff it induces an isomorphism on the homology modules (z') (p. 458) – which sounds reasonable precisely because we are working with Kan complexes. If this is so, the homotopy category $\mathrm{Hot}_1(k)$ of 1-connected homotopy types over k may be identified with a category of Kan complexes "up to homotopy", as usual (but working now with complexes of unipotent bundles over k). Third thing, still over arbitrary ground ring, would be a development of the usual homotopy formalism in the unipotent context, including (one hopes) homotopy fibers of maps, and Postnikov dévissage. Again, it is hard to imagine how to get such dévissage, without getting hold inductively of homotopy invariants π_i which are k-modules. This should come out if we are able to define homotopy fibers as for an $(n-1)$-connected k-homotopy type (defined here as one whose homology invariants H_i are zero for $i \le n-1$), π_n should be no more, no less than H_n, which is indeed a k-module. Coming back to $k = \mathbb{Z}$ again, this should imply that for $n \geq 1$ at any rate, the canonical functor

$$(***) \qquad \mathrm{Hot}_n(\mathbb{Z}) \to \mathrm{Hot}_n$$

induces a bijection between isomorphism classes of schematic and ordinary n-connected homotopy types – and it will be hard to believe this can be so, without this functor being actually an equivalence of categories – the expected apotheosis of the theory! Maybe to this end, one may even be able to introduce reasonable internal **Hom**'s within the category of schematic Kan complexes, in a way compatible with the familiar notion in the discrete set-up.

If it is possible indeed to construct Postnikov dévissage of a schematic Kan complex over any ground ring k, it is clear that this is compatible with restriction of ground ring, hence it would seem that formation of the homotopy invariants π_i is compatible with restriction of rings (whereas, as we noticed yesterday, the same does definitely *not* hold for the homology invariants H_i). Taking restriction to the ground ring $\mathbb{Z}$, this shows that the canonical functor (***) from schematic to discrete homotopy types is compatible with taking homotopy groups (but not with homology) – thus, the relation between X_* and the complex of sections $X_*(k)$ seems to be a rather close one, via the homotopy groups, which are the same (and thus, the homotopy groups of $X_*(k)$ seem to turn out to be k-modules after all!). By the way, speaking of "restriction of ground ring" for Kan complexes was a little hasty, in view of the projectivity condition on the components, which a priori seems to oblige us to assume k to be a projective k_0-module (for a given ring homomorphism [p. 465]

$$k_0 \to k \quad).$$

Still, the remark about the sections functor $X_* \to X_*(k)$ makes sense, without having to assume k to be a projective $\mathbb{Z}$-module! Also, we feel that, by analogy of what can be done in the linear set-up, when we define a total derived functor

$$\mathrm{D}_\bullet(\mathrm{Ab}_k) \to \mathrm{D}_\bullet(\mathrm{Ab}_{k_0})$$

without any assumption on the ring homomorphism $k_0 \to k$, a notion of ring restriction for schematic homotopy types should make sense without any restriction, as was surmised yesterday. As in the linear case, we should allow ourselves to work with schematic complexes which are *not* projective, but be prepared to take "resolutions" (in some sense) of such general complexes by the more restricted ones (with projective components).

There is no such difficulty in the case of the ring extension functor, which transforms projective bundles over k_0 into projective bundles over k. The reflections above suggest that, whereas ring extension is compatible with taking total homology invariants $\mathrm{LH}_\bullet$, via the corresponding functor

$$\mathrm{D}_\bullet(\mathrm{Ab}_{k_0}) \to \mathrm{D}_\bullet(\mathrm{Ab}_k),$$

it is compatible too with taking homotopy invariants π_i separately.

[p. 466] 27.8.

113 *"Soft" versus "hard" Postnikov dévissage, π_1 as a group scheme.*

What I was thinking of last night (see last sentence) is that whereas for total homology (not for the separate H_i's) we have the comprehensive formula

$$\text{(A)} \qquad \mathrm{LH}_\bullet(X_* \otimes_k k') \simeq \mathrm{LH}_\bullet(X_*) \overset{\mathrm{L}}{\otimes}_k k' \quad (\text{in } \mathrm{D}_\bullet(k')),$$

(where $\overset{\mathrm{L}}{\otimes}_k$ denotes the left derived functor of the ring extension functor for $k \to k'$, and $X'_* = X_* \otimes_k k'$ denotes ring extension for the semisimplicial unipotent bundle X_*), for homotopy modules we should have the term-by-term isomorphisms

$$\text{(B)} \qquad \pi_i(X_* \otimes_k k') \leftarrow \pi_i(X_*) \otimes_k k'.$$

This however was pretty rash indeed (it was time to go to sleep I guess!). Whereas the map on sections

$$X_*(k) \to X'_*(k')$$

does induce a map

$$\text{(B')} \qquad \pi_i(X_*) \simeq \pi_i(X_*(k)) \to \pi_i(X'_*) \simeq \pi_i(X'_*(k'))$$

which surely is k-linear, and hence induces a map (B), this map is certainly *not* an isomorphism without some flatness restriction either on k' over k, or on the k-modules $\pi_j(X_*)$ for $j < i$, as we had noted already three days ago when looking at the case when X_* comes from a chain complex $M_\bullet$ in (Ab_k) (with projective coefficients say), and hence X'_* comes from

$$M'_\bullet = M_\bullet \otimes_k k'.$$

If we look at the description of the *k-module* $\pi_i(X_*) = \pi_i$ in terms of a Postnikov dévissage of X_*, we should recall that the semisimplicial group object $M(i)_*$ which enters into the picture as the i'th step fiber is *not* the one defined *directly* (via the Kan-Dold-Puppe functor) by π_i placed in degree i, but rather by the chain complex $M(i)_\bullet$ *with projective components*, obtained by taking first a shifted projective resolution of π_i. Thus, by ring extension we get from this dévissage of X_* another one of X'_*, whose successive fibers are

$$M(i)'_* = M(i)_* \otimes_k k',$$

corresponding to the chain complexes

$$M(i)'_\bullet = M(i)_\bullet \otimes_k k'.$$

[p. 467] The latter has $\pi_i \otimes_k k'$ as homology module in degree i (and zero homology in degree $j < i$), but the homology modules in degree $j > i$ need not be zero. In other words, there is a canonical *augmentation*

$$\text{(B'')} \qquad M(i)'_* \to K(\pi_i \otimes_k k', i),$$

(where the second member is the semisimplicial k'-module defined by an i-shifted projective resolution of $\pi_i(X) \otimes_k k'$), but this augmentation need not be a quasi-isomorphism, unless we make the relevant flatness assumptions. To sum up, *the dévissage of*

$$X'_* = X_* \otimes_k k'$$

deduced from a Postnikov dévissage of X_* is not *a Postnikov dévissage of X'_**, unless we assume either k' flat over k, or the k-modules $\pi_j(X_*)$ flat. This at the same time solves the puzzle raised on page 444, and points towards a *serious shortcoming of the (usual) Postnikov dévissage – namely that it is not compatible with ground ring extension*, or, as we would say in the language of algebraic geometry, that this construction is not “geometric” – a harsh thing to say indeed!

At this point the idea comes up that we may define another dévissage, a lot more natural in the spirit of a theory of “abelianization” of homotopy types it would seem, and which is “geometric”, namely compatible with ring extension. Here, we'll have to work, though, with the “prohibitive” abelianization functor

$$U(k) \to \mathrm{Ab}_k, \quad X \mapsto L(X) \, (= \Gamma_k^{\wedge}(M))$$

(where M is a k-module “representing” the object X in $U(k)$), as we'll need the functorial embedding [see also page 474]

(C) $$X \to W(L(X))$$

(where

$$W : \mathrm{Ab}_k \to U(k)$$

denotes the canonical functor from k-modules to unipotent k-bundles); this is licit anyhow, if we admit that the canonical map

$$L_{\mathrm{pt}}(X_*) \to L(X_*)$$

(cf. page 454 (w)) is a quasi-isomorphism, i.e., a weak equivalence, for the complexes X_* we are working with. (This of course should hold over an arbitrary ground ring, not just $\mathbb{Z}$.) Applying (C) componentwise, we get for a complex of bundles X_* a canonical map into its abelianization [p. 468]

(C') $$X_* \to W(L(X_*)).$$

Postnikov's construction, for $(n-1)$-connected X_*, consists in composing this map with the “augmentation”

$$W(L(X_*)) \to K(\pi_n, n)$$

(where $\pi_n \simeq \mathrm{H}_n$ is the first possibly non-trivial homology module of the chain complex corresponding to $L(X_*)$), and after this only take the homotopy fiber, and iterate (the homotopy fiber will be n-connected now). This process, in a way, breaks the natural abelianization into pieces, a brutal thing to do one will admit, all the more so as we start with a beautiful complex with projective components, and kind of destroy its unmarred harmony by tearing out of breaking off the most showy part, H_n to name it, which now looks so lost and awkward we really can't just leave it as it is, we first have to take a projective resolution of it, choosing it as we may…But we won't do all this, will we, and rather keep abelianization and (C') as God gave them to us, and take the homotopy fiber (we hope God will give this too…), and repeat the process, without even having to care at any stage which π_i's vanish and which not. Let's call this the "*soft Postnikov dévissage*", in contrast to the "brutal one". In describing the process (which of course makes sense in the discrete context as well as in the schematic one), I implicitly admitted that the X_* we start with is 1-connected, or for the very least has abelian π_1 (a notion we'll have to come back to, in the schematic set-up). But we may as well apply it to a (discrete) $K(G,1)$ type, G any discrete group, then it amounts to taking the descending filtration of G by iterated commutator groups, which is a finite filtration iff G is solvable. Maybe it would be more natural still to take the similar descending filtration, suitable for the study of *nilpotent* groups rather than solvable ones, with

[this second superscript in this equation is rather hard to read in the typescript…]

(D) $$G^{(n+1)} = [G, G^{(n-1)}],$$

where $[A,B]$ denotes the subgroup of G generated by commutators

$$(a,b) = aba^{-1}b^{-1},$$

with a in A, b in B. It doesn't seem there is a similar distinction to make in case we start with a 1-connected X_*, more specifically if $X_0 = X_1 = e$. There may be some extra caution needed, however, when we assume [p. 469] only $X_0 = e$ without assuming 1-connectedness, even when π_1 is abelian, because of the possibility of operation of π_1 upon the π_i's. Maybe, when trying to modelize usual homotopy types by complexes of unipotent bundles over $\mathbb{Z}$, we should restrict to homotopy types which are not only 0-connected and have abelian π_1, but moreover with π_1 operating trivially (or for the very least, in a unipotent way) upon the higher π_i's. At any rate, as soon as π_1 operates non-trivially (on itself, or on the higher π_i's) there will presumably be two non-equivalent ways for defining soft Postnikov dévissage, corresponding to the two standard descending commutator group series in a discrete group G The more relevant in view of unipotent schematization would seem to be the "nilpotent" one.

Restricting for simplicity to the 1-connected case $X_0 = X_1 = e$, I would expect soft Postnikov dévissage to be the key for an understanding as well of the behavior of the $\pi_i(X_*)$ modules with respect to ring extension, as of the full relationship between these invariants, and the homology invariants $\mathrm{H}_i(X_*)$.

* * *

I still should have a look upon complexes X_* in $U(k)^\bullet$ satisfying (as always in this game) $X_0 = e$, but not necessarily $X_1 = e$. Even when we make the Kan assumption (plus “*smoothness*” of the components, by which I mean that their k-linearization is projective), I don’t feel too sure yet if they fit into a good formalism, for instance (when $k = \mathbb{Z}$) if they satisfy the “linearization theorem” (quasi-isomorphy for the two maps in (w) p. 454). If we start for instance with a group object G of $U(k)$ and let X_* be the corresponding semisimplicial complex in $U(k)^\bullet$, then we get an isomorphism

$$\pi_1(X_*) \stackrel{\text{def}}{=} \pi_1(X_*(k)) \simeq G(k), \tag{E}$$

which shows us that $\pi_1(X_*)$ need not be abelian even when $k = \mathbb{Z}$. If we assume that $X_1 = G$ is “of finite presentation” (namely the projective k-module which describes X_1 is of finite presentation), or what amounts to the same, representable by an actual (group) scheme, it is true, however, that G and hence $\pi_1 = G(k)$ is *nilpotent* (this holds for any k). It looks an intriguing question whether π_1 is nilpotent under the only assumption that X_* is a smooth Kan complex with X_1 a scheme (without assuming anymore X comes from a group object). At any rate, it follows from the [p. 470] Kan condition that π_1 may be interpreted as a quotient set of $E \stackrel{\text{def}}{=} X_1(k)$ (without having to pass to the full free group generated by this set), with a set of relations

$$z_i = x_i y_i, \quad i \text{ in } X_2(k) = I, \tag{F}$$

indexed by $X_2(k)$, where

$$i \mapsto x_i, \quad i \mapsto y_i, \quad i \mapsto z_i$$

are the three boundary maps, remembering moreover the Kan condition that the three maps

$$I \rightrightarrows E \times E, \quad i \mapsto (x_i, y_i), \quad i \mapsto (x_i, z_i), \quad i \mapsto (y_i, z_i)$$

are *surjective*, which implies indeed that any element of the group π described by the set of generates E and relations (F) comes from an element in E. Replacing k by any k-algebra k', we see that we have a presheaf

$$k' \mapsto \pi_1(X_*(k')) = \pi_1(X_* \otimes_k k')$$

on the category $\mathrm{Aff}_{/k}$ of affine schemes over k, with values in the category of groups, which may be viewed (as a presheaf of sets) as a quotient presheaf of the presheaf on $\mathrm{Aff}_{/k}$ defined by X_1. We feel that this presheaf will fit into a reasonable “schematic” set-up, only if it turns out to be a *sheaf*, and more exactingly still, if this sheaf is isomorphic (as a sheaf of sets) to one stemming from an object of $U(k)$, i.e., if it is isomorphic to a sheaf $W(M)$, for suitable k-module M (not necessarily a projective one). If we denote by G this object of $U(k)$, it will be endowed with a group structure, and *it is this group object of* $U(k)$*, rather than just the set-theoretic group of its sections*, i.e., *of* k*-valued “points”, which merits to be viewed as the “true”* $\pi_1(X_*)$. To say it differently, whereas the higher

$\pi_i(X_*) \overset{\text{def}}{=} \pi_i(X_*(k))$ (for $i \geq 2$), in the cases considered so far, should be viewed as being not mere abelian groups, but moreover endowed with a natural k-module structure, in the case when $i = 1$, i.e., for the fundamental group $\pi_1(X_*)$, the natural structure to expect on this (possibly non-commutative) group is a "unipotent schematic" structure, namely essentially a pointed "parametrization" of this group by elements of a suitable k-module M_1, in such a way that the composition law is expressed in terms of a polynomial law, making sense therefore not only for k-valued points, i.e., for elements of M, but for k'-valued
[p. 471] points as well (for any k-algebra k'), namely defining a group law on $W(M)(k') = M \otimes_k k'$. If henceforth we denote by $\pi_1(X_*)$ this group object of $U(k)$, the relevant formula now is

(G) $$\pi_1(X_*(k')) \simeq \pi_1(X_*)(k'),$$

a group isomorphism functorial with respect to variable k-algebra k', which will imply the corresponding isomorphism

(G') $$\pi_1(X_* \otimes_k k') \simeq \pi_1(X_*) \otimes_k k'$$

of groups objects in $U(k')$, i.e., formation of the "schematic" π_1 is compatible with ground ring extension $k \to k'$ (provided π_1 exists, for a given X_*).

We will expect the map of passage to quotient

(H) $$X_1 \to \pi_1(X_*) = G$$

to be "epimorphic" in a very strong sense, stronger even than just in the sense of presheaves, the first thought that comes to mind here is that it should be a "*submersion*", in the sense suggested in yesterday's reflections in connection with the description of the Kan condition. If, however, we want to be able to get for $\pi_1(X_*(k)) = \pi_1(X_*)(k)$ *any* abelian group beforehand, in the case $k = \mathbb{Z}$ say, without demanding that it be a projective k-module, and still get it via an X_* with *smooth* components, this shows that when defining a notion of "submersion" for objects of $U(k)$ which may not be smooth, we should not be quite as demanding as suggested yesterday (cf. page 463), but find a definition which will include also any map $X \to Y$ coming from an epimorphism $M \to N$ of k-modules (which will allow us to take X_1 as associated to a projective k-module admitting the given π_1 as its quotient). One idea that comes to mind here, is to take this property as the *definition* of a submersion, as an arrow in $U(k)$ which is isomorphic to one obtained from an epimorphism in Ab_k. This, of the three definitions that have come to my mind so far for this notion, is the one which looks the most convincing to me. I wouldn't expect too much from a complex X_*, even a smooth one and satisfying the Kan condition, unless (in terms of the three boundary maps from X_2 to X_1) it gives rise, as just explained, to a group object $G = \pi_1(X_*)$ in $U(k)$, together with a *submersion* (H). Thus, definitely, when defining a schematic model category $\mathcal{M}_n(k)$ of n-connected ss complexes of unipotent bundles over k, I feel like insisting

[in the typescript there is here a tautological equation, but I think this is what was meant…]

in case $n = 0$ at least upon this extra condition (plus of course $X_0 = e$). The still more stringent condition one may think of, in order to have schematic models as close as one may wish to k-modules, is to demand that moreover G is isomorphic to the object of $U(k)$ defined by a k-module M_1, the group law moreover coming from the addition law in M. This condition is stronger still than merely demanding that G be commutative, even when M_1 is free of rank one, because one knows that over a non-perfect field k there may be "forms" of the additive group $\mathbb{G}_a$ which are not isomorphic to $\mathbb{G}_a$; presumably, there should be similar examples over $\mathbb{Z}$ too, with rank larger than one, however. [p. 472]

We feel, however, that the case when $G = \pi_1(X_*)$ is a non-linear or even a non-abelian group object of $U(k)$ is still worthy of interest. The first test it would seem, to check if we do have a good notion indeed, is to see if it does satisfy to the "linearization theorem" in case $k = \mathbb{Z}$, i.e., the maps (w) on page 454 are quasi-isomorphisms. Another key test, which now makes sense for arbitrary k, is whether for a smooth Kan complex in $\mathcal{M}_0(k)$ (i.e., satisfying the extra assumption involving G), X_* is homotopic to a bundle over $K(G,1)$, with a 1-connected fiber, or more specifically, a fiber Y_* satisfying $Y_0 = Y_1 = e$. Among other features to expect is a natural operation of the group object G on the k-modules $\pi_i(X_*)$, as well as $\mathrm{H}_i(X_*)$, If however we wish, for $k = \mathbb{Z}$, to use models in $\mathcal{M}_0(\mathbb{Z})$ for describing possibly homotopy types with nilpotent π_1 say, and devise a corresponding equivalence between suitable homotopy categories, we should first investigate the question of the relationship between nilpotent discrete groups, and group objects of $U(k)$ – a question already touched on earlier in our reflection on linearization (see end of section 94), and of separate interest.

and moreover unipotent action on the π_i's

114 Outline of a program.

114 During these four days of reflection on schematization of homotopy types, a relatively coherent picture has gradually been emerging from darkness. How far this image reflects substantial reality, not just day-dreaming, I would be at a loss to tell now. Maybe some substantial corrections will have to be made still, besides getting in other ideas for a more complete picture – I would be amazed at any rate if everything should turn out as just nonsense! If it doesn't, there is surely a lot of work ahead to get everything straightened out and ready-to-use. I will leave it at that I suppose, for the time being – maybe just finish this digression by a quick review of the set-up, and of some main questions which have come out.

For a given ground ring k, the basic category we'll use of "schematic" objects over k is the category of *unipotent bundles* over k, which may be defined as the category of functors from $\mathrm{Alg}_{/k}$ to (Sets) isomorphic to functors of the type [p. 473]

$$W(M) = (k' \mapsto M \otimes_k k'),$$

where M is any k-module. We do not restrict, here, M to be projective or flat, as we definitely want to have, for a ring homomorphism $k \to k'$, a problemless functor "*restriction of scalars*"

$$U(k') \to U(k),$$

inserting in the commutative diagram

(I)
$$\begin{tikzcd} \mathrm{Ab}_{k'} \arrow[r, "\text{restr.}"] \arrow[d, "W_{k'}"'] & \mathrm{Ab}_k \arrow[d, "W_k"] \\ U(k') \arrow[r, "\text{restr.}"] & U(k) \end{tikzcd} .$$

Another reason is that we want that the k-modules of the type $\pi_i(X_*)$ which will come out should be eligible for defining objects in $U(k)$. We are more specifically interested though in "*smooth*" objects of $U(k)$, namely those that correspond to projective k-modules. (We prefer to call them "smooth" rather than "projective", in order to avoid confusion with the notion of a projective object in the usual categorical sense for $U(k)$.) Another relevant notion is the notion of a *submersion*, namely a map in $U(k)$ isomorphic to one coming from an epimorphism $M \to N$ in Ab_k. (If the latter can be chosen to have a projective kernel, we may speak of a smooth submersion.) The ring restriction functor transforms submersions into submersions, and also smooth objects into smooth ones provided k' is projective as a module over k. We also have a *ring extension functor* from $U(k)$ to $U(k')$, giving rise to a diagram (I') similar to (I) above, it transforms submersions into submersions, smooth objects into smooth ones.

The smoothness condition is likely to come in in two ways, one is via *flatness* (we may call an object of $U(k)$ "flat" when it isomorphic to some $W(M)$ with M a flat k-module), whereas projectivity is needed in order to ensure that in certain cases, weak equivalences are homotopisms. Flatness is the kind of condition which ensure the validity of "naive" universal coefficients formulæ for homotopy or homology objects, whereas projectivity may be needed in case of such formulæ for cohomology rather than homology.

[p. 474] The description I just recalled of $U(k)$ is the one most intuitive to my mind, other people may prefer the more computational one on page 451 in terms of $\Gamma_k^{\wedge}(M)$ (endowed with its augmentation to k and its diagonal map), which is of importance in its own right. It shows the existence of a canonical *k-linearization functor*

(J)
$$L : U(k) \to \mathrm{Ab}_k,$$

giving rise to the commutative diagram (up to can. isom.)

(J')
$$\begin{tikzcd} \mathrm{Ab}_k \arrow[rr, "W"] \arrow[dr, "\Gamma_k^{\wedge}"'] & & U(k) \arrow[dl, "L"] \\ & \mathrm{Ab}_k & \end{tikzcd} ,$$

where

$$\Gamma_k^{\wedge}(M) = \prod_{i \geq 0} \Gamma_k^i(M).$$

This linearization is not quite compatible with ring extension, it becomes so only when we view it as a functor with values, not just in the category Ab_k of k-modules, but of separated and complete linearly topologized k-modules, the ring extension functor for these being the *completed* tensor product. This is a little (or big?) technical drawback for this notion of linearization. We have a canonical embedding

(K) $$x \mapsto \exp(x) : X \to W^{\wedge}(L(X)) \quad \text{(cf. p. 450, (n))}$$

functorial in X, where for a topological k-module M as above, described as a filtering inverse limit of discrete ones M_i, we define

(K') $$W^{\wedge}(M) = (k' \mapsto \varprojlim_i M_i \otimes_k k').$$

The map (K) has a universal property with respect to all possible maps $X \to W^{\wedge}(M)$ with M a linearly topologized separated and complete k-module, which accounts for its role as "linearization". It should be noted here that the map (C) on page 467 doesn't quite exist, we have corrected this point here – definitely we cannot in (K) replace $W^{\wedge}$ by W. Of course, linearization L (or its variant L_{pt}) doesn't commute in any sense whatever to restriction of ground ring.

The image of X in $W^{\wedge}(L(X))$ is characterized by the simple formulæ (q) p. 451. Maps from X to Y may be described as just continuous k-linear maps from $L(X)$ to $L(Y)$, compatible with augmentations and diagonal maps.

We'll more specifically work in the category $U(k)^{\bullet}$ of *pointed* objects of $U(k)$, namely objects endowed with a section over the final object e, the so-called *pointed unipotent bundles*. We now have a functor [p. 475]

(L) $$W^{\bullet} : \mathrm{Ab}_k \to U(k)^{\bullet}$$

deduced from W using the fact that $W(0) = e$, and a "*pointed linearization functor*"

(M) $$L^{\bullet} \text{ or } L_{\mathrm{pt}} : U(k)^{\bullet} \to \mathrm{Ab}_k,$$

giving rise to a commutative diagram similar to (J')

(M') $$\begin{tikzcd} \mathrm{Ab}_k \arrow[rr, "W^{\bullet}"] \arrow[dr, "\Gamma_k"'] & & U(k)^{\bullet} \arrow[dl, "L^{\bullet}"] \\ & \mathrm{Ab}_k & \end{tikzcd} \quad ;$$

the notation $L^{\bullet}$ seems here the most coherent one, but may bring about confusion with the similar notation for some cochain complex say, therefore we had first used the alternative notation L_{pt}, to which one may still come back if needed. This time the functor $L^{\bullet}$ commutes to ring extension without any grain of salt. We have of course a canonical embedding

(N) $$L^{\bullet}(X) \hookrightarrow L(X)$$

defined via the corresponding embedding for an object M of Ab_k

(N') $$\Gamma_k(M) \hookrightarrow \Gamma_k^{\wedge}(M),$$

by which $L(X)$ may be viewed as the completion of $L^\bullet(X)$ with respect to the topology it induces on it, which is a *canonical topology* on $L^\bullet(X)$. Maps in $U(k)^\bullet$ correspond to *k-linear maps*

$$L^\bullet(X) \to L^\bullet(Y)$$

which are moreover *continuous, and compatible with coaugmentation* (i.e., transforms 1 into 1) *as well as with augmentations and diagonal maps*. I wonder if there is any simple characterization of submersions in $U(k)^\bullet$ in terms of the corresponding map between the linearizations. At any rate, an object X of $U(k)^\bullet$ is smooth resp. flat iff $L^\bullet(X)$ is a projective resp. a flat k-module.

For any natural integer, we want now to define a model category

(O) $$\mathcal{M}_n(k) \subset \mathbf{Hom}(\triangle^{\mathrm{op}}, U(k)^\bullet),$$

[p. 476] which should be a full subcategory of the category of semisimplicial objects in $U(k)^\bullet$. We'll get a functor

(P) $$X_* \mapsto X_*(k) : \mathcal{M}_n(k) \to \mathbf{Hom}(\triangle^{\mathrm{op}}, \mathrm{Sets}^\bullet)$$

from this category to the category of semisimplicial pointed sets. For $n \geq 1$, the only condition, it seems, to impose upon X_* in the second member of (O), i.e., upon a semisimplicial pointed unipotent bundle over k, in order to belong to $\mathcal{M}_n(k)$, is

($\mathrm{Q_n}$) $$X_i = e \quad \text{for} \quad i \leq n.$$

This, for $n = 0$, reduces to the common condition

($\mathrm{Q_0}$) $$X_0 = e,$$

which definitely is not enough, though, to get a category of "models" $\mathcal{M}_0(k)$ whose objects should have the kind of properties we are after. There are various kinds of extra restrictions one may want to impose, according to the type of situations one wants to describe, some hints along these lines are given on pages 469–472. For a preliminary study, the case $n \geq 1$, and more specifically, the case $n = 1$, is quite enough, the latter corresponding to the restrictions

($\mathrm{Q_1}$) $$X_0 = X_1 = e.$$

From P we get a functor

(P') $$\mathcal{M}_n(k) \to \mathrm{Hot}_n^\bullet = \text{category of pointed } n\text{-connected homotopy types},$$

we define a map in $\mathcal{M}_n(k)$ to be a *weak equivalence* if its image by (P) is, i.e., its image by (P') is an isomorphism, and localizing by weak equivalences we get the category

(R) $$\mathrm{Hot}_n(k)$$

of “n-connected schematic homotopy types over k”, together with a “sections functor” induced by (P)

(R’) $$\mathrm{Hot}_n(k) \to \mathrm{Hot}_n.$$

One main point in our definitions is that *we hope this functor to be an equivalence of categories, in the case when* $k = \mathbb{Z}$, and of course $n \geq 1$.

The description just given of categories $\mathrm{Hot}_n(k)$ is suitable for defining functors of restriction of ground ring for $k \to k'$

(S) $$\mathrm{Hot}_n(k') \to \mathrm{Hot}_n(k),$$

compatible with the sections functor (R’) for k and k'. It isn’t directly suited, though, for describing ring extension – as a matter of fact, ring extension for homotopy types (an operation of greater interest than ring restriction surely) is *not* expressed, in general, by just performing the trivial ring extension operation [p. 477]

$$X_* \mapsto X_* \otimes_k k'$$

on models in $\mathcal{M}_n(k)$, unless we assume k' to be flat over k say – but even in this case it is by no means clear a priori that the operation above transforms weak equivalences into weak equivalences. This is very clearly shown by the linear analogon, the categories $\mathcal{M}_n(k)$ being replaced by the categories of chain complexes in Ab_k say, or by $\mathrm{Comp}^-(\mathrm{Ab}_k)$ or the like. In order to correctly describe ground ring extension on homotopy types, we’ll have first to take a suitable “resolution” of X_*, namely replace X_* by some K_* say, endowed with a weak equivalence

$$K_* \to X_*,$$

and K_* satisfying some extra assumptions. Maybe flatness of the components would be enough here. For other purposes, we may have to use resolutions which are even smooth (componentwise), or which satisfy a suitable *Kan condition* (or a type outlined on page 463), or both. Our expectation is that, when we restrict to the subcategory

$$\mathrm{sK}\mathcal{M}_n(k)$$

of the model category $\mathcal{M}_n(k)$ made up with smooth Kan complexes, *that the category* $\mathrm{Hot}_n(k)$ *may be described simply in terms of such sK-complexes “up to homotopy”*, as usual. If this is so, the ground ring extension functor follows trivially from a corresponding functor on the sK-model categories

(T) $$\mathrm{sK}\mathcal{M}_n(k) \to \mathrm{sK}\mathcal{M}_n(k'), \quad X_* \mapsto X_* \otimes_k k',$$

hence

(T’) $$\mathrm{Hot}_n(k) \to \mathrm{Hot}_n(k').$$

From the sections functor (R’) we get homotopy invariants π_i for an object in $\mathrm{Hot}_n(k)$, but the relevant k-module structure on these is not apparent on this definition. We have a better hold, via linear algebra over k, upon homology invariants of X_*, which are k-linear objects, and are definitely distinct (unless $k = \mathbb{Z}$) from the homology invariants of $X_*(k)$, which for general k are definitely of little interest it seems. The definition of homology goes via the pointed linearization functor (M)

$$\text{(U)} \qquad \begin{cases} \mathrm{LH}_\bullet(X_*) \overset{\text{def}}{=} L_{\mathrm{pt}}(X_*), \text{ viewed as an object in } \mathrm{D}_\bullet(\mathrm{Ab}_k), \\ \mathrm{H}_i(X_*) \overset{\text{def}}{=} \mathrm{H}_i(\mathrm{LH}_\bullet(X_*)) = \pi_i(L_{\mathrm{pt}}(X_*)) \text{ in } \mathrm{Ab}_k, \end{cases}$$

[p. 478] where the L in $\mathrm{LH}_\bullet$ suggests that we are taking something similar to a total left derived functor, and where definitely in the right-hand member we had to write L_{pt} and not $L^\bullet$, in order not to get sunk into a morass of confusion! In the formulæ (U) we should assume however that X_* is a smooth Kan complex, which will imply (if indeed $\mathrm{Hot}_n(k)$ may be described in terms of $\mathrm{sK}\mathcal{M}_n(k)$ as said above) that $\mathrm{LH}_\bullet$ may be viewed as a functor

$$\text{(U')} \qquad \mathrm{Hot}_n(k) \to \mathrm{D}_\bullet(\mathrm{Ab}_k),$$

and likewise the H_i's are functors from $\mathrm{Hot}_n(k)$ to Ab_k. In order to compute these homology invariants for an arbitrary complex in $\mathcal{M}_n(k)$, we'll first have to resolve it by a sK complex, and then apply (U).

We expect *that a map $X_* \to Y_*$ in $\mathcal{M}_n(k)$ is a weak equivalence iff the corresponding map for* $\mathrm{LH}_\bullet$ *is a quasi-isomorphism*, in other words we expect the functor (U') to be "conservative": a map in the first category is an isomorphism iff its image in the second one is. A second main feature we expect from linearization, is that in the case $k = \mathbb{Z}$ it corresponds to the usual abelianization of homotopy types. This statement, when made more specific as in (w) page 454, decomposes into two distinct ones. One is of significance over an arbitrary ring k, and states that *for a* sK *complex X_*, the inclusion* (coming from (N))

$$\text{(V)} \qquad L_{\mathrm{pt}}(X_*) \to L(X_*)$$

from L_{pt} into its completion, when viewed as a map of chain complexes in Ab_k (using the simplicial differential operator, or passing to the corresponding "normalized" chain complexes first) *is a quasi-isomorphism*. Whether this is always so or not, or whether noetherian conditions on k or some finiteness conditions for the components of X_* are needed, looks like a rather standard question of linear homological algebra! On the other hand, using the exponential embedding (K) for sections, we get another map of semisimplicial k-modules

$$\text{(V')} \qquad k^{(X_*(k))} \to L(X_*),$$

and here *the question again* (the expectation I might say?) *is whether this is a quasi-isomorphism*. This would just mean (if coupled with quasi-isomorphy of (V)) that the homology invariants (U) are just the usual homology invariants of the discrete homotopy type modeled by $X_*(k)$, but with coefficients not in $\mathbb{Z}$, but in k. We certainly do expect this to be true for $k = \mathbb{Z}$ – which was the content of "question 1" on page 452 (taken up again on page 454 and following). Of course, in case we don't assume $k = \mathbb{Z}$, writing $\mathbb{Z}^{(X_*(k))}$ instead of $k^{(X_*(k))}$ as we did in (w) p. 454 now looks kind of silly, and the idea in this k-linear context to take k-valued homology of $X_*(k)$ rather than $\mathbb{Z}$-valued one is evident enough! However, I was confused by the misconception that the *internal* homology of $X_*(k)$ should carry k-linear structure, as this was what I expected too for the invariants $\pi_i(X_*(k))$ (which seems to turn out to be correct). This misconception was corrected a few pages later (p. 457) but still I kept dragging along the silly first member of (w). Anyhow, it now just occurs to me that except in case $k = \mathbb{Z}$, it is definitely *false* that (V') is a quasi-isomorphism, except in some wholly trivial cases. Indeed, let H_i be the first non-vanishing homology invariant (U) of X_* (or more safely still, take $i = n+1$), then we definitely expect to have a canonical isomorphism of k-modules [p. 479]

(V) $$\pi_i(X_*(k)) \simeq \mathrm{H}_i(X_*) = \mathrm{H}_i$$

but we equally have by Hopf's theorem, as the lower π_j's of $X_*(k)$ are zero

$$\pi_i(X_*(k)) = \mathrm{H}_i(X_*(k), \mathbb{Z}),$$

hence

(V') $$\mathrm{H}_i \simeq \mathrm{H}_i(S, \mathbb{Z}), \quad \text{where } S = X_*(k),$$

which is not compatible with the guess that $\mathrm{H}_i \simeq \mathrm{H}_i(S, k)$ $(\simeq \mathrm{H}_i \otimes_{\mathbb{Z}} k)$. Presumably, the isomorphism (V') above is induced by the first map in (w) above, but (except for $k = \mathbb{Z}$) we should expect in (w) to have an isomorphism only for the lowest dimensional homology groups which are occurring in the two first members. Anyhow, it appears after all that this map in (w) is the more reasonable one compared to (V') above, as it yields an isomorphism on homology in the key dimension $n+1$, whereas (V') apparently will practically *never* give an isomorphism.

What is mainly lacking still in this review of the expected main features of schematization of homotopy types, is description of the k-module structure on the homotopy groups

(W) $$\pi_i(X_*) \stackrel{\text{def}}{=} \pi_i(X_*(k)),$$

or preferably still, a direct description of those invariants as k-modules, working within the model category $\mathcal{M}_n(k)$. This, as suggested in yesterday's notes (p. 464), may be achieved *by developing a theory of Postnikov dévissage within* $\mathcal{M}_n(k)$ *and using* (V) in order to pull ourselves by the bootstraps, defining homotopy finally in terms of homology. At this point it should be noted that the dévissage we'll have to use here is the "brutal" one, which we frowned upon earlier today! To develop such a formalism, it seems essential to work with smooth Kan complexes and projective resolutions of the k-modules π_i as they appears one by one. Whether we want to describe "hard" or "soft" Postnikov dévissage (see [p. 480]

p. 468 for the latter), one common key step is the *linearization map* coming from the exponential map (K) applied componentwise

$$\text{(W)} \qquad X_* \to W\hat{\ }(L(X_*)),$$

which we would like to look upon as defining a homotopy class of maps in $\mathcal{M}_n(k)$

$$\text{(W?)} \qquad X_* \to W(L_{\mathrm{pt}}(X_*)),$$

where the second member moreover is endowed with its natural abelian group structure (its components are abelian group objects of $U(k)$ and the simplicial maps are additive). To pass from (W) to (W?), it is felt that the essential step is that (V) above be a quasi-isomorphism, hence, applying the functor W, we should get a weak equivalence, hence an isomorphism in the derived category $\mathrm{Hot}_n(k)$, hence (W) implies (W?). The main flaw in this "argument" comes from the $W\hat{\ }$ in the second member of (W), which isn't quite the same as W definitely. Thus, some further amount of work will be needed, presumably, to get (W?) from (W). Of course, we can't possibly just keep (W) as it is, as for getting dévissage we need a map in $\mathcal{M}_n(k)$, whereas the map (W) is just a map of semisimplicial sheaves on $\mathrm{Aff}_{/k}$, where the second member is *not* in $\mathcal{M}_n(k)$, i.e., its components are not in $U(k)$. Once we got (W?) factoring (W) up to homotopy (NB of course we assume X_* to be an sK complex in all this), we still need a reasonable notion of homotopy fibers of maps in $\mathcal{M}_n(k)$, in order to push through the inductive step.

Thus, a large part of the weight of the work ahead may well lie upon *developing the standard homotopy constructions within the model category* $\mathcal{M}_n(k)$, as contemplated on page 464. This should be fun, if it can be done indeed! One difficulty here seems to be that Quillen's standard machines won't work, not "telles quelles" at any rate, because of the category $U(k)$ failing to be stable under finite limits – it doesn't even have fiber products. But I think I'll stop my ponderings on schematization here…

[p. 481] 28.8.

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$L(X)$ as the pro-quasicoherent substitute for $\mathcal{O}_k(X)$.

For the last four days, while reflecting on "schematization", each time I think I am going to be through with that unforeseen green apple within an hour or two, and get back to "l'ordre du jour" – and overnight something else still appears I feel I should still look into just a little; and there I am again, sure enough, with some extra reflection on "schematic linearization" which I hadn't quite understood yet, it appears to me now. These last days I had given up numbering formulas as usual by Arab ciphers (1), (2), etc., as I didn't want to "cut" the numbering of that unending "review" of section 104 to 109 which wasn't quite finished yet, got it only till formula (136). But now I will stop this nonsense with numberings (a), (b) and (A), (B), after all even if there are in-between "Arab" formulas now, this doesn't prevent me, when it comes to it, to start a "review" section with formula (137) and go on till (1000) if I like... And now to the schematic linearization functor again, for unipotent bundles!

When writing up that schematization program yesterday, some technical difficulties appeared at the end (see page before) for a proper understanding of the relationship between the two linearizations L_{pt} and L, in order to define, in a suitable derived category, a map

$$X_* \to W(L_{\mathrm{pt}}(X_*))$$

using the canonical term-by-term exponential map

$$X_* \to W\,\hat{}\,L(X_*).$$

It seems to me that the exact significance of the objects $L(X_*)$ or $W\,\hat{}\,L(X_*)$ isn't quite understood yet, and that the confusion which occurred between which usual kind of linearization we should compare this with, whether $X \mapsto \mathbb{Z}^{(X)}$ as I did first, or $X \mapsto k^{(X)}$ as it occurred to me yesterday (pages 478–479) , is quite typical of this lack of understanding. It now occurred to me that neither term, for a general ground ring k (namely, not assuming $k = \mathbb{Z}$), is reasonable, whereas the reasonable "usual" kind of linearization comparing with $L(X)$ (when X is a unipotent bundle or a ss complex of such) is

(1) $$X \mapsto \mathcal{O}_k^{(X)},$$

where $\mathcal{O}_k$ is the basic *quasicoherent* sheaf of rings over k, i.e., over $\mathrm{Aff}_{/k}$, given by the tautological functor

(2) $$\mathcal{O}_k : (\mathrm{Aff}_{/k})^{\mathrm{op}} \approxeq \mathrm{Alg}_{/k} \to (\mathrm{Rings}), \quad k' \mapsto k',$$

associating to any affine scheme $S = \mathrm{Spec}(k')$ over k the ring of sections of its usual Zariski structure sheaf, which ring is canonically isomorphic to k' itself! The operation (1) is the usual linearization operation with respect to this sheaf of rings, working in the topos of fpqc sheaves of sets over k which we described at some length in section 111 (p. 447). As I was fearing that working in such a thing would cause anguish to a number of prospective readers, I took pains to translate unipotent bundles from the geometric language which is the suggestive one, to the language of commutative algebra which is more liable to hide than to disclose geometrical meaning; so much so that in the process I myself lost contact somewhat with the geometric flavor, and more specifically still with this basic fact, that in our context of unipotent bundles and complexes of such, the "natural" coefficients for cohomology (such as the $\mathrm{H}^{n+2}(X(n)_*, \pi_{n+1})$ groups occurring in Postnikov dévissage) are by no means "*discrete*" ones such as $\mathbb{Z}$ or k, but *quasi-coherent* ones, namely provided by quasi-coherent sheaves of $\mathcal{O}_k$-modules or complexes of such. Thus, in the above Postnikov obstruction group, π_{n+1} does *not* stand as a constant group of coefficients (if it was, this would drag us into the niceties and difficulties of étale cohomology for the components $X(n)_i$ of the semisimplicial unipotent bundle $X(n)_*$); but using the k-module

[p. 482]

structure of π_{n+1} for defining a *quasi-coherent* sheaf of modules $W(\pi_{n+1})$ "over k", i.e., over $\mathrm{Aff}_{/k}$, it is this "continuous" sheaf (or "vector bundle") over k, lifted of course to the various components $X(n)_i$, which yields the correct answer. This was kind of clear in my mind the very first day when I started reflecting on schematization, even before introducing formally unipotent bundles (pages 443–444), but this instinctive understanding later became dulled somewhat, largely due, it seems to me, to the concession I had made to algebra, giving up to some extent the language of geometry.

Let's recall that the operation (1) may be defined as the solution of a universal problem, namely sending the non-linear object X into a "linear" one, namely into a sheaf of $\mathcal{O}_k$-modules (or a $\mathcal{O}_k$-module, as we'll simply say). This is expressed by a canonical map of sheaves of sets

$$X \to \mathcal{O}_k^{(X)} \tag{3}$$

(which I am tempted to call the *"exponential" map* for X, and denote by a corresponding symbol such as $\exp_X$), giving rise, for every module

[p. 483] (over $\mathcal{O}_k$) to a corresponding map which is *bijective*

$$\mathrm{Hom}_{\mathcal{O}_k}(\mathcal{O}_k^{(X)}, F) \simeq \mathrm{Hom}(X, F) \quad (\simeq \Gamma(X, F_X)), \tag{4}$$

where in the last member (included as a more geometric interpretation of the second) F_X denotes the restriction of F to the object X, more accurately to the topos (or site) induced on X by the ambient topos (or site) we are working in. Thus, we may indeed view (1) as the most perfect notion of linearization, as far as generality goes – it makes sense of course in any ringed topos (without even a commutativity assumption!). The only trouble is that, even for such a down-to-earth X as a unipotent bundle, the standard affine line E_k^1 say, the sheaf $\mathcal{O}_k^{(X)}$ in (1) is not quasi-coherent and therefore not too amenable it seems to computations – thus, we get easily from (4) a canonical map (for general X)

$$k^{(X(k))} \to \Gamma(k, \mathcal{O}_k^{(X)}) \quad (= \mathcal{O}_k^{(X)}(k))$$

(where the Γ in the second member denotes sections over k, i.e., value of a functor on $\mathrm{Alg}_{/k}$ on the initial object k, and remembering in the first member that the ring of sections of $\mathcal{O}_k$ is k), but I would be at a loss to make a guess as for reasonable conditions for this map to be an isomorphism! This may seem a prohibitive "contra" against using at all such huge sheaves as $\mathcal{O}_k^{(X)}$, the point though is that in most questions where such linearizations are introduced (mainly questions where interest lies in computing cohomology invariants), one is practically never interesting in taking the groups of sections of these, but rather in looking at their maps into sheaves of modules F precisely, which is achieved by (4), or taking more generally their Ext^i with such an F, which is achieved by the similar familiar formula

$$\mathrm{Ext}^i_{\mathcal{O}_k}(\mathcal{O}_k^{(X)}, F) \simeq \mathrm{H}^i(X, F_X), \tag{5}$$

more neatly

$$\mathrm{RHom}_{\mathcal{O}_k}(\mathcal{O}_k^{(X)}, F) \simeq \mathrm{R}\Gamma_X(F_X), \tag{5'}$$

valid of course again for any ringed topos. In the present context however, the "coefficients" F we are interested in, as was just pointed out, are not arbitrary $\mathcal{O}_k$-modules, but rather *quasi-coherent* ones. Thus, if we get a variant of (1)

$$X \to L(X) \tag{6}$$

with $L(X)$ some *quasi-coherent* module, giving rise to (4), or even to (5) and (5'), this would be for us a perfectly good substitute for (3), which would deserve the name of a "*quasicoherent envelope*" of X. Of course, this module $L(X)$ would be unique up to unique isomorphism, as the solution of a universal problem embodied by (4), namely as the quasi-coherent module representing the functor [p. 484]

$$F \mapsto \mathrm{Hom}(X, F) \simeq \Gamma(X, F_X) \tag{7}$$

on the category this time of all *quasicoherent* $\mathcal{O}_k$-modules.

For the unipotent schematization story, we are more specifically interested in the case when X comes from a quasicoherent module itself, by forgetting its module structure. Now, as well-known, the functor

$$M \mapsto W(M) : \mathrm{Ab}_k = (k\text{-Mod}) \to \text{category of quasicoherent } \mathcal{O}_k\text{-modules, Qucoh}(k) \text{ say} \tag{8}$$

is an equivalence of categories. Thus, for X defined by such an M, the question of representability of (7) within the category of quasicoherent modules, amounts to the similar question in Ab_k for the functor

$$N \mapsto \mathrm{Hom}(W(M), W(N)), \tag{7'}$$

where the Hom denotes homomorphisms of sheaves *of sets* of course. Now, as suggested first, somewhat vaguely still, in section 111 (page 450), we have an alternative expression of this functor, via

$$\begin{aligned} \mathrm{Hom}(W(M), W(N)) &\simeq \mathrm{Homcont}_k(\Gamma_k^{\wedge}(M), N) \\ &\simeq \varinjlim_i \mathrm{Hom}_k(\Gamma_k(M)(i), N), \end{aligned} \tag{9}$$

where in the second member, $\mathrm{Homcont}_k$ denotes the set of k-homomorphisms which are continuous on

$$\Gamma_k^{\wedge}(M) = \prod_{i \geq 0} \Gamma_k^i(M) \tag{10}$$

(endowed with the product of discrete topologies), and in the third we have written

$$\Gamma_k(M)(i) = \prod_{j \leq i} \Gamma_k^j(M) \tag{10'}$$

for the product of the i first factors occurring in (10). The map (9) is deduced in the evident way from the exponential map

$$M \to W\,\hat{}\,\Gamma_k^{\wedge}(M) \overset{\text{def}}{=} \varprojlim_i W(\Gamma_k(M)(i)). \tag{11}$$

(NB The relation between the description (9) of maps $W(M) \to W(N)$ with the description given p. 451 in terms of maps (p) from $\Gamma_k^{\wedge}(M)$ to $\Gamma_k^{\wedge}(N)$, is by associating to such a map f its composition with the projection of the target upon its factor $N\ldots$). An incorrect way of expressing (9), which I slipped into in section 111 and kind of remained in till now,
[p. 485] is by pretending that the k-module $\Gamma_k^{\wedge}(M)$ represents the functor (7'), this is clearly false, as we do not have any canonical map from $W(M)$ into $W(\Gamma_k^{\wedge}(M))$, only into $W\,\hat{}\,\Gamma_k^{\wedge}(M)$ – we have an embedding

$$W(\Gamma_k^{\wedge}(M)) \hookrightarrow W\,\hat{}\,\Gamma_k^{\wedge}(M), \tag{12}$$

but it is clear that in general, the exponential map (11) does not factor through the first term in (12). (It does of course when we look at sections over k only, but when we go over to a general k', we hit into the trouble that formation of inverse limits does not commute with ring extension $\otimes_k k'$!) We may however express (9) by stating that the functor (7') is "*prorepresentable*" by the *pro-object*

$$\operatorname{Pro}\Gamma_k(M) \overset{\text{def}}{=} (\Gamma_k(M)(i))_{i\geq 0} \quad \text{in } \operatorname{Pro}(\mathrm{Ab}_k), \tag{13}$$

this is even a *strict* pro-object (the transition morphisms are epimorphisms), which implies that the functor it prorepresents is representable iff this projective system is "essentially constant" in the most trivial sense, which means here

$$\Gamma_k^i(M) = 0 \quad \text{for large } i,$$

a condition which presumably is satisfied only for $M = 0$! Thus, the "correct" interpretation of non-pointed quasi-coherent linearization seems to me to be the corresponding functor, which I would like now to call L_k or simply L as before but with slightly different meaning:

$$L \text{ or } L_k : U(k) \to \operatorname{Pro}(\mathrm{Ab}_k) \approxeq \operatorname{Pro}(\mathrm{Qucoh}(k)), \tag{14}$$

where $\mathrm{Qucoh}(k)$ is defined in (8). In computational terms, I would like to view $L(X)$ (for a unipotent bundle X) to be a pro-k-module, but in terms of geometric intuition, I would see it rather as a pro-$\mathcal{O}_k$-module, i.e., essentially as an inverse system of quasicoherent modules. It is in these latter terms that the construction we just gave generalizes to unipotent bundles over arbitrary ground schemes, not necessarily affine ones. As for the "pointed" quasicoherent linearization functor

$$L_{\mathrm{pt}} \text{ or } L_{k\,\mathrm{pt}} : U(k)^{\bullet} \to (\mathrm{Ab}_k) \approxeq \mathrm{Qucoh}(k), \tag{15}$$

which I like best to view as taking values $L_{\mathrm{pt}}(X)$ which are quasicoherent sheaves, it maps into L by

$$L_{\mathrm{pt}}(X) \hookrightarrow L(X), \tag{16}$$

interpreting objects of a category as special cases of pro-objects. We'll denote by

$$(17) \qquad W L(X) \in \operatorname{Pro}(U(k))$$

the pro-unipotent bundle defined in terms of $L(X)$ via the canonical extension [p. 486]

$$\operatorname{Pro}(W) \text{ or simply } W : \operatorname{Pro}(\mathrm{Ab}_k) \to \operatorname{Pro}(U(k))$$

of W (cf. (8)) to pro-objects. Thus, instead of the map (6) which doesn't quite exist, we get a canonical "exponential" map

$$(18) \qquad X \to W L(X)$$

in $\operatorname{Pro} U(k)$, which has of course little chance to factor through

$$(16') \qquad W L_{\mathrm{pt}}(X) \to W L(X)$$

deduced from (16) by applying W. It is via this map (18) that we may declare that $L(X)$ prorepresents the functor (7') – it may be viewed as the universal map of the type

$$X \to W(N),$$

where now N is (not just a k-module, but) a variable object in $\operatorname{Pro}\mathrm{Ab}_k$. Whereas the pro-object $L(X)$ is of a "k-linear" nature and may be viewed as the (*quasi-coherent*) *k-linearization of the unipotent bundle X*, the pro-object $W L(X)$ of $U(k)$ has lost its k-linear nature, we would rather view it as the canonical "abelianization" of X, retaining mainly its additive structure (plus maybe operation of k on it, which is a lot weaker, though, than structure of an $\mathcal{O}_k$-module...).

I would like now to examine if the quasi-coherent pro-object $L(X)$, which has been obtained as the suitable quasi-coherent substitute for $\mathcal{O}_k^{(X)}$ in order to get the basic isomorphism (4) for quasicoherent F, may serve the same purpose for $\mathrm{RHom}_{\mathcal{O}_k}$, in analogy to (5), (5'). Quite generally, if

$$\Gamma = (\Gamma_\alpha)$$

is any pro-$\mathcal{O}_k$-module, let's define for any module F

$$(19) \qquad \mathrm{RHom}_{\mathcal{O}_k}(\Gamma, F) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{O}_k}(\Gamma, C^\bullet(F)),$$

where $C^\bullet(F)$ is an injective resolution of F – thus, the definition extends the usual one when Γ is just an $\mathcal{O}_k$-module. Our expectation now would be

$$(20) \qquad \mathrm{RHom}_{\mathcal{O}_k}(L(X), F) \xrightarrow{\sim} \mathrm{R}\Gamma_X(F_X),$$

giving rise to

$$(20') \qquad \mathrm{Ext}^n_{\mathcal{O}_k}(L(X), F) \xrightarrow{\sim} \mathrm{H}^n(X, F_X),$$

[p. 487] for any unipotent bundle X over k, and any quasicoherent sheaf

$$F = W(N),$$

where N is any k-module. Of course, (19) yields (for general Γ)

$$\mathrm{Ext}^i_{\mathcal{O}_k}(\Gamma, F) \simeq \varinjlim_{\alpha} \mathrm{Ext}^i_{\mathcal{O}_k}(\Gamma_\alpha, F), \tag{19'}$$

so that (20') may be rewritten more explicitly, if $X \simeq W(M)$, as

$$\mathrm{H}^n(X, W(N)_X) \simeq \bigoplus_{i \geq 0} \mathrm{Ext}^n_k(\Gamma^i_k(M), N). \tag{21}$$

At any rate, we have a canonical map (20) in $\mathrm{D}^+(\mathrm{Ab}_k)$, hence maps (20'), in view of the isomorphism (5) and the canonical map

$$\mathcal{O}_k^{(X)} \to L(X) \tag{*}$$

deduced from (18), and the question now is whether these are isomorphisms. We may of course assume in (21)

$$X = W(M), \quad n \geq 1.$$

If M is projective, so are the modules $\Gamma^i_k(M)$, and hence the second member in (21) is zero, so we should check the first member is too. This is clear when M is of finite type, hence X representable by an affine scheme, whose quasicoherent cohomology is well-known therefore to vanish in dim. $n > 0$. The general case should be a consequence of this, representing X as the filtering direct limit of its submodules which are projective of finite type – this should work at any rate when M is free with a basis which is at most countable, using the standard so-called "Mittag-Leffler" argument for passage to limit. Thus, in case M projective, (21) and hence (20) seems OK indeed. When M is *not* projective, however, there must be some k-module N such that $\mathrm{Ext}^1_k(M, N) \neq 0$, and hence the second member of (19) is non-zero for $n = 1$, which should imply rather unexpectedly

$$\mathrm{H}^1(X, F) = \mathrm{H}^1(W(M), W(N)_{W(M)}) \neq 0,$$

whereas till this very moment I had been under the impression that quasi-coherent cohomology of unipotent bundles should be zero, just as for affine schemes! Maybe it has been familiar to Larry Breen for a long time that this is *not* so? Maybe also for what we want to do it isn't really basic to find out whether (21) is true in full generality, as for the purpose
[p. 488] of studying Postnikov type dévissage, the unipotent bundles X we are going to work with will be smooth, i.e., M projective (and we may even get away with free M's, if we need so). The natural idea here for getting (20) via (21) in full generality, is to use a projective resolution of M (even a free one), but I'll not try to work this out now. The main impression which remains is that for the more relevant cases (involving cohomology groups of a *smooth* unipotent bundle X at any rate, with

quasicoherent coefficients), the quasicoherent "pro"-linearization $L(X)$ is just as good for computing cohomology invariants, as the forbidding $\mathcal{O}_k^{(X)}$ modules we were shrinking from.

If now we take an X_* instead of just X, namely a ss complex in $U(k)$, and assuming the components X_n to be smooth (to be safe), the isomorphisms (20') should give rise to isomorphisms (for $F = W(N)$)

$$\text{(22)} \qquad \mathrm{H}^n(X_*, F) \simeq \mathrm{Ext}^n_{\mathcal{O}_k}(L(X_*), F) \simeq \mathrm{Ext}^n_k(L(X_*), N),$$

where the Ext^n should be viewed as hyperext functors (not term-by-term), and where in the last member $L(X_*)$ may be interpreted as a chain complex in $\mathrm{Pro}(\mathrm{Ab}_k)$. The first member of (22) is the kind of group occurring as obstruction group in the Postnikov-type dévissage of X_* into linear structures $W(M(i)_*)$. The chain-pro-complex $L(X_*)$ may still look a little forbidding, our hope, though, now is that in the "pointed" case we are really interested in, we may replace $L(X_*)$ by $L_{\mathrm{pt}}(X_*)$, which is just a true honest chain complex in Ab_k. Now, from (16) we get indeed a canonical map

$$\text{(23)} \qquad \mathrm{Ext}^n_k(L(X_*), N) \to \mathrm{Ext}^n_k(L_{\mathrm{pt}}(X_*), N)$$

and we hope that this is an isomorphisms, under suitable assumptions on X_*, the most basic one I can think of now being

$$X_0 = e.$$

The map (23) was defined as the transposed of a map of chain complexes in $\mathrm{Pro}\,\mathrm{Ab}_k$

$$\text{(24)} \qquad L_{\mathrm{pt}}(X_*) \to L(X_*),$$

deduced from (16) by applying it componentwise. We recognize here, but with a different interpretation (which seems to me "the correct" one), the second map in the often referred-to diagram (w) of page 454, or (V) in yesterday's reflections (p. 478). To say that it gives rise to isomorphisms (23), for any n and any module N, should be equivalent to saying that (24) is a quasi-isomorphism – but to make sure I should demand a little work on foundations matters on pro-complexes I guess; also, to see if the assumption that (24) is a quasi-isomorphism should [p. 489] imply the same statement with $L(X_*)$ replaced by its componentwise projective limit – namely that (V) on p. 478 is, which we'll need of course in case $k = \mathbb{Z}$ for the so-called "linearization theorem". Thus, we get three isomorphism or quasi-isomorphism statements, concerning (23), (24) and (V) in yesterday's notes, which are at any rate closely related, and which one hopes to be true, because this seems needed for a schematization theory of homotopy types to work. But I should confess I have not tried even to get any clue as to why this should be true, under the only assumptions, say, that the components of X_* should be smooth (and possibly the Kan assumption?), plus $X_0 = e$ say.

Now to the second ingredient of the looked-for "linearization theorem", which previously was the first map in (w) p. 454, or (V') on page 478, involving maps of

$$\text{either } \mathbb{Z}^{(X_*(k))} \text{ or } k^{(X_*(k))}$$

into what was previously called $L(X_*)$, and which we would now rather denote by

$$\varprojlim L(X_*) \quad (\simeq \Gamma_k^{\wedge}(M_*) \text{ if } X_* = W(M_*)).$$

We made sure that, unless $k = \mathbb{Z}$, none of the two had any chance to be a quasi-isomorphism. The only positive thing that came out in this direction was that the first one of these maps would induce an isomorphism on H_i in the critical dimension (namely $i = n+1$ in the n-connected case). We now understand why, for $k \neq \mathbb{Z}$, we would not get any actual quasi-isomorphism – namely, the "correct" naive linearization which compares reasonably with $L(X_*)$ should not be relative to a *constant* ring such as $\mathbb{Z}$ or k, but relative to $\mathcal{O}_k$, via the map (*) (p. 487) giving rise now to

$$\mathcal{O}_k^{(X_*)} \to L(X_*). \tag{25}$$

The more reasonable question now, making good sense really for any ground ring k, is whether this map (under the usual assumptions say on X_*) is a quasi-isomorphism. I wouldn't really but it is, as I have some doubts as to whether the homology sheaves of the first member (both members of course being viewed as chain complexes of $\mathcal{O}_k$-modules or "pro" such) are quasi-coherent – but for the time being I am not sure either if those of the second member are essentially constant pro-objects! But even if (25) isn't a quasi-isomorphism, it does behave like one for all practical purposes of computing quasi-coherent cohomology it would seem, as this boils down indeed to the isomorphisms (20) or (20').

[p. 490] 7.9.

The need and the drive.

116 For ten days I haven't written any notes, and the time when I stopped looks a lot more remote still. For two days still after I last wrote on the notes, I kept pondering about schematization of homotopy types – it were rather lively days, first day I found the amazingly simple description of the homotopy groups in the schematic set-up, which got me quite excited; next day, from a phone call to Illusie, it turned out that the key assumption in all my ponderings on schematization, namely the "abelianization theorem" (sic) asserting isomorphism between discrete and schematic (namely quasi-coherent) homology (or equivalently, cohomology) invariants, was definitely false: consequently, the canonical functor from schematic to discrete homotopy types turns out definitely not to be an equivalence. This completely overthrows the idyllic picture in my mind about the relationship between schematic and discrete homotopy types – but the reflection on schematic homotopy types "in their own right" had by then proceeded far enough, the very day before, so that my faith in the relevance of schematic homotopy types wasn't seriously shaken – rather, I got excited at drawing a systematic "bilan" from the evidence now at hand, about the prospects of developing a theory of "schematic homotopy types" satisfying some basic formal properties, whether or not such theory be based on semisimplicial unipotent bundles as models, or on any other kind of models making sense over arbitrary ground rings. Before reverting to a review of the more formal properties of abelianization in the context of the basic modelizer (Cat), I would like still to write down with some care what had thus come up Monday and Tuesday last week. . .

["bilan" translates as "assessment", "results", "balance sheet", or even "death toll". . .]

The next days I felt a great fatigue in all my body, and I then stopped (till yesterday) any involvement in mathematical reflection. I am glad I followed this time the hint that had come to me through my body, rather than brush it aside and go on rushing ahead with the work I was so intensely involved in, as had been a rule in my life for many years. This time, I understood that the reluctance of the body to follow that forward rush, even though I was taking good care of myself with sleep and food, had strong reasons, which had nothing to do with neither sleep or food nor with my general way of life. Rather, during the weeks before and also during those very days, a number of things had occurred in my life, not all visibly related and of differing weight and magnitude, to none of which I had really devoted serious reflection, nor even a minimum of time and attention needed for giving me a chance to let these things and their meaning "enter". In lack of this, there was little chance my response to current events would be any better than purely mechanical, and my interaction with some of the people I love would be in any way creative. There was this need for being attentive, an urgent need springing from life itself and which I was about to ignore – and there was this drive, this impatience driving me recklessly ahead, with no look left nor right. Of course, I did know about the need, "somewhere" – and in my head too I kind of knew, but the head was prejudiced as usual and would take no notice, not of the need and not of the conflict between an urgent need and a powerful, ego-invested passion. The head was prejudiced and foolish – so it was my body finally which told me: now you stop this nonsense and you take care of what you well know you better take care *first*, and *now*! And its language was strong and simple enough and cause me to listen. [p. 491]

Thus, the main work I was involved in for these last seven days was to let a number of things "enter" – mainly things that were being revealed through the death of my granddaughter Ella. It surely *was* "work", taking up the largest part of my nights and my days, – so much so that I can't really say there is any less fatigue now than seven days ago. It seems to me, though, it isn't quite the same fatigue – this time it is the fatigue coming from work done, not from work shunned. The "work done" wasn't really done by me, I feel, rather work taking place within me, and "my" main contribution has been to allow it to take place, by providing the necessary time and quietness; and of course, also, to allow the outcome of this work to become conscious knowledge, rather than burying it away in some dark corner of the mind. The rough material, as well as the outcome of this work, have not been this time new facts or new insights; rather, things which I had come to perceive and notice, for some time already, over the last two years, without granting them the proper weight and perspective – somehow as if I didn't quite believe what I was unmistakingly perceiving, or didn't take it quite seriously. Such a thing, I noticed, happens quite often, not only with me, and takes care of making even the most lively perceptions innocuous, by disconnecting them at all price from the image of reality and of ourselves we are carrying around with us, that the image remains static, unaffected by any kind of "information" flowing around or through us.

[p. 492] 10.9.

117 *"Schematic" versus "formal" homology and cohomology invariants.*

Maybe the best will be to write up (and possibly develop some) my reflections (of the two days after I stopped with the notes) roughly in the order as they occurred.

There were some somewhat technical afterthoughts. One was about the logical difficulty coming from the site $\mathrm{Aff}_{/k}$ on page 446 not being a $\mathfrak{U}$-site (where $\mathfrak{U}$ is the universe we are working in), hence strictly speaking, the category of all sheaves on this site is possibly not even a $\mathfrak{U}$-category (i.e., the Hom objects need not be small, i.e., with cardinal in $\mathfrak{U}$), still less a topos, and hence the standard panoply of notions and constructions in a topos does not apply. This doesn't look really serious, though, one way out is to limit beforehand the "size" of the unipotent bundles we are allowing, i.e., of the k-modules describing them, in terms say of cardinality of a family of generators of the latter – and then restrict accordingly the size of the k-algebras k' taken as "arguments" for our sheaves, i.e., as objects of the basic site we are working on. For instance, when working with unipotent bundles of finite type only (i.e., corresponding to k-modules of finite type – a rather interesting and natural finiteness condition anyhow on the components of a schematic model X_*), it is appropriate to work on the "fppf site", where the arguments k' are k-algebras of *finite presentation*. If we should be unwilling to be limited by a fixed size restriction on the unipotent bundles we are working with (and hence also on the corresponding homotopy types), we may have to work with a hierarchy of size restrictions and passage from one to any other less stringent one – a technical nuisance to be sure, if we don't find a more elegant way out, but surely not a substantial difficulty. At the present heuristic stage of reflections, it doesn't seem worth while really to dwell on such questions any longer.

["fidèlement plat de présentation finie"]

Another afterthought is about the functor

$$M \mapsto W(M) : \mathrm{Ab}_k = (k\text{-Mod}) \to (\mathcal{O}_k\text{-Mod})$$

from k-modules to $\mathcal{O}_k$-modules – a fully faithful functor we know, whose essential image by definition consists of the so-called *quasi-coherent* sheaves. A little caution is needed, as this functor is right exact, but *not exact*, i.e., it does not commute with formation of kernels, because for a k-algebra k' which isn't flat, the functor

$$M \mapsto M \otimes_k k'$$

doesn’t. When we wrote down a formula such as (21) on page 487, we were implicitly making use of the assumption that for k-modules M,N we have a canonical isomorphism [p. 493]

$$\mathrm{RHom}_k(M,N) \xleftarrow{\sim} \mathrm{RHom}_{\mathcal{O}_k}(W(M),W(N)), \tag{1}$$

at any rate such a formula is needed if we want to view (21) as a more explicit way of writing (20) or (20’). Using a free resolution $L_\bullet$ of M, and the fact that

$$\mathrm{Ext}^i_{\mathcal{O}_k}(\mathcal{O}_k, W(N)) \simeq \mathrm{H}^i(\mathrm{Spec}(k), W(N)) = 0 \quad \text{if } i > 0,$$

stemming from the cohomology properties of flat descent, we easily get a map (1), but I did not check that this map is an isomorphism, the difficulty coming from the fact that $W(L_\bullet)$ need not be a resolution of $W(M)$, unless M is flat. This perplexity already arises in the fppf context – surely Larry Breen should know the answer. For what we are after here it doesn’t seem to matter too much, as the computations of loc. cit. were of interest mainly (maybe exclusively) in the case when M is projective or at any rate flat, i.e., when working with flat unipotent bundles – in which case (1) is indeed an isomorphism.

The interpretation of polynomial maps between k-modules M, N in terms of the topological augmented coalgebras $\Gamma_k^\wedge$ associated to these may seem a little forbidding to some readers. In the all-important and typical case when M and N are projective and of finite type, things become quite evident, though, by just dualizing the more familiar concepts around polynomial functions and homomorphisms between rings of such. Thus, $W(M)$ is just a usual vector-bundle, hence also a true honest affine scheme over k, whose affine ring is the ring of polynomial functions on M, which can be identified with $\mathrm{Sym}_k^*(M^{\mathrm{op}})$, the symmetric algebra on the dual module:

[To make sense of these formulæ I have added some $^{\mathrm{op}}$s to denote dual modules. I hope this matches AGs intentions. . .]

$$W(M) \simeq \mathrm{Spec}(\mathrm{Sym}_k^*(M^{\mathrm{op}})), \tag{2}$$

and similarly for $W(N)$. Polynomial maps from M to N, i.e., maps from the k-scheme $W(M)$ to the k-scheme $W(N)$, just correspond to k-algebra homomorphisms

$$\mathrm{Sym}_k^*(N^{\mathrm{op}}) \to \mathrm{Sym}_k^*(M^{\mathrm{op}}) \tag{3}$$

(irrespective of the graded structures). As each of these algebras, as a k-module, is a filtering direct limit of its projective submodules of finite type such as

$$\mathrm{Sym}_k(M^{\mathrm{op}})(i) = \bigoplus_{j\leq i} \mathrm{Sym}_k^j(M), \tag{4}$$

the dual module [p. 494]

$$\Gamma_k^\wedge(M) \simeq (\mathrm{Sym}_k^*(M))^\vee \simeq \prod_j (\mathrm{Sym}_k^j(M) \simeq \Gamma_k^j(M))$$

may be viewed as the inverse limit of the duals of those submodules (compare p. 484 (11)), and topologized accordingly; linear maps (3) may be interpreted in terms of continuous maps between the dual structures (or, equivalently, between the corresponding pro-objects)

$$\Gamma_k^{\wedge}(M) \to \Gamma_k^{\wedge}(N), \tag{3'}$$

and compatibility of (3) with multiplication and units is expressed by compatibility of (3') with "comultiplication", i.e., diagonal maps, and with augmentations. On the other hand, maps $W(M) \to W(N)$ respecting the "pointed structures" coming from zero sections, correspond to map (3') transforming 1 into 1, besides the other requirements. Such maps, it turns out, automatically induce a map between the submodules

$$\Gamma_k(M) \to \Gamma_k(N) \tag{4}$$

which is rather evident indeed, if we remind ourselves of the fact that the submodule $\Gamma_k(M)$ may be viewed as the (topological) dual of $\mathrm{Sym}_k^*(M)$, topologized by the powers of the augmentation ideal

$$\mathrm{Sym}_k^+(M) = \bigoplus_{i>0} \mathrm{Sym}_k^i(M),$$

or equivalently of the corresponding adic completion

$$\mathrm{Sym}_k^{*\wedge}(M) = \varprojlim_i \mathrm{Sym}_k^*(M)(i) \simeq \prod_{j\geq 0} \mathrm{Sym}_k^j(M), \tag{5}$$

and correspondingly for N. The "pointed" assumption on a map $W(M) \to W(N)$ in terms of the corresponding homomorphism of k-algebras (3), just translates into compatibility with the augmentations, or equivalently, with the corresponding ideals Sym_k^+, which implies that it induces a homomorphism of the corresponding adic rings, and hence by duality a homomorphism (4) on their duals.

These reminders bring near that working with the (discrete) k-algebras $\mathrm{Sym}_k^*(M)$, or equivalently, with the *topological* coalgebras $\Gamma_k^{\wedge}(M)$, amounts to working with "unipotent bundles" (projective and of finite type), which are just usual schemes (of a rather particular structure of course), whereas working with the *topological* k-algebras $\mathrm{Sym}_k^{*\wedge}(M)$, or equivalently, with the discrete coalgebras $\Gamma_k(M)$, amounts to working with *formal schemes*, namely essentially, with the formal completions of the former along the zero sections. (Of course, the topological objects just
[p. 495] considered may be equally viewed as being *pro-modules* endowed with suitable extra structure.) Correspondingly, we will expect the cohomology invariants constructed in terms of (the apparently more forbidding) $\Gamma_k^{\wedge}$ to express quasi-coherent cohomology of the corresponding *schemes*, or semisimplicial systems of such; whereas, working with the (apparently more anodyne!) Γ_k will lead to cohomology invariants of *formal schemes* and semisimplicial systems of such. Both types of invariants are of interest it would seem, the one however which looks the more relevant in connection with studying ordinary homotopy types in terms

of schematic ones, is surely the first. On the other hand, there isn't any reason whatever to believe that under fairly general conditions, these two types of invariants are going to be isomorphic, by the evident map from "schematic" to "formal" quasicoherent cohomology. To say it differently, I do no longer expect that under reasonably wide assumptions, the map (24) of p. 488

$$L_{\mathrm{pt}}(X_*) \to L(X_*), \tag{6}$$

is a quasi-isomorphism, nor behaves like one with respect to taking Ext^i's with values in a quasicoherent module, as I was hastily surmising for about one week, while loosing track of the geometric meaning of the algebraic objects I was playing around with. To give just one example, take X_* to be the standard semisimplicial unipotent bundle associated to the group-object $\mathbb{G}_{\mathrm{a}}$ in $\mathcal{U}(k)$, namely just the usual affine line with addition law. The Ext^i's of the two members of (6) with values in the k-module k may be interpreted as either schematic or formal Hochschild cohomology of the additive group, with coefficients in $\mathcal{O}_k = \mathbb{G}_{\mathrm{a}}$. The map of the former into the latter is not always an isomorphism, already in dimension 1, where the two groups to be compared are just the groups of endomorphisms of $\mathbb{G}_{\mathrm{a}}$, and of the corresponding formal group. If k is of char. $p > 0$, a prime, then the latter group can be described as the group of all formal power series of the type

$$F(t) = \sum_{i \geq 0} c_i t^{(p^i)},$$

whereas for the first group we must restrict to those F which are polynomials, i.e., only a finite number of the coefficients c_i are non-zero.

One may of course object to this example, because the X_* we are working with is not simply connected, and because the example does not apply over a ring such as $\mathbb{Z}$, which is the one we are interested in most of all. I am convinced now, however, that even when assuming $X_1 = X_0 = e$ and $k = \mathbb{Z}$, (6) is very far from being a quasi-isomorphism, even for such basic structures as $K(\mathbb{Z}, n)$ with $n \geq 2$. At any rate, the [p. 496]
"way-cut" argument I have finally been thinking of, in order to check (6) is a quasi-isomorphism, rests on vanishing assumptions which (as I was informed by Illusie the next day) are wholly unrealistic. This finally clears up, it would seem, a tenacious misconception which has been sticking to my first heuristic ponderings about the homology and cohomology formalism for schematic homotopy types: *one should be very careful not to substitute the "pointed" linearization $L_{\mathrm{pt}}(W(M)) \simeq \Gamma_k(M)$ for the non-pointed one $L(W(M)) \simeq \Gamma_k^{\wedge}(M)$, in computing homology and cohomology invariants of schematic homotopy types*. To say it differently, in order to be able to compute (or just define) the "schematic" homology and cohomology invariants, we do need as a model a full-fledged semisimplicial unipotent bundle, not just the corresponding formal one, giving rise to invariants of it's own, namely "formal" homology and cohomology, which are definitely distinct from the former.

118
The homotopy groups π_i as derived functors of the "Lie functors". Lack of satisfactory models for S^2 and S^3.

It is all the more remarkable, in view of the preceding findings, that the homotopy invariants

$$\pi_i(X_*), \quad i \geq 0,$$

of a pointed semisimplicial unipotent bundle X_* (still assuming the components X_n to be smooth, i.e., to correspond to projective modules) turn out to be invariants of the corresponding "formal" object, and, more startling still, of the corresponding "infinitesimal" object of order 1. More specifically, consider the "Lie functor" or "tangent space at the origin"

$$\text{(7)} \qquad \mathrm{Lie} : U(k)^\bullet \to \mathrm{Ab}_k, \quad X = W(M) \mapsto M,$$

which we'll need only for the time being for smooth X, when the geometric meaning of it is clear. This functor transforms semisimplicial pointed bundles into semisimplicial k-modules $\mathrm{Lie}(X_*)$, thus we should get, besides abelianization, another remarkable functor, from $\mathcal{M}_1(k)$ say to $\mathrm{D}_\bullet(\mathrm{Ab}_k)$:

$$\text{(8)} \qquad \mathrm{Lie} : \mathcal{M}_1(k) \to \mathrm{D}_\bullet(\mathrm{Ab}_k), \quad \text{via } X_* \mapsto \mathrm{Lie}(X_*),$$

granting that $X_* \mapsto \mathrm{Lie}(X_*)$ transforms quasi-isomorphisms into quasi-isomorphisms. Now, that this must be so follows from the really startling formula

$$\text{(9)} \qquad \pi_i(X_*) \simeq \pi_i(\mathrm{Lie}(X_*)) \quad (\simeq \mathrm{H}_i \text{ of the associated chain complex in } \mathrm{Ab}_k),$$

where the left-hand side, I recall, is defined as

$$\text{(10)} \qquad \pi_i(X_*) = \pi_i(X_*(k)).$$

[p. 497] I don't have, I must confess, any direct description of such an isomorphism (9), valid for any semisimplicial bundle X_*, say, satisfying the assumptions

$$X_0 = X_1 = e, \quad X_n \text{ smooth for any } n \text{ (maybe flat is enough)},$$

plus possibly (if needed) a Kan type condition. However, we have such isomorphisms (9) in a tautological way, when X_* comes in the usual way from a chain complex in Ab_k with projective components, hence also when X_* admits a Postnikov-type dévissage into "abelian" pieces as above. If we admit that any X_* satisfying the assumptions is homotopic to one admitting such a dévissage, the isomorphisms (9) should follow, except of course that extra work would be still needed to get naturality of (9).

I have the feeling however that, besides the specific abelianization functor in the schematic context, *formula* (9) *should be made a cornerstone of a theory of schematic homotopy types, and serve as "the" natural definition of the homotopy invariants* of a model X_*, within the context of schematic models and without any need a priori to tie them up with, let one subordinate their study to, invariants of the corresponding discrete homotopy type $X_*(k)$. Accordingly, weak equivalences should be defined (for semisimplicial bundles satisfying the suitable assumptions at any rate) as maps inducing isomorphisms for the π_i invariants, namely inducing quasi-isomorphisms for the corresponding Lie chain complexes. It is immediately checked that this implies that the corresponding map for "formal homology", namely

$$L_{\mathrm{pt}}(X_*) \to L_{\mathrm{pt}}(X_*)$$

is then a quasi-isomorphism too, when viewed as a map of chain complexes and hopefully the same should hold for "schematic homology"

$$L(X_*) \to L(X_*),$$

and of course one would expect converse statements to hold too.

I would like to comment a little on the significance of formula (9). As far as I know, this is the only fairly general formula, not reducing to an "abelian" case, where the homotopy groups π_i appear as just the H_i invariants of a suitable chain complex, defined up to unique isomorphism in the relevant derived category. This chain complex comes here, moreover, with an amazingly simple description, of immediate geometrical significance, and suggestive of relationships with the homology [p. 498] invariants a lot more precise, presumably, than those currently used so far. Of course, the significance of (9) for the study of the usual, "discrete" homotopy types, will be subordinated to how difficult it will turn out for such a homotopy type to be a) realizable by a schematic over $\mathbb{Z}$, and b) to get hold of a more or less explicit description of such a schematic homotopy type, via say a semisimplicial bundle as a model. One is of course thinking more specifically of the case of the spheres S^n, the first case (besides the trivial S^1 case) being S^2, the sequence of homotopy groups of which (as far as I know) it not understood yet. Viewing the spheres S^n as successive suspensions of S^1, where S^1 is fitting nicely into the formalism of schematic homotopy types as a $K(\mathbb{Z}, 1)$ (except that the 1-connectedness condition $X_1 = e$ is not satisfied), this brings near the question of defining the suspension operation in a relevant derived category $\mathcal{M}_0(k)$ or $\mathcal{M}_1(k)$ (whereas before we had met with the "dual" question of constructing homotopy fibers of maps, such as loop spaces). Thus it would seem that *a breakthrough in getting hold of the standard homotopy constructions within the schematic context*, assuming that these constructions do still make a sense, *may well mean a significant advance for the understanding of the homotopy groups of spheres*. This looks like a very strong motivation for trying to carry through those constructions (possibly even construct a corresponding "derivator" embodying any kinds of finite "integration" and "cointegration" operations on schematic homotopy types) and at the same time warns us that such work, if at all feasible, will most probably be a highly non-trivial one.

The question of "schematizing" the homotopy types of S^2 and S^3 reminds me of a fact which struck me a long time ago (maybe J. P. Serre or someone else pointed it out to me first). Namely, in some respects

[I'm guessing that SL(2, $\mathbb{J}$) is another notation for SU(2)...]

there doesn't seem to be really satisfactory algebraic models for these homotopy types, taking into account the basic relationship between the two, namely: the 2-sphere (or, equivalently, the projective complex line) is a *homogeneous space* under the quaternionic *group* S^3 (or, equivalently, under the complex linear group $\mathrm{SL}(2, \mathbb{J})$). This relationship, and its manifold "avatars" in the realms of discrete groups, Lie groups, algebraic groups or group schemes etc., is one of the few key situations met with, and of equal basic significance, in the most diverse quarters in mathematics, from topology to arithmetic. Thus, SL(2) as a simple Lie (or algebraic) group of minimum rank 1, plays the role of *the* basic building block for building up the most general semisimple groups, whereas P^1 may be viewed as being the most significant homogeneous space under this group, namely the first and most elementary case of flag manifolds. In view of this significance of S^2 and S^3, it is all the more reasonable that no simple, non-plethoric semisimplicial model say in [unreadable], in terms say of a semisimplicial group having the homotopy type of

[p. 499]

$$S^3 \sim \mathrm{SL}(2, \mathbb{C}) = G,$$

and a subgroup having the homotopy type of

$$S^1 \sim \mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C}) \sim K(\mathbb{Z}, 1)$$

and playing the part of a Borel subgroup or a maximal torus, in such a way that the quotient will have the homotopy type of

$$S^2 = P^1_{\mathbb{C}} \simeq S^3/S^1 \simeq G/B.$$

It would be tempting now to try and construct such a model of the situation in terms of semisimplicial unipotent bundles over $\mathbb{Z}$ – which would at the same time display the homotopy groups of S^2 and S^3 (not much of a difference!) via formula (9). All the more tempting of course, as it is felt that the geometric objects and their relationship, the homotopy shadow of which we want to modelize schematically, are themselves already, basically, most beautiful schemes over $\mathrm{Spec}(\mathbb{Z})$!

Another way of getting a display of the homotopy groups of S^2 and S^3 would be in terms of a (discrete) model of the situation above, in terms of "hemispherical complexes" rather than semisimplicial ones. On the other hand, there is no reason why a theory of schematic homotopy types could not be carried through as well, using hemispherical complexes rather than semisimplicial ones. The latter kind of complexes have the advantage that they have become thoroughly familiar through constant use by topologists and homotopy people for thirty years or so – the former however are newcomers, have the advantage of still greater formal simplicity (just two boundary operations, and just one degeneracy), and more importantly still, of allowing for a direct computational description of the homotopy invariants, in the discrete set-up. When working with hemispherical complexes of unipotent bundles as models for schematic homotopy types, we'll get then *two* highly different descriptions of the homotopy invariants π_i, one by the "infinitesimal" formula (9) interpreting them as derived functors of the Lie functor, the other one via the hemispherical set $X_*(k)$, handled as if it came from an actual ∞-groupoid (by taking its source and target operations etc.) even if it does not. (I confess I did not check that this process does correctly describe the homotopy groups of a hemispherical set, even without assuming it comes from an ∞-groupoid or an ∞-stack, but never mind for the time being. . .) Among the interesting things still ahead (once we get a little accustomed to working with hemispherical complexes) is to try and understand how these two descriptions relate to each other, which may be one means for a better understanding of the basic formula (9), in the context of hemispheric schematic models. [p. 500]

26.9.

119 After the last notes (of September 10) I was a little sick for a few days, then I was taken by current tasks from professional and family life, which left little leisure for mathematical reflection, except once or twice for a couple of hours, by way of recreation. It would seem now that in the days and weeks ahead, there will be more time to go on with the notes, and I feel eager indeed to push ahead. Also, I more or less promised the publisher, Pierre Bérès, that a first volume would be ready for the printer by the end of this calendar year, and I would like to keep this promise.

Breakdown of an idyllic picture – and a tentative next best "binomial" version of the "comparison theorem" for schematic versus discrete linearization.

I still have to tie in with the reflections and happenings of the end of last month, as I started upon with the last notes (of Sept. 10). Next thing then to report upon is the "coup de théâtre" occurring through the phone call to Luc Illusie. When I told him about what by then still looked to me as the key assumption for a theory of schematization of homotopy types, namely that the homology of $K(\pi, n)$ should be computable in terms of derived functors of the "divided power algebra"-functor $\Gamma_{\mathbb{Z}}$, he at once felt rather skeptical, and later he called me back to tell me it was definitely false. He could not give me an explicit counterexample for $\mathrm{H}_i(n, \mathbb{Z}; \mathbb{Z})$, say, with given n and i, rather he said that when suitably "stabilizing" the assumption I had in mind, it went against results of Larry Breen on Ext^i functors of $\mathbb{G}_\mathrm{a}$ with itself over prime fields $\mathbb{F}_p$. I don't know if I am going some day to give into Illusie's argument and into Larry Breen's results – however, even before I got Illusie's confirmation that definitely my assumption was wrong, I convinced myself that at any rate it was false for $n = 1$. This is a non-simply connected case and hence not entirely conclusive maybe, but still it was enough to shake my confidence that the assumption was OK. The counterexample is in terms of cohomology [p. 501]

[a sudden or unexpected event in a play. . .]

$$\mathrm{H}^2(1, \mathbb{Z}; \mathbb{Z}) = \mathrm{H}^2(\mathbb{Z}, \mathbb{Z}) = \mathrm{H}^2(B_{\mathbb{Z}}, \mathbb{Z})$$

rather than homology, as usual. As the classifying space $B_{\mathbb{Z}} = S^1$ of $\mathbb{Z}$ is one dimensional, its cohomology is zero in dimensions $i \geq 2$. On the other hand, H^2 classifies central extensions of $\mathbb{Z}$ by $\mathbb{Z}$, and an immediate direct argument shows indeed that such extensions (indeed, any extension of $\mathbb{Z}$ by any group) split. If we take the schematic H^2, defined by Hochschild cochains which are polynomial functors, we get the classification of central extensions of $\mathbb{G}_\mathrm{a}$ by $\mathbb{G}_\mathrm{a}$, as group schemes over the ring of integers. Now, it is easy to find such an extension which does *not* split, the first one one may think of being the group scheme representing the functor

$$k \mapsto W_2(k) = \text{group of truncated power series } 1 + at + bt^2 \text{ in } k[t]/(t^3),$$

where a, b are parameters in the commutative ring k, the group structure being multiplication. These parameters define in an evident way a structure of an extension of $\mathbb{G}_\mathrm{a}$ by $\mathbb{G}_\mathrm{a}$ upon W_2, a splitting of which would correspond to a group homomorphism

$$\mathbb{G}_\mathrm{a} \to W_2, \quad a \mapsto 1 + at + P(a)t^2,$$

with

$$P \in \mathbb{Z}[t].$$

Expressing compatibility with the group laws gives the condition

$$P(a + a') = P(a) + P(a') + aa',$$

which has, as unique solutions in $\mathbb{Q}[t]$, expressions

$$P(t) = t(t-1)/2 + ct$$

with c in $\mathbb{Q}$, none of which has coefficients in $\mathbb{Z}$. This argument shows in fact that for given ring k, $(W_2)_k$ is a split extension iff 2 is invertible in k, in which case a splitting is given by

$$a \mapsto 1 + at + (a(a-1)/2)t^2.$$

This example brings near one plausible "reason" why the expected comparison statement about discrete and schematic linearization could not reasonably hold true, and in particular why we shouldn't expect discrete and schematic Hochschild cohomology (for group schemes over
[p. 502] $\mathbb{Z}$ such as $\mathbb{G}_\mathrm{a}$ or successive extensions of such) to give the same result. Namely, the latter is computed in terms of cochains which are polynomial functions *with coefficients in* $\mathbb{Z}$, whereas there exist polynomial functions *with coefficients in* $\mathbb{Q}$ (not in $\mathbb{Z}$) which, however, give rise to integer-valued functions on the group of integer-valued points. Such are the binomial expressions

$$P_n(t) = t(t-1)\dots(t-n+1)/n! \quad (\text{for } n \in \mathbb{N}).$$

These (in the case of just one variable t) are known to form a basis of the $\mathbb{Z}$-module of all integer-valued functions on $\mathbb{Z}$, and these is a corresponding basis for integer-valued functions on $\mathbb{Z}^r$, for any natural integer r. Thus, the hope still remains that a sweeping comparison theorem for discrete versus "schematic" linearization might hold true, provided it is expressed in such a way that the "schematic models" we are working with should be built up with "schemes" (of sorts) described in terms of spectra not of polynomial algebras $\mathbb{Z}[t]$ and tensor powers of these, but rather of "binomial algebras" $\mathbb{Z}\langle t\rangle$ built up with the binomial expressions above, and tensor powers of such. If we want to develop a corresponding notion of homotopy types over a general ground ring k, we should then require upon k an extra structure of a "binomial ring" (as introduced in the Riemann-Roch Seminar SGA 6 in some talks of Berthelot), namely a ring endowed with operations

[Berthelot, Grothendieck, and Illusie (SGA 6)]

$$x \mapsto \binom{x}{n} : k \to k \quad (n \in \mathbb{N}),$$

satisfying the formal properties of the binomial functions $x \mapsto P_n(x)$ in the case $k = \mathbb{Z}$ or $\mathbb{Q}$. Whereas linearization of homotopy types via De Rham complexes with divided powers relies on a "commutative algebra with divided powers" (which was developed extensively by Berthelot and others for the needs of crystalline cohomology), linearization via unipotent bundles (assuming it can be done in such a way as to ensure that any discrete homotopy type can be "schematized" in an essentially unique way) might well rely on the development of a "binomial commutative algebra" and a corresponding notion of "binomial schemes". There should be a lot of fun ahead developing the necessary algebraic machinery, which may prove of interest in its own right. It should be realized, however, that for a ring k to admit a binomial structure is a rather strong restriction – thus, for a given prime p, no field of char. p (except possibly the prime field?) admits such a structure. This remark may temper somewhat the enthusiasm for pushing in this direction, even granting that a "binomial comparison theorem" for discrete versus "binomial" linearization holds true.

See comments next section p. 506–507.

Maybe it is worthwhile to give a down-to-earth formulation of such a comparison statement. For any free $\mathbb{Z}$-module M of finite type, let [p. 503]

$$\mathrm{Symbin}_{\mathbb{Z}}(M) \subset \mathrm{Sym}^*_{\mathbb{Q}}(M_{\mathbb{Q}}) \quad (\text{where } M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q})$$

be the subalgebra of the algebra of polynomial functions on $M^{\vee}_{\mathbb{Q}} \simeq (M^{\vee})_{\mathbb{Q}}$ which are integral-valued on $M^{\vee}$ = dual module of M. Now let L_* be any semisimplicial $\mathbb{Z}$-module whose components are free of finite type, and consider the canonical map of cosemisimplicial $\mathbb{Z}$-modules

$$\mathrm{Symbin}_{\mathbb{Z}}(L^{\vee}_*) \to \mathrm{Maps}(L_*, \mathbb{Z}),$$

described componentwise in an obvious way. The question is whether this is a weak equivalence, i.e., induces a quasi-isomorphism for the associated cochain complexes, under the extra assumption that L_* is 0-connected, i.e., the associated chain complex has zero H_0 (and possibly, if needed, assuming even 1-connectedness, i.e., H_0 *and* H_1 of the associated chain complex are both zero). Presumably, by easy dévissage arguments one should be able to reduce to the case when L_* is a $K(\mathbb{Z}, n)$ type, and more specifically still, that it is the semisimplicial abelian complex associated to the chain complex reduced to $\mathbb{Z}$ placed in degree n (where $n \geq 1$). Thus, the question is whether Eilenberg-Mac Lane cohomology (with coefficients in $\mathbb{Z}$) for $K(\pi, n)$ types (or more specifically, $K(\mathbb{Z}, n)$ types) can be expressed in terms of derived functors of the Symbin($M^\vee$) functor. At any rate, whether Symbin is just the right functor to fit in or not, it looks like an interesting question whether Eilenberg-Mac Lane cohomology (or, more relevantly still, homology) can be expressed in terms of the derived functor of a suitable non-additive contravariant (resp. covariant) functor B from (Ab) (or from the subcategory of free $\mathbb{Z}$-modules) to itself. If so, there should be a way of defining a (possibly somewhat sophisticated) notion of "B-schematic homotopy types" (over a ground ring k endowed with suitable extra structure, such as a binomial structure), in terms of "unipotent B-bundles", in such a way that any "discrete", namely usual homotopy type, satisfying a suitable 1-connectedness restriction, admits an essentially unique "B-schematization".

I don't feel like pursuing these questions here, which would take me too far off the main line of investigation I've been out for. At any rate, whether or not Eilenberg-Mac Lane homology may be expressed in terms of the total left derived functor of a suitable functor from (Ab) to itself, it would seem that the somewhat naive approach towards schematic homotopy types we have been following, valid over an arbitrary (commutative) ground ring k without any extra structure needed on it, is
[p. 504] worthwhile pursuing even for the mere sake of studying ordinary homotopy types. The main reason for feeling this way is the amazingly simple description of the homotopy modules π_i of a homotopy type defined in terms of a semisimplicial (or hemispherical) unipotent bundle, as derived functors (so to say) of the Lie functor (cf. previous section 118). The main test for deciding whether there is indeed a rewarding new tool to be dug out, is to see whether or not in the model categories $\mathcal{M}_n(k)$ we have been working with so far, the standard homotopy constructions (around loop spaces and suspensions) make sense, and in such a way of course that the canonical functor from schematic to discrete homotopy types should commute to these operations. It may well turn out that to get a handy formalism, one will have still to modify more or less the conceptual set-up of unipotent bundles I've been tentatively working with so far. I already lately hit upon suggestions of such modifications, and presumably I'm going to discuss this still, before leaving the topic of schematizations.

Another reason which makes me feel that there should exist a notion of homotopy types over more general ground rings k than $\mathbb{Z}$, is that for a number of rings, such a notion has been known for quite a while. If I got it right, already in the late sixties (even before I withdrew from the mathematical milieu) I heard about such things as homotopy types over residue class rings $\mathbb{Z}/n\mathbb{Z}$, or over rings (such as $\mathbb{Q}$) which are localizations of $\mathbb{Z}$, or over rings such as $\mathbb{Z}^\wedge$ or $\mathbb{Z}_p$ (p-adic integers) which are completions of $\mathbb{Z}$ with respect to a suitable linear topology. Last week, which was the first time I was at a university after the Summer
[Bousfield and Kan (1972)] vacations, I took from the library the Bousfield-Kan Lecture Notes book

on homotopy limits (which had been pointed out to me by Tim Porter in June, when he had taken the trouble to tell me about "shape theory" and its relations to (filtering) homotopy limits), and while glancing through it, I noticed there is a systematic treatment of such homotopy types.

At this very minute I had a closer look upon the introduction of part I, it turns out that Bousfield and Kan are working with an arbitrary (commutative) ground ring k, and they are defining corresponding k-completion $k_\infty X$ of a homotopy type X, rather than a notion of "homotopy type over k". But the two kinds of notions are surely closely related, the k-completion of BK presumably should have more or less the meaning of ground ring extension $\mathbb{Z} \to k$.[†] At any rate, for 1-connected spaces and $k = \mathbb{Z}$ the completion operation seems to be no more no less than just the identity, thus it would seem that the implicit notion of "homotopy type over $\mathbb{Z}$" should be just the ordinary "discrete" notion of homotopy types – unlike the notion of schematic homotopy types (over $k = \mathbb{Z}$) defined via semisimplicial unipotent bundles. Definitely, an understanding of schematic homotopy types will have to include the (by now classical) Bousfield-Kan ideas, and these are also relevant for my reflections on "integration" and "cointegration" operations (in connection with the notion of a derivator (section 69)), called in their book "homotopy direct limits" and "homotopy inverse limits" (in the special case of the derivator associated to ordinary homotopy types, if I got it right). It came as a surprise that in their book, these operations are developed mainly as technical tools for developing their theory of k-completions, whereas in my own reflection they appeared from the start as "the" main operations in homotopical as well as homological algebra. There had been quite a similar surprise when Tim Porter had sent me a reprint (in July, just before I stopped with my notes for a month or so) of Don Anderson's beautiful paper "Fibrations and Geometric Realizations" (Bulletin of the Amer. Math. Soc. September 1978), where a very general and (as I feel) quite basic existence theorem for integration and cointegration operations in the set-up of closed model categories of Quillen of the type precisely I was after, is barely alluded to at the end of the introduction, and comes more or less as just a by-product of work done in view of a result on geometric realizations which (to an outsider like me at any rate) looks highly technical and not inspiring in the least!

[†] definitely not, in general!

[p. 505]

[Anderson (1978)]

It is becoming clear that I cannot put off much longer getting acquainted with the main ideas and results of Bousfield-Kan's book, which definitely looks like one of the few basic texts on foundational matters in homotopy theory. Still, before doing some basic reading, I would like to write down the sporadic reflections on schematic homotopy types I went into during the last weeks, while they are still fresh in my mind!

[p. 506] 28.9.

120 *Digression on the Lazare "analyzers" for "binomial" commutative algebra, λ-commutative algebra, etc.*

Yesterday (prompted by the reflections from the day before, cf. section 119), I pondered a little on the common features of the various set-ups for "commutative algebra" (possibly, too, for corresponding notions of "schemes") one gets when introducing extra operations on commutative rings or algebras, such as divided power structure (on a suitable ideal) or a λ-structure with operations λ^i paraphrasing exterior powers, or a binomial structure with operations $x \mapsto \binom{x}{n}$ paraphrasing binomial coefficients, or an S-structure with operations S^i of Adams' type paraphrasing sums of i^{th} powers of roots of a polynomial (a weakened version of a λ-structure). It seems that the unifying notion here is the notion of an "analyzer" (analyseur) of Lazare, "containing" the Lazare analyzer for commutative rings (not necessarily with unit), so that the components Ω_n ($n \geq -1$) are commutative rings (not necessarily with unit). In case the extra operations we want to introduce on commutative rings are to be defined on rings *with units* (not just on a suitable ideal of such a ring, as is the case for the divided power structure), and they all can be defined in terms of operations involving just one argument, the reasonable extra axiom on the corresponding analyzer (as suggested by the examples at hand) is that for $n \geq 1$, Ω_n can be recovered in terms of Ω_0 and $\Omega_{-1} = k_0$ (the latter acting as a ground ring for the theory) as the $(n+1)$-fold tensor power of Ω_0 over k_0. Thus, the whole structure of the analyzer may be thought of as embodied in the system $\Omega = (k_0, \Omega_0)$, where k_0 is a commutative ring (with unit, now), Ω_0 a commutative k_0-algebra with unit, endowed moreover with a composition operation $(F, G) \mapsto F \circ G$, satisfying a bunch of simple axioms I don't feel like writing down here. The simplest case of all of course (corresponding to usual commutative algebra "over k_0" as a ground ring, with no extra structure on commutative algebras with unit over k_0) is $\Omega_0 = k[T]$, with the usual composition of polynomials, T acting as the two-sided unit for composition. In the general case, Ω_0 and its tensor powers Ω_n over k_0 are going to play the part played by polynomial rings in ordinary commutative algebra. There should be a ready generalization, in this spirit, of taking the symmetric algebra of a k_0-module (which, for a module free and of finite type will yield an Ω-structure isomorphic to one of the Ω_n's). An Ω-structure on a set k amounts to giving a structure of a commutative k_0-algebra with unit on k, plus a map

$$\Omega_0 \to \mathrm{Maps}(k, k), \quad F \mapsto (x \mapsto F(x))$$

[p. 507] compatible with the structures of k_0-algebras as well as composition operations, and satisfying moreover two conditions for

$$F(x+y), \quad \text{resp. } F(xy)$$

in terms of two diagonal maps

$$\Omega_0 \underset{\Delta_{\mathrm{m}}}{\overset{\Delta_{\mathrm{a}}}{\rightrightarrows}} \Omega_1 = \Omega_0 \otimes_{k_0} \Omega_0$$

(which may be described in terms of the composition structure on Ω_0, as expressing the compositions $F \circ (G' + G'')$ resp. $F \circ (G'G'')$), and moreover one trivial condition for $F(\lambda)$ when λ in k comes from k_0, namely compatibility of the map $k_0 \to k$ with the operations of Ω_0 on both k_0 and k. (NB the operation of Ω_0 on k_0 is defined by

$$F(\lambda) = F \circ \lambda,$$

where k_0 is identified to a subring of Ω_0.)

I didn't pursue much further these ponderings, just one digression among many in the main line of investigation! I also read through the preprint of David W. Jones on Poly-T-complexes, which Ronnie Brown (acting as David Jones' supervisor) had sent me a while ago. There he develops a notion of polyhedral cells, with a view of using these instead of simplices or cubes for doing combinatorial homotopy theory. As I had pondered a little along this direction (cf. sections 91, topic 8), and section 93), I was hoping that some of the perplexities I had been meeting would be solved in David Jones' notes – for instance that there would be handy criteria for a category M made up with such polyhedral cells to be a weak test category, namely that objects of $M^\vee$ may be used as models for homotopy types; also, that the "standard" chain complex constructed in section 93 is indeed an "abelianizator" for M, i.e., may be used for computing homology of objects of $M^\vee$. David Jones' emphasis, however, is a rather different one – he seems mainly interested in generalizing the theory of "thin" structures of M. K. Dakin from the simplicial to the more general polyhedral set-up and prove a corresponding equivalence of categories. Thus, my perplexities remain – they are admittedly rather marginal in the main line of thought, and I doubt I'll stop to try and solve them.

[Jones (1983)]

[Dakin (1977)]

The basic pair of adjoint functors $\widetilde{K} : \mathrm{Hotab}_0 \leftrightarrows \mathrm{Hot}_0 : \mathrm{L}\widetilde{\mathrm{H}}_*$.

[p. 508]

121 The tentative approach towards defining and studying "schematic" homotopy types I have been following lately relies heavily on a suitable notion of "linearization" of such homotopy types. One can imagine that many different approaches (for instance via De Rham complexes with divided powers, or additive small categories with diagonal maps) may be devised for "schematic" homotopy types, but in any case it seems likely that a suitable notion of linearization will play an important role. It may be worthwhile therefore to try and pin down the wished-for main features of such a theory, with the hope maybe of getting an axiomatic description for it, with a corresponding unicity statement. Before doing so, the first thing to do seems to review some main formal features of linearization for ordinary ("discrete") homotopy types.

Recall the definition

(1) $\mathrm{Hotab} \overset{\text{def}}{=} \mathrm{D}_\bullet(\mathrm{Ab}) =$ derived category of the category of abelian chain complexes, with respect to quasi-isomorphisms,

and the two canonical functors

$$\text{Hot} \underset{K_\pi}{\overset{\text{LH}_\bullet}{\rightleftarrows}} \text{Hotab}, \tag{2}$$

Where $\text{LH}_\bullet$ is the "abelianization functor", and K_π is defined via the Kan-Dold-Puppe functor, associating to a chain complex the corresponding semisimplicial abelian group. The diagram (2) may be viewed (up to equivalence) as deduced from the corresponding diagram

$$\mathbb{\Delta}^\wedge \underset{\text{DP}}{\overset{W}{\rightleftarrows}} \mathbb{\Delta}^\wedge_{\text{ab}} \quad (\cong \text{Ch}_\bullet(\text{Ab})) \tag{2'}$$

by passing to the suitable localized categories Hot and Hotab. In the diagram (2') W is left adjoint to DP, which is now just the forgetful functor. One main fact about (2) is

$$K_\pi \text{ is right adjoint to } \text{LH}_\bullet, \tag{3}$$

which is just a neater way for expressing the familiar fact that for given abelian group π and natural integer n, the object $K(\pi,n)$ in Hot (namely the image by K_π of the chain complex $\pi[n]$ reduced to π in degree n) represents the cohomology functor

$$X \mapsto \text{H}^n(X,\pi) \simeq \text{Hom}_{\text{Hotab}}(\text{LH}_\bullet(X), \pi[n]).$$

[p. 509] The functor K_π may be called the "*Eilenberg-Mac Lane functor*", as its values are immediate generalizations of the Eilenberg-Mac Lane objects $K(\pi,n)$. As any object in Hotab is isomorphic to a product of objects $\pi[n]$, it follows that in order to check the adjunction formula between $\text{LH}_\bullet$ and K_π it is enough to do so for objects in Hotab of the type $\pi[n]$, which is the "familiar fact" just recalled. The notation K_π in (2) is meant to suggest the Eilenberg-Mac Lane $K(\pi,n)$ object generalized by the objects $K_\pi(L_\bullet)$, and also to recall that we recover the homology invariants of $L_\bullet$ from $K_\pi(L_\bullet)$ via the π_i invariants, by the formula

$$\pi_i(K_\pi(L_\bullet)) \simeq \text{H}_i(L_\bullet), \tag{4}$$

which implies by the way that the functor K_π is "conservative", i.e., a map in Hotab which is transformed into an isomorphism is an isomorphism. This should not be confused with the stronger property of being fully faithful, or equivalently of the left adjoint $\text{LH}_\bullet$ being a localization functor, or equivalently still, the adjunction morphism

$$\text{LH}_\bullet(K_\pi(L_\bullet)) \to L_\bullet \tag{5}$$

being an isomorphism in Hotab, which is definitely false!

The other adjunction morphism

$$X \to K_\pi(\text{LH}_\bullet(X)) \overset{\text{def}}{=} X_{\text{ab}} \tag{6}$$

is still more interesting, its effect on the homotopy invariants π_i are the Hurewicz homomorphisms

(7) $$\pi_i(X) \to \mathrm{H}_i(X) \overset{\text{def}}{=} \pi_i(X_{\mathrm{ab}}) \simeq \mathrm{H}_i(\mathrm{LH}_\bullet(X));$$

introducing the homotopy fiber of (6) (in the case of pointed homotopy types) and denoting by $\gamma_i(X)$ its homotopy invariants, we get the exact sequences of J. H. C. Whitehead (as recalled in a letter from R. Brown I just got)

(7') $$\cdots \to \gamma_i(X) \to \pi_i(X) \to \widetilde{\mathrm{H}}_i(X) \to \gamma_{i-1}(X) \to \cdots,$$

where $\widetilde{\mathrm{H}}_i$ denotes the "reduced" homology group of a *pointed* homotopy type, equal to H_i for $i \neq 0$ and to $\mathrm{Coker}(\mathrm{H}_0(\mathrm{pt}) \to \mathrm{H}_0(X)) \simeq \mathrm{H}_0(X)/\mathbb{Z}$ for $i = 0$.

The case of pointed homotopy types seems of importance for schematic homotopy types, and deserves some extra mention and care. We may factor diagram (2) into

(8) $$\triangle^\wedge \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \triangle^{\wedge\bullet} \underset{\widetilde{\mathrm{DP}}}{\overset{\widetilde{W}}{\rightleftarrows}} \triangle^\wedge_{\mathrm{ab}}\ ,$$

where $\triangle^{\wedge\bullet}$ is the category of *pointed* semisimplicial complexes, β the forgetful functor from these to non-pointed complexes, and α its left adjoint, which may be interpreted as [p. 510]

(9) $$\alpha(X_*) = X_* \amalg e_*,$$

where e_* is the final object of $\triangle^\wedge$, and the second member is pointed by its summand e_*. The functor $\widetilde{\mathrm{DP}}$ comes from applying componentwise the obvious functor from (Ab) to (Sets$^\bullet$) (pointed sets), $\widetilde{W}$ is its left adjoint. Passing to the suitable localized categories, we get from (8)

(10) $$\begin{tikzcd} \mathrm{Hot} \arrow[r, shift left, "\alpha"] \arrow[rr, bend left=40, "\mathrm{LH}_\bullet"] & \mathrm{Hot}^\bullet \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\mathrm{L}\widetilde{\mathrm{H}}_\bullet"] & \mathrm{Hotab} \arrow[l, shift left, "\widetilde{K}_\pi"] \arrow[ll, bend left=40, "K_\pi"] \end{tikzcd}\ ,$$

factoring (2), where α is now defined by the formula similar to (9)

(9') $$\alpha(X) = X \amalg e,$$

where e denotes the final object of Hot, and is used for defining the pointed structure of the second member. The functor $\widetilde{W}$ in (8) can be described (as is seen componentwise) as

(11) $$\widetilde{W}(X_*) = W(\beta(X_*))/\operatorname{Im} W(e_*),$$

where
$$W(e_*) \to W(X_*), \quad \text{i.e.,} \quad \mathbb{Z}^{(e)} \to \mathbb{Z}^{(X)}$$

is deduced from the pointing map $e_* \to X_*$. Accordingly, we get an expression

$$\mathrm{L}\widetilde{\mathrm{H}}_\bullet(X) \simeq \mathrm{LH}_\bullet(X)/\,\mathrm{LH}_\bullet(e),$$

more accurately, an exact triangle

$$\text{(12)} \qquad \begin{tikzcd} & \mathrm{L}\widetilde{\mathrm{H}}_\bullet(X) \arrow[dl] & \\ \mathrm{LH}_\bullet(e) = \mathbb{Z}[0] \arrow[rr] & & \mathrm{LH}_\bullet(X) \arrow[ul] \end{tikzcd} ,$$

where X is any object in $\mathrm{Hot}^\bullet$ and $\mathrm{LH}_\bullet(X)$ is short for $\mathrm{LH}_\bullet(\beta(X))$. Of course, the functors $\mathrm{L}\widetilde{\mathrm{H}}$ and $\widetilde{K}_\pi$ are still adjoint (one hopes!), hence for any pointed homotopy type X an adjunction map in $\mathrm{Hot}^\bullet$

$$\text{(6')} \qquad X \to \widetilde{K}_\pi(\mathrm{LH}_\bullet(X)),$$

and (7) and (7') are deduced from (6') and its homotopy fiber, rather than from (6) where it doesn't really make sense because of lack of canonical base points for taking γ_i's and homotopy fibers.

[p. 511] 2.10.

122 *Rambling reflections on $\mathrm{L}\widetilde{\mathrm{H}}_*$, Postnikov invariants, $S(H,n)$'s – and on the non-existence of a "total homotopy"-object $\mathrm{L}\pi_*(X)$ for ordinary homotopy types.*

Since the last notes, I have been doing three days' scratchwork (including today's) on various questions around abelianization in general (for discrete homotopy types) and on Postnikov dévissage, in connection with the review on some main formal properties of abelianization, started upon in the previous section 121. I didn't get anything really new for me, rather it was just part of the necessary rubbing against the things, in order to get a better feeling of what they are like, or what they are likely to be like – what is likely to be true, and what not. The most interesting, maybe, is that I got an inkling of a fairly general version of a Kan-Dold-Puppe kind of relationship, in terms of derived categories, valid presumably for any local test category, and in particular for categories like $\Delta_{/X}$, with X in $\Delta^\wedge$. It would be untimely, though, to build up still more the (already pretty high) tower of digressions, and for the time being I'll stick to what is relevant strictly to my immediate purpose – namely getting through with the wishful thinking about schematization! Thus, I'll be content to work with the category Δ in order to construct models for homotopy types and perform constructions with them (such as abelianization), without getting involved at present in looking up how much is going over (and how) to the case of more general small categories A...

It occurred to me that the variant $\widetilde{W}$ or $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ for the abelianization functors W and $\mathrm{LH}_\bullet$, introduced in the previous section using a pointed structure for the argument X in $\Delta^\wedge$ or $\mathrm{Hot}^\bullet$, could be advantageously defined without this extra structure. The construction for $\widetilde{W}$ which follows goes through indeed in any topos whatever (not only $\Delta^\wedge$). Let X be an object in $\Delta^\wedge$, then there is a canonical augmentation map

$$\varepsilon : W(X) = \mathbb{Z}^{(X)} \to \mathbb{Z}_\Delta$$

towards the constant semisimplicial group $\mathbb{Z}_{\triangle}$, corresponding to the constant map

$$X \to \mathbb{Z}_{\triangle}$$

with value 1. We thus get a functorial exact sequence

$$(1) \qquad 0 \to \widetilde{W}(X) \to W(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\widetilde{W}(X)$ denotes the kernel of the augmentation above, hence an extension of $\mathbb{Z}$ by $\widetilde{W}(X)$, or (what amounts essentially to the same) a torsor under the group object $\widetilde{W}(X)$, which we'll denote by

$$(2) \qquad W(X)(1) = \varepsilon^{-1}(1).$$

Moreover, the canonical map [p. 512]

$$X \to W(X) = \mathbb{Z}^{(X)}$$

factors (by construction) through

$$(3) \qquad X \to W(X)(1) \quad (\hookrightarrow W(X)),$$

and this map is universal for all maps of X into torsors under abelian group objects of $\triangle\hat{}$. It is an immediate consequence of Whitehead's theorem that a weak equivalence $X \to Y$ in $\triangle\hat{}$ induces weak equivalences in $\triangle\hat{}$

$$(4) \qquad \widetilde{W}(X) \to \widetilde{W}(Y), \quad W(X)(1) \to W(Y)(1)$$

(and also $W(X)(n) \to W(Y)(n)$ for any $n \in \mathbb{Z}$), and accordingly that the map between the normalized chain complexes corresponding to $\widetilde{W}(X)$ and $\widetilde{W}(Y)$, namely by definition

$$(4') \qquad \mathrm{L}\widetilde{\mathrm{H}}_\bullet(X) \to \mathrm{L}\widetilde{\mathrm{H}}_\bullet(Y)$$

is a quasi-isomorphism. Thus, we get a functor

$$\mathrm{L}\widetilde{\mathrm{H}}_\bullet : \mathrm{Hot} \to \mathrm{Hotab},$$

whose composition $\mathrm{L}\widetilde{\mathrm{H}}_\bullet \circ \beta$ with the forgetful functor

$$\beta : \mathrm{Hot}^\bullet \to \mathrm{Hot}$$

is canonically isomorphic to the functor $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ of p. 510 (formula 9). More specifically, when X is in $\triangle\hat{}^\bullet$, then the map

$$\mathbb{Z} = \mathbb{Z}^{(e)} \to \mathbb{Z}^{(X)} \quad \text{deduced from } e \to X$$

defines a splitting of the extension (1), i.e., an isomorphism

$$(5) \qquad \mathbb{Z}^{(X)} = W(X) \simeq \widetilde{W}(X) \oplus \mathbb{Z},$$

and accordingly, $\widetilde{W}(X)$ may be interpreted, at will, as a quotient group of $W(X)$ (which we did on p. 510), or as a subobject of $W(X)$, which is the better choice, because it makes good sense without using any pointed structure. Again, without using the pointed structure, we get a canonical exact triangle in D(Ab) interpreting (1)

(6)
$$\begin{tikzcd} & \mathbb{Z} \arrow[dl] & \\ \mathrm{L}\widetilde{\mathrm{H}}_\bullet(X) \arrow[rr] & & \mathrm{LH}_\bullet(X) \arrow[ul] \end{tikzcd}$$

replacing ((12) p. 510), which (in the case considered there, namely X pointed) should be replaced by the more precise relationship

(5') $$\mathrm{LH}_\bullet(X) \simeq \mathrm{L}\widetilde{\mathrm{H}}_\bullet(X) \oplus \mathbb{Z},$$

an isomorphism functorial for X in $\mathrm{Hot}^\bullet$.

[p. 513] Truth to tell, these generalities are more interesting still for an argument X in a category like ${\triangle^\wedge}_{/Y} \approx (\triangle_{/Y})^\wedge$, where Y is an arbitrary object in $\triangle^\wedge$, rather than in $\triangle^\wedge$ itself. In the latter case, X always admits a pointed structure, i.e., a section over the final object (provided X is non-empty), hence there always exists a splitting for (1), and hence one for (6); whereas for an object X over Y, there does not always exist a section over Y, and accordingly it may well happen that the extension similar to (1) (but taken "over Y") does not split, i.e., that the torsor $W_{/Y}(X)(1)$ is not trivial. The class of this torsor is an element

(7) $$c(X/Y) \in \mathrm{H}^1(Y, \widetilde{W}_{/Y}(X)),$$

a very interesting invariant indeed, giving rise to the Postnikov invariants when Y is one of the X_n's occurring in the Postnikov dévissage of X. It was while trying to understand the precise relationship between a cohomology group as in (7), with "continuous" coefficients (by which I mean to suggest that $\widetilde{W}_{/Y}(X)$ is viewed intuitively as a fiberspace over the "space" X, whose fibers are topological abelian groups which are by no means discrete, but got a bunch of non-vanishing π_i's!), and a cohomology group with "discrete" coefficients, more accurately with coefficients in a complex of chains in the category of abelian sheaves over X, i.e., over $\triangle_{/X}$ whose homology sheaves are definitely of a "discrete" nature (for instance, they are locally constant if $X \to Y$ is a Kan fibration, and their fibers are $\mathbb{Z}$-modules of finite type provided we make a mild finite-type assumption on the fibers of $X \to Y$), that I got involved in a more general reflection on a Kan-Dold-Puppe type of relationship. I hope to come back to this when getting back to the general review of linearization begun in part V of these notes, and abruptly interrupted after section 109, when getting caught unwittingly by the enticing mystery of schematization!

Another afterthought to the previous notes is that I am going to denote by $K, \widetilde{K}$, the functors

(7) $$K : \mathrm{Hotab} \to \mathrm{Hot}, \quad \widetilde{K} : \mathrm{Hotab} \to \mathrm{Hot}^\bullet, \quad K = \beta \circ \widetilde{K},$$

denoted by K_π, $\widetilde{K}_\pi$ in section 121. This gives a formula such as

$$K(\pi[n]) = K(\pi, n), \tag{8}$$

where $\pi[n]$ denotes the chain complex in (Ab) which is π placed in degree n. I was intending to complement the former notation K_π by a similar notation K_{H} (where H stands for "homology", as π was standing for "homotopy", and K means "complex"), but I finally found the notation S (suggestive of "spheres") more congenial, giving rise to [p. 514]

$$S(\mathrm{H}[n]) = S(\mathrm{H}, n) = \text{sphere-like homotopy type whose } \mathrm{L}\widetilde{\mathrm{H}}_\bullet \text{ is isomorphic to } \mathrm{H}[n]. \tag{8'}$$

But I am anticipating somewhat on some of the wishful thinking still ahead, involving the would-be description (preferably in the schematic set-up, but more elementarily maybe in the discrete one of ordinary homotopy types) of a functor

$$S : \mathrm{Hotab} = \mathrm{D}_\bullet(\mathrm{Ab}) \to \mathrm{Hot},$$

whose most important formal property should be

$$\mathrm{LH}_\bullet(S(L_\bullet)) \simeq L_\bullet,$$

compare with formula (8') for $L_\bullet = \pi[n]$. However, such a formula cannot hold for any $L_\bullet$, i.e., an arbitrary chain complex in (Ab) is not quasi-isomorphic to an object $\mathrm{LH}_\bullet(X)$ with X in (Hot), as a necessary condition for this is that $\mathrm{H}_0(L_\bullet)$ should be free. Thus, we better restrict to the 0-connected case, and take $L_\bullet$ subject to

[in (8'), the $\mathrm{H}[n]$ was originally written $\pi[n]$. I'm not sure what to make of this...]

$$\mathrm{H}_0(L_\bullet) = 0, \quad \text{i.e., } L_\bullet \text{ in } \mathrm{D}_{\geq 1}(\mathrm{Ab}) \overset{\text{def}}{=} \mathrm{Hotab(0\text{-}conn)},$$

and describe S, or better still $\widetilde{S}$, as a functor

$$\widetilde{S} : \mathrm{Hotab(0\text{-}conn)} \to \mathrm{Hot(0\text{-}conn)} \hookrightarrow \mathrm{Hot}^\bullet, \tag{9}$$

subject to the condition

$$\mathrm{L}\widetilde{\mathrm{H}}_\bullet(\widetilde{S}(L_\bullet)) \simeq L_\bullet, \tag{10}$$

giving rise to

$$\widetilde{\mathrm{H}}_i(\widetilde{S}(L_\bullet)) \simeq \mathrm{H}_i(L_\bullet), \tag{10'}$$

which should be compared to the familiar formula

$$\pi_i(\widetilde{K}(L_\bullet)) \simeq \mathrm{H}_i(L_\bullet). \tag{11}$$

Maybe formula (10) is not quite enough for characterizing the functor $\widetilde{S}$ up to unique isomorphism, and I confess I didn't try to construct in some canonical way a functor (9) satisfying (10) – and I am not quite sure even whether such a functor exists. One main evidence for existence of such a functor would be the existence and unicity (up to

unique isomorphism) of the homotopy types $S(H,n)$ in (8') – and I have a vague remembrance of having looked through a paper, fifteen or twenty years ago, where such spaces were indeed introduced and studied. Thus, a functor $\widetilde{S}$ is perhaps nowadays more or less standard
[p. 515] knowledge in homotopy theory. I was led to postulate the existence of a functor $\widetilde{S}$ in the context of *schematic* homotopy types, by reasons of symmetry in the $\mathrm{LH}_\bullet$ and $\mathrm{L}\pi_\bullet$ formalism, which I'll try to get through a little later. It should be clear from the outset, though, that in the schematic set-up, even the mere existence of (schematic) homotopy types $S(\mathbb{Z},n)$ (corresponding to ordinary spheres), and even for $n=2$ or 3 only, is very far from trivial, and as a matter of fact isn't even known (in the set-up of semisimplicial unipotent bundles). This shows at the same time that if such a functor $\widetilde{S}$ can actually be constructed in the schematic case, it is likely to be a lot more interesting still than in the discrete case, as it will presumably give information on homotopy groups of spheres (via the description of the π_i's of a semisimplicial unipotent bundle via the Lie functor (cf. section 118)).

There is a fourth tentative functor in between the categories Hotab and Hot, or more accurately, from a suitable subcategory $\mathrm{Hot}^\bullet(0)$ of $\mathrm{Hot}^\bullet$ (corresponding to 0-connected pointed homotopy types whose π_1 is abelian and acts trivially on the higher π_i's) to Hotab(0-conn) = Hotab(0), strongly suggested by the schematic case of section 118, namely a functor

$$(*)\qquad \mathrm{L}\pi_\bullet : \mathrm{Hot}^\bullet(0) \to \mathrm{Hotab}^\bullet(0)???,$$

whose main functorial property should be

$$(12)\qquad \mathrm{H}_i(\mathrm{L}\pi_\bullet(X)) \simeq \pi_i(X).$$

To play really safe, one might hope such a functor to be defined at any rate for 1-connected pointed homotopy types. Another equally important property, suggested by the schematic set-up as well as by (11) and (12), is the formula

$$(13)\qquad \mathrm{L}\pi_\bullet(\widetilde{K}(L_\bullet)) \simeq L_\bullet,$$

similar to formula (10) (with the pair $(\widetilde{S},\mathrm{L}\widetilde{\mathrm{H}}_\bullet)$ replaced by the pair $(\widetilde{K},\mathrm{L}\pi_\bullet)$). Of course, for X of the type $\widetilde{K}(L_\bullet)$, (12) follows from (13)

3.10.

We'll see, though, that a functor $\mathrm{L}\pi_\bullet$ as in (*), satisfying the properties above, *does not exist*. Indeed, applying $\mathrm{L}\pi_\bullet$ to the adjunction morphism

$$X \to \widetilde{K}(\mathrm{L}\widetilde{\mathrm{H}}_\bullet(X))$$

(cf. p. 509, (6)), and using (13), we should get a Hurewicz homomor-
[p. 516] phism

$$(14)\qquad \mathrm{L}\pi_\bullet(X) \to \mathrm{L}\widetilde{\mathrm{H}}_\bullet(X)$$

more precise than the separate homomorphisms

$$\pi_i(X) \to \widetilde{\mathrm{H}}_i(X),$$

and applying this to an object $\widetilde{K}(L_\bullet)$ and applying (13) again, we should get a functorial homomorphism

$$L_\bullet \to \mathrm{L}\widetilde{\mathrm{H}}_\bullet(\widetilde{K}(L_\bullet)) \tag{15}$$

in opposite direction from the adjunction morphism

$$\mathrm{L}\widetilde{\mathrm{H}}_\bullet(\widetilde{K}(L_\bullet)) \to L_\bullet \tag{16}$$

(p. 509, (5)), the composition of the two being the identity in $L_\bullet$. In other words, (15) should be a canonical splitting of the natural adjunction morphism (16). Take for instance $L_\bullet = \pi[n]$, then the first member of (16) is the Eilenberg-Mac Lane homology $\mathrm{L}\widetilde{\mathrm{H}}_\bullet(\pi, n)$, whose first non-vanishing homology group is π, and (16) is just the canonical augmentation

$$\mathrm{L}\widetilde{\mathrm{H}}_\bullet(\pi, n) \to \pi[n],$$

and taking the $\mathrm{Ext}^i_{\mathbb{Z}}$ of both members with $M[n]$ (M any object in (Ab)) yields the transposed canonical homomorphism

$$\mathrm{Ext}^i_{\mathbb{Z}}(\pi, M) \to \mathrm{H}^{n+i}(\pi, n; M), \tag{17}$$

which is an isomorphism for $i = 0$ (whereas for $i < 0$ both members are zero). For $i = 1$, we get an exact sequence ("universal coefficients")

$$0 \to \mathrm{Ext}^i_{\mathbb{Z}}(\pi, M) \to \mathrm{H}^{i+1}(\pi, n; M) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_{i+1}(\pi, n), M) \to 0. \tag{18}$$

A canonical splitting of (15) would yield a canonical splitting of this exact sequence, which definitely looks somehow as "against nature"! Surely, it shouldn't be hard to find, for any given integer $n \geq 1$, suitable abelian groups π and M, such that there does not exist a splitting of (18) stable under the action of the group

$$G = \mathrm{Aut}(\pi) \times \mathrm{Aut}(M)$$

(acting by "transport de structure" on the three terms of (18)); presumably even, looking at $\mathrm{Aut}(\pi)$ should be enough. As I am not at all familiar with Eilenberg-Mac Lane cohomology, except a little in the case $n = 1$, i.e., for usual group cohomology, I didn't try to construct an example for any n, only one for $n = 1$, i.e., in a non-simply connected case, which again is a little less convincing as if it was one for $n = 2$...

Thus, let's take $n = 1$, a vector space of finite dimension over the prime field $\mathbb{F}_2$, and $M = \mathbb{F}_2$, in this case standard calculations show [p. 517]
that the exact sequence (18) can be identified with the familiar exact sequence

$$0 \to V' \to \mathrm{Sym}^2(V') \to \bigwedge^2 V' \to 0, \tag{18'}$$

where the first arrow associates to every linear form on V its square, i.e., the same form (!) but viewed as being a quadratic form on V; and the second arrow associates to every quadratic form on V the associated bilinear form, which is alternate because of char. 2. It is well-known I guess that for $\dim V \geq 2$, there does not exist a splitting of (18') which is stable under $\mathrm{Aut}(V) \simeq \mathrm{Aut}(V')$.

Remark. If we admit that a similar example can be found for non-splitting of (18), with $n \geq 2$ (which shouldn't be hard I guess for someone familiar with Eilenberg-Mac Lane homology), this shows that there does *not* exist (as was contemplated at the beginning, cf. section 111) an equivalence of categories between the schematic 1-connected homotopy types (defined via the model category $\mathcal{M}_1(\mathbb{Z})$ of semisimplicial unipotent bundles over $\mathbb{Z}$ satisfying $X_0 = X_1 = e$) and 1-connected pointed usual homotopy types, as this would imply the existence of a functor $\mathrm{L}\pi_\bullet$, hence of a canonical splitting of (18). The same argument will show that the once hoped-for comparison theorem between usual Eilenberg-Mac Lane homology (or cohomology), and the schematic one, is false, because in the schematic set-up $\mathrm{L}\pi_\bullet$ does exist, and the direct construction of (15) (a functorial splitting of (16)) is then anyhow a tautology. Thus after all, we don't have to rely on delicate results of Breen's on $\mathrm{Ext}^i(\mathbb{G}_\mathrm{a}, \mathbb{G}_\mathrm{a})$ over the prime fields $\mathbb{F}_p$ (as suggested by Illusie), in order to get this "negative" result, causing unreasonable expectations to crash...

[p. 518] 4.10.

123 *The hypothetical complexes ${}^n\Pi_* = \mathrm{L}\pi_*(S^n)$, and comments on homotopy groups of spheres.*

For the last few days – since I resumed the daily mathematical ponderings (interrupted more or less for nearly one month), the wind in my sails has been rather slack I feel. It has been nearly six weeks from now that I started pondering on schematization and on schematic homotopy types – in the process I got rid of some misplaced expectations, very well. Still, the unpleasant feeling remains of not having really any hold yet on those would-be schematic homotopy types, due mainly to my incapacity so far of performing (in the model category of semisimplicial unipotent bundles say) the basic homotopy operations of taking homotopy fibers and cofibers of maps. In lack of this, I am not ("morally") sure yet that there does exist indeed a substantial reality of the kind I have been trying to foreshadow. I am not even wholly clear yet of how to define the notion of "weak equivalence" in t he model category $\mathcal{M}_0(k)$ (or in the smaller one $\mathcal{M}_1(k)$, to play safe, as even the definition of $\mathcal{M}_0(k)$ isn't too clear yet) – there are three ways to define weak equivalences, using $\mathrm{LH}_\bullet$, or $\mathrm{L}\pi_\bullet$, or the sections functor from $\mathcal{M}_1(k)$ to ordinary semisimplicial complexes, and it isn't clear yet whether these are indeed equivalent. We may of course call "weak equivalence" in $\mathcal{M}_1(k)$ a map which becomes an isomorphism by any of these three functors. But I wouldn't say I am wholly confident yet that there exists a localization $\mathrm{Hot}_1(k)$ of $\mathcal{M}_1(k)$, with respect to this notion of weak equivalence or some finer one, in such a way that in $\mathrm{Hot}_1(k)$ one may introduce the two types of "fibration" and "cofibration" sequences with the usual properties, and the sections functor

$$\mathrm{Hot}_1(k) \to \mathrm{Hot}_1$$

should respect this structure, and moreover give rise to functorial isomorphisms

$$\mathrm{H}_i(\mathrm{L}\pi_\bullet(X_*)) \simeq \pi_i(X_*(k)),$$

nor even do I feel wholly confident that a theory of this kind can be developed, possibly with different kinds of models for k-homotopy types from the ones I have been using. At any rate, it seems to me that the two kinds of conditions or features I have just been stating, are indeed the crucial ones, plus (I should add) existence in $\mathrm{Hot}_1(k)$ of an object

$$S_k^2 = S(2,k),$$

standing for the 2-sphere, with

$$\mathrm{L}\widetilde{\mathrm{H}}_\bullet(S_k^2) \simeq \mathbb{Z}[2],$$

whose image in Hot_1 (in case $k = \mathbb{Z}$ say) should be the (homotopy type of the) ordinary 2-sphere. Taking suspensions, we'll get from this the n-spheres $S_k^n = S(n,k)$ over k, for any $n \geq 2$. (NB In case the relevant homotopy theoretic structures can be introduced in a suitable larger $\mathrm{Hot}_0(k)$, containing the objects $\widetilde{K}(L_\bullet)$ for $L_\bullet$ a 0-connected chain complex of k-modules, $\mathrm{Hot}_0(k)$ contains an object $S(1,k) = \widetilde{K}(1,k)$, and S_k^2 may be simply described as its suspension.) One main motivation for trying to push through a theory of schematic homotopy types may be the hope that this may provide new insights into the homotopy groups of spheres. A first interesting consequence would be that for any given sphere S^n (in the set-up now of ordinary homotopy theory), the series of all its homotopy groups $\pi_i(S^n) = \mathrm{Hom}_{\mathrm{Hot}}(S^i, S^n)$ may be viewed as being the homology modules of an object of $\mathrm{D}_\bullet(\mathrm{Ab})$, namely $\mathrm{L}\pi_\bullet(S_{\mathbb{Z}}^n)$, canonically associated to S^n. As was seen by an example in the previous section, the existence of such a *canonical* representation is by no means a trivial or innocuous fact, it is indeed definitely false for arbitrary homotopy types – here then it would turn out as a rather special feature of the full subcategory of $\mathrm{Hot}^\bullet$ made up with the spheres S^n $(n \geq 2)$. Thus, for any $\alpha \in \pi_i(S^n)$, i.e., for any map [p. 519]

$$\alpha : S^i \to S^n$$

in $\mathrm{Hot}^\bullet$, there should be an induced map

$$\mathrm{L}\pi_\bullet(\alpha) : \mathrm{L}\pi_\bullet(S^i) \to \mathrm{L}\pi_\bullet(S^n)$$

with associativity condition for a composition

$$S^i \xrightarrow{\alpha} S^n \xrightarrow{\beta} S^m.$$

If we write

$${}^n\Pi_\bullet \overset{\mathrm{def}}{=} \mathrm{L}\pi_\bullet(S^n),$$

this defines on the set of chain complexes (more accurately, abelian homotopy types) ${}^n\Pi_\bullet$ ($n \geq 2$) quite a specific structure, which merits to be investigated a priori (under the assumption, of course, of the existence of a reasonable theory of schematic homotopy types, satisfying the criteria above). Maybe after all, it will turn out that the homotopy groups of spheres *cannot* be fitted into such an encompassing structure – so much the better, as this will show that the kind of theory I started trying to dig out doesn't exist, which will clear up the situation a great deal! But if there does exist such a canonical structure, it surely shouldn't be ignored, and it should be pleasant work to try and pin down exactly
[p. 520] what extra information on homotopy groups of spheres is involved in such extra structure. It may be noted for instance that, when taking $i = n$ hence $\pi_i(S^n) = \mathbb{Z}$, we get an operation of the multiplicative monoid $\mathbb{Z}^{\mathrm{mult}}$ on ${}^n\Pi_\bullet$, whose action on the homology groups of ${}^n\Pi_\bullet$, i.e., on the homotopy groups $\pi_j(S^n)$, is surely an important extra structure on these, which I hope has been studied by the homotopy people extensively. Surely it has been known, too, for a long time (at any rate since Artin-Mazur's foundational work on profinite homotopy types) that when $\pi_j(S^n)$ is finite (i.e., practically in all cases except $j = n$) this action comes from a continuous action of the multiplicative monoid

$$M = \mathbb{Z}^{\wedge\,\mathrm{mult}},$$

where $\mathbb{Z}^\wedge$ is the completion of $\mathbb{Z}$ with respect to the topology defined by subgroups of finite index:

$$\mathbb{Z}^\wedge = \lim \mathbb{Z}/n\mathbb{Z} \simeq \prod_p \mathbb{Z}_p,$$

where in the last member the product is taken over all primes, and $\mathbb{Z}_p$ denotes p-adic integers. Taking homotopy types over the ring $k = \mathbb{Z}^\wedge$, we should get that this monoid M operates on the object

$${}^n\Pi_\bullet\Big(\bigoplus^l{}_{\mathbb{Z}}\mathbb{Z}^\wedge\Big) \quad \text{in } \mathrm{D}_\bullet(\mathrm{Ab}_{\mathbb{Z}^\wedge}),$$

as this may be interpreted as $\mathrm{L}\pi_\bullet(S^n_k)$, and k^{mult} operates on $S^n_k = S(n,k)$ for any ring k (if indeed $S(n,\pi)$ depends functorially on the variable k-module π). At any rate, it is surely well-known that $M = \mathbb{Z}^{\wedge\,\mathrm{mult}}$ operates on the profinite completions of all spheres. For the odd-dimensional spheres, and taking the restrictions of this operation to the largest subgroup

$$M^\times = \mathbb{Z}^{\wedge\,\times} \simeq \prod_p \mathbb{Z}_p^\times$$

of M, this group may be viewed as the profinite Galois group of the maximal cyclotomic extension of the prime field $\mathbb{Q}$ (deduced by adjoining all roots of unity), and its action on the profinite completion $S^{n\,\wedge}$ of S^n may be viewed as the canonical Galois action, when $S^{n=2m-1}$ is interpreted as the homotopy type of affine (complex) m-space minus the origin (which makes sense as a scheme over the prime field $\mathbb{Q}$). This was the interpretation in my mind, when stating that the action of $\mathbb{Z}^{\text{mult}}$ and of its completion $\mathbb{Z}^{\wedge\,\text{mult}}$ on the homotopy groups of spheres was in important extra structure on these.

Outline of a program (second version): an autodual formulaire for the basic "four functors" $\mathrm{L}\widetilde{\mathrm{H}}_*$, $\mathrm{L}\pi_*$, $\widetilde{K}$, $\widetilde{S}$.

[p. 521]

124 I guess I'll skip giving a more or (rather!) less complete axiomatic description of $\mathrm{Hot}_0^\bullet$ or of $\mathrm{Hot}_0^\bullet(k)$, in terms of the pair of adjoint functors $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ and $\widetilde{K}$ and Postnikov dévissage, although I did go through this exercise lately – it doesn't really shed new light on the approach we are following here towards "schematic" homotopy types over arbitrary ground rings. In contrast with the two other approaches I have been thinking of before (namely De Rham complexes with divided powers, and small categories with diagonal maps, cf. sections 94 and 109 respectively), the characteristic feature of this approach seems to be that it takes into account the existence of the canonical functor $\widetilde{K} : \mathrm{D}_\bullet(\mathrm{Ab}) \to (\mathrm{Hot})$, paraphrased by a functor

$$\widetilde{K} : \mathrm{D}_\bullet(\mathrm{Ab}_k)(0\text{-conn}) \to \mathrm{Hot}_0(k) \tag{1}$$

compatible with ring extension $k \to k'$; while all three approaches have in common that they yield a paraphrase

$$\mathrm{L}\widetilde{\mathrm{H}}_\bullet : \mathrm{Hot}_0(k) \to \mathrm{D}_\bullet(\mathrm{Ab}_k)(0\text{-conn}) \tag{2}$$

of the (total) homology functor, in a way again compatible with extension of ground rings. In the two earlier approaches this latter functor comes out in a wholly tautological way, whereas in the present approach via semisimplicial schemes, it isn't quite so tautological indeed. If we want to define a functor $\widetilde{K}$ in the context say of De Rham algebras with divided powers, one may think of associating first to a chain complex in Ab_k the corresponding semisimplicial k-module, view this as a semisimplicial set, and take its De Rham complex with divided powers and coefficients in k (as we want an object over k). But this is visibly silly, except possibly for $k = \mathbb{Z}$, as the result doesn't depend on the k-module structure of the chain complex we start with. It is doubtful, anyhow, that in this context, the tautological functor (2) admits a right adjoint (which we then would call $\widetilde{K}$ of course). – Another noteworthy difference between the present approach towards defining "schematic" homotopy types, and the two earlier ones ("earlier" in these notes, at any rate!) is that for $k = \mathbb{Z}$ say, the category of "homotopy types over $\mathbb{Z}$" we get here maps into the category of ordinary homotopy types, whereas it was the opposite with the two other approaches.

What, however, I still would like to do, is a little more daydreaming about the expected formal properties of the basic functors in between the two categories appearing in (1) and (2) – namely, essentially, between "schematic homotopy types" and "abelian" ones. It will be convenient to denote the category of the latter by

[p. 522]

$$\mathrm{Hotab}_0(k) \tag{3}$$

rather than $\mathrm{D}_\bullet(\mathrm{Ab})(0\text{-conn})$, the subscript 0 denoting the 0-connectedness condition. The reason for this change in notation is that, according to the choices made in the basic definitions, the "k-linear" algebraic interpretation of this category may vary still. Strictly speaking, in the approach we have been following in terms of semisimplicial unipotent bundles, the category (3) cannot really be described as the subcategory of 0-connected objects of $\mathrm{D}_\bullet(\mathrm{Ab}_k)$, we have seen, rather, that in order to define a functor $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$, we have to work with chain complexes in $\mathrm{Pro}(\mathrm{Ab}_k)$ rather than in (Ab_k); cf. section 115, where this point is made rather forcefully. (Recall that taking projective limits instead wouldn't help, because then the linearization functor $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ would no longer commute to ground ring extension!) The trouble then is that in order to define

$$\widetilde{K} : \mathrm{Hotab}_0(k) \to \mathrm{Hot}_0(k),$$

we are obliged to enlarge accordingly the category of models used for describing $\mathrm{Hot}_0(k)$, namely take semisimplicial *pro*-unipotent bundles, rather than just semisimplicial unipotent bundles. This sudden invasion of the picture by pro-objects may appear forbidding – but maybe it will appear less so, or at any rate kind of natural and inescapable, if we recall that when it will come to working with homotopy types of general topoi, these are anyhow "prohomotopy types" rather, and they can be described only by working systematically in terms of pro-objects of various categories. We'll see, however, that there may be a way out of the "pro"-mess, by using a slightly different, more or less dual approach towards the idea of "unipotent bundles". Of course, one may also think of using the "pointed linearization" L_{pt} for $\mathrm{LH}_\bullet$, instead of L, but as was pointed out in section 117, this will lead to "formal" homology and cohomology invariants rather than to "schematic" ones, and the relation between these and corresponding invariants for ordinary homotopy types will be looser still; at any rate, it is suited for describing Postnikov dévissage in the set-up of "formal schematic" homotopy types only, not for schematic homotopy types. Still, the theory of the former may have an interest in its own right, even though its relation to ordinary homotopy types isn't so clear at present. Thus, we may state that there are around at present three or four different candidates for possibly
[p. 523] fitting the daydreaming I want to write down. With this in mind, we shouldn't be too specific (for the time being) about the exact meaning to be given to the symbols $\mathrm{Hotab}_0(k)$ and $\mathrm{Hot}_0(k)$. It may be safer to replace these by $\mathrm{Hotab}_1(k)$ and $\mathrm{Hot}_1(k)$ (where the subscript 1 means "1-connected"), all the more so as we haven't been able yet, in the context of unipotent bundles, to give even a tentative precise definition of $\mathrm{Hot}_0(k)$, i.e., of $\mathcal{M}_0(k)$ (mimicking the condition of abelian π_1 with trivial action on the higher π_i's), except by expressing things via the associated ordinary homotopy type (passing over to $X_*(k)$), which looks kind of stupid indeed! But nothing is "safe" here anyhow, and the subscript 0 looks definitely more natural here than subscript 1, so we may as well keep it!

5.10.

What we are mainly interested in, for expressing the relationships between the non-additive category $\mathrm{Hot}_0(k)$ and its additive counterpart $\mathrm{Hotab}_0(k)$, is an array of four functors in between these, two of which being $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ ((total) "homology", or "linearization") and $\widetilde{K}$ (Eilenberg-Mac Lane functor) in (1) and (2), the two remaining ones being $\mathrm{L}\pi_\bullet$ ("total homotopy") and $\widetilde{S}$ (the "spherical functor", compare p. 514 (9)), fitting into the following diagram

(4)
$$\begin{tikzcd} \mathrm{Hotab}_0(k) \arrow[r, shift left, "\widetilde{K}"] \arrow[r, shift right, "\widetilde{S}"'] & \mathrm{Hot}_0(k) \arrow[l, bend right=40, "\mathrm{L}\widetilde{\mathrm{H}}_\bullet"'] \arrow[l, bend left=40, "\mathrm{L}\pi_\bullet"] \end{tikzcd} \quad .$$

I would now like to discuss the main formal properties to be expected from this set of functors.

A) Adjunction properties:

(5)
$$\begin{cases} \text{a)}\ \ (\mathrm{LH}_\bullet, \widetilde{K}) \text{ is a pair of adjoint functors,} \\ \text{b)}\ \ (\widetilde{S}, \mathrm{L}\pi_\bullet) \text{ is a pair of adjoint functors.} \end{cases}$$

Thus, we get adjunction homomorphisms (for $L_\bullet$ in $\mathrm{Hotab}_0(k)$ and X in $\mathrm{Hot}_0(k)$)

(6a)
$$\begin{cases} X \to \widetilde{K}(\mathrm{L}\widetilde{\mathrm{H}}_\bullet(X)) \qquad (\overset{\text{def}}{=} \widetilde{W}(X) \text{ ("pointed linearization"))} \\ \mathrm{L}\widetilde{\mathrm{H}}_\bullet(\widetilde{K}(L_\bullet)) \to L_\bullet \quad , \end{cases}$$

and [p. 524]

(6b)
$$\begin{cases} \widetilde{S}(\mathrm{L}\pi_\bullet(X)) \to X \\ L_\bullet \to \mathrm{L}\pi_\bullet(\widetilde{S}(L_\bullet)) \quad . \end{cases}$$

The adjunction formulas for the two pairs in (5) are

(7)
$$\begin{cases} \text{a)} \quad \mathrm{Hom}(\mathrm{L}\widetilde{\mathrm{H}}_\bullet(X), L_\bullet) \simeq \mathrm{Hom}(X, \widetilde{K}(L_\bullet)) \\ \text{b)} \quad \mathrm{Hom}(L_\bullet, \mathrm{L}\pi_\bullet(X)) \simeq \mathrm{Hom}(\widetilde{S}(L_\bullet), X) \quad . \end{cases}$$

The first-hand side of a) should be viewed as *cohomology* of X with coefficients in $L_\bullet$. More specifically, as $\widetilde{\mathrm{H}}^0(X, L_\bullet)$, and in case $L_\bullet = \pi[n]$, posing

(8a)
$$K(\pi[n]) \overset{\text{def}}{=} K(\pi, n),$$

we get that the Eilenberg-Mac Lane objects $K(\pi, n)$ represents the cohomology functor
$$X \mapsto \widetilde{\mathrm{H}}^0(X, \pi[n]) = \mathrm{H}^n(X, \pi).$$

Symmetrically, the first-hand side of (7b) should be viewed as a "mixed homotopy module" of X (relative to the "co-coefficient" $L_\bullet$), I am tempted to denote it as

$$\pi_0(X, L_\bullet), \quad \text{resp. } \pi_n(X, H) \text{ if } L_\bullet = H[n];$$

thus, posing

(8b) $\quad \widetilde{S}(H[n]) \stackrel{\text{def}}{=} S(H, n) \quad$ (sphere-like object for H, n),

we get that the H-sphere $S(H, n)$ over k represents the (covariant) functor on $\mathrm{Hot}_0(k)$

$$X \mapsto \pi_n(X, H).$$

In case $H = k$, we get

$$\pi_n(X, k) = \mathrm{Hom}(k[n], \mathrm{L}\pi_\bullet(X)) = \mathrm{H}_n(\mathrm{L}\pi_\bullet(X)) \stackrel{\text{def}}{=} \pi_n(X),$$

and we get that the "usual" homotopy module functor

$$X \mapsto \pi_n(X)$$

is represented by the "usual" n-sphere (over k however!) $S(n, k)$, as it should. (This is of course the main justification why we expect $(\widetilde{S}, \mathrm{L}\pi_\bullet)$ to be a pair of adjoint functors, whereas adjunction for the pair $(\mathrm{L}\widetilde{\mathrm{H}}_\bullet, \widetilde{K})$ is already fairly familiar from the set-up of ordinary homotopy types.)

The adjunction formulæ (7) show that the objects $K(L_\bullet)$ in $\mathrm{Hot}_0(k)$ are group objects, whereas the objects $\widetilde{S}(L_\bullet)$ are co-group objects. This gives
[p. 525] some inner justification for calling $\widetilde{W}(X) = \widetilde{K}(\mathrm{L}\widetilde{\mathrm{H}}_\bullet(X))$ the (pointed) "linearization" of the schematic homotopy type X, which maps into the former by the first adjunction morphism in (6a). Dually, we may call $\widetilde{S}(\mathrm{L}\pi_\bullet(X))$ the (pointed) "co-linearization" of X, it maps into X by the first adjunction morphism in (6b).

The source $\mathrm{L}\widetilde{\mathrm{H}}_\bullet(\widetilde{K}(L_\bullet))$ in the second adjunction map (6a) may be viewed as Eilenberg-Mac Lane type (total) homology, corresponding to the chain complex $L_\bullet = \pi[n]$ (but in the schematic sense of course!), whereas the target $\mathrm{L}\pi_\bullet(\widetilde{S}(L_\bullet))$ in the second adjunction map (6b) may be viewed as "total homotopy" of a sphere-type space, reducing in case $L_\bullet = \pi[n]$ to the total homotopy of the standard n-sphere over k, $S(n, \pi)$. In case $k = \mathbb{Z}$, we expect of course the homology groups of this chain complex to be the ordinary homotopy groups of the usual sphere S^n – cf. G) below.

B) Inversion formulæ: two functorial isomorphisms

$$(9) \qquad \begin{cases} \text{a)} \quad \mathrm{L}\pi_\bullet(\widetilde{K}(L_\bullet)) \xrightarrow{\sim} L_\bullet & , \\ \text{b)} \quad \mathrm{L}\widetilde{\mathrm{H}}_\bullet(\widetilde{S}(L_\bullet)) \xrightarrow{\sim} L_\bullet & . \end{cases}$$

Maybe we should have begun with these, as they kind of fix the meaning of the two functors $\widetilde{K}, \widetilde{S}$ in terms of $\mathrm{L}\widetilde{\mathrm{H}}_\bullet, \mathrm{L}\pi_\bullet$, which may be viewed as embodying respectively the two main sets of invariants of a homotopy type, namely homology and homotopy.

C) Hurewicz map:

$$\mathrm{L}\pi_\bullet(X) \to \mathrm{L}\widetilde{\mathrm{H}}_\bullet(X). \tag{10}$$

We get such a map by applying $\mathrm{L}\pi_\bullet$ to the linearization map in (6a), and using (9a); symmetrically, we may apply $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ to the colinearization map (6b), and use (9b). We get a priori two maps (10), and the statement is that these two maps are the same. Moreover, we have the all-important *Hurewicz theorem*: The first non-vanishing homology objects for $\mathrm{L}\pi_\bullet(X)$ and $\mathrm{L}\widetilde{\mathrm{H}}_\bullet(X)$ occur in the same dimension, n say, and (10) induces an isomorphism for H_n, an epimorphism for H_{n+1}.

D) Exactness properties: they can be stated shortly by saying that in each one of the two pairs of adjoint functors (5), the left adjoint one respects cofibration sequences, whereas the right adjoint respects fibration sequences. This may be detailed as four exactness statements, one for each one of the four basic functors.

Thus, $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ respects cofibration sequences, which means essentially that for such a sequence in $\mathrm{Hot}_0(k)$ [p. 526]

$$Y \to X \to Z, \quad Z \text{ the homotopy cofiber of } Y \to X,$$

we get a corresponding exact triangle in $\mathrm{Hotab}_0(k)$

$$\begin{tikzcd}[column sep=small] & \mathrm{L}\widetilde{\mathrm{H}}_\bullet(Z) \arrow[dl, "i"'] & \\ \mathrm{L}\widetilde{\mathrm{H}}_\bullet(Y) \arrow[rr] & & \mathrm{L}\widetilde{\mathrm{H}}_\bullet(X) \arrow[ul] \end{tikzcd}, \tag{11b}$$

which is the most complete and elegant way, I guess, for expressing behaviour of homology with respect to homotopy cofibers and suspensions. Dually, $\mathrm{L}\pi_\bullet$ respects fibration sequences, which means that for such a sequence in $\mathrm{Hot}_0(k)$

$$Z \to X \to Y, \quad Z \text{ the (connected) homotopy fiber of } X \to Y,$$

we get a corresponding exact triangle in $\mathrm{Hotab}_0(k)$

$$\begin{tikzcd}[column sep=small] & \mathrm{L}\pi_\bullet(Z) \arrow[dl, "i"'] & \\ \mathrm{L}\pi_\bullet(X) \arrow[rr] & & \mathrm{L}\pi_\bullet(Y) \arrow[ul] \end{tikzcd}, \tag{11a}$$

which is a more complete and elegant way of stating the exact homotopy sequence of a fibration, in terms of total homotopy.

Formula (11b) implies that $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ commutes with suspension functors Σ, the latter in $\mathrm{Hotab}_0(k)$ (visualized as a derived chain complex category) is just the shift functor

$$\Sigma_{\mathrm{ab}} : L_\bullet \mapsto L_\bullet[1] \quad (L[1]_n = L_{n-1},$$

which is a *fully faithful* functor. Dually, formula (11a) implies that $\mathrm{L}\pi_\bullet$ commutes to the loop-space functors Ω^0; the latter in $\mathrm{Hotab}_0(k)$ cannot be described as just a shift in opposite direction

$$L_\bullet \mapsto L_\bullet[-1],$$

as this will get us out of 0-connected chain complexes, we have to truncate, moreover, at dimension 1 (afterwards, or at dimension 2 beforehand). This functor is not fully faithful therefore – we loose something when passing from $L_\bullet$ to $\Omega^0_{\mathrm{ab}}(L_\bullet)$.

As for the two functors $\widetilde{K}, \widetilde{S}$ in the opposite direction, from $\mathrm{Hotab}_0(k)$ (i.e., essentially chain complexes) to $\mathrm{Hot}_0(k)$, that the first one respects fibration sequences is surely quite familiar a fact (and kind of tautological) in the set-up of ordinary homotopy types. That the second, less [p. 527] familiar functor $\widetilde{S}$ should respect cofibration sequences should be useful in order to give a more or less explicit construction of $\widetilde{S}(L_\bullet)$, for given $L_\bullet$, in terms of the "spheres over k" $S(H,n)$.

Remarks. 1) The superscript 0 for the loop functor Ω^0 or Ω^0_{ab} should not be confused with the subscript n in the iterated loop-space functor Ω_n, it is added here to suggest that we are taking the *neutral* connected component of the "true" full loop-space, this being imposed by the restriction of working throughout with 0-connected objects; likewise, the "homotopy fiber" operation in the present context should be viewed as meaning "connected component at the marked point of the full homotopy fiber". These necessary readjustments of the usual notions is being felt as an unwelcome feature (of which I have become aware only at this very moment, I confess, through the writing of the notes). Intuitively, the restriction to 0-connected (pointed) homotopy types doesn't seem to imperative, technically, however, when working with semisimplicial schematic models, we had felt like introducing the condition $X_0 = e$, which may be viewed as a strong form of a 0-connectedness condition. Truth to tell, it isn't so clear that this condition is going to be of great utility – anyhow, it isn't enough to ensure what we're really after at present (namely abelian π_1 and trivial action on the higher π_i's), a condition which is anyhow independent of any 0-connectedness type assumption, and is moreover (it would seem) stable under the basic fiber and cofiber operations. To sum up, it may well turn out that we better replace the categories of 0-connected homotopy types $\mathrm{Hotab}_0(k)$ and $\mathrm{Hot}_0(k)$, by slightly larger ones, so as to get rid of the 0-connectedness restriction. This, however, is at present a relatively minor point, and therefore we'll leave the notations as they are.

2) Behaviour of $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ with respect to *fibration* sequences (instead of cofibration sequences) is a relatively delicate matter, it is governed by the Leray spectral sequence, whose initial term is the homology of the base Y with "coefficients" in the homology of the fiber. I wonder if there is anything similar for the behavior of $\mathrm{L}\pi_\bullet$ with respect to cofibration sequences?

3) In the display of the main expected properties of the four basic functors, there is a striking *symmetry*, which we tried to stress by the

way of presented the main formulæ. One way to express this symmetry is to say that things look as if there was an auto-duality in the pair of categories $(\mathrm{Hotab}_0(k), \mathrm{Hot}_0(k))$, namely a pair of contravariant involutive functors (D_{ab}, D), each of which interchanges fibration and cofibration sequences, i.e., transforms fibers into cofibers and vice versa, and the pair of functors interchanging $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ and $\mathrm{L}\pi_\bullet$ on the one hand, $\widetilde{K}$ and $\widetilde{S}$ on the other. This heuristic formulation is compatible with all we have stated so far, except one thing – namely, the suspension functor Σ_{ab} in $\mathrm{Hotab}_0(k)$ is fully faithful, whereas the (supposedly "dual") loop-space functor Ω^0_{ab} is not. Thus, it will be more accurate to say that, if we view the formulaire developed so far as being the description of a certain structure type, whose basic ingredients are two categories $\mathcal{H}_{\mathrm{ab}}$ and $\mathcal{H}$, endowed with fiber and cofiber operations, tied by four functors as above satisfying a bunch of properties, then the axioms are autodual in an obvious sense; namely if they are satisfied for a system $(\mathcal{H}_{\mathrm{ab}}, \mathcal{H}, \mathrm{L}\widetilde{\mathrm{H}}_\bullet, \widetilde{K}, \widetilde{S}, \mathrm{L}\pi_\bullet)$, they are equally satisfied by the system $(\mathcal{H}^{\mathrm{op}}_{\mathrm{ab}}, \mathcal{H}^{\mathrm{op}}, \mathrm{L}\pi_\bullet, \widetilde{S}, \widetilde{K}, \mathrm{L}\widetilde{\mathrm{H}}_\bullet)$. (Of course, among the properties of the structure type, we are *not* going to include that the suspension functor in $\mathcal{H}_{\mathrm{ab}}$ is fully faithful!) As already stated before, the main reason for introducing the fourth functor $\widetilde{S}$ in the picture, was because it was felt that this was lacking in order to round it up. Thus for any kind of notion or statement in this set-up, suggested by some kind of geometric insight, it becomes automatic to look at its dual and see whether it makes sense. [p. 528]

As for formal autoduality, we already noticed there is none, even in $\mathrm{Hotab}_0(k)$ just by itself. Thinking of this category as $\mathrm{D}_\bullet(\mathrm{Ab}_k)$, namely as a full subcategory of the derived category $\mathrm{D}(\mathrm{Ab}_k)$, we cannot help, though, but thinking of the standard "dualizing" functor

$$L_\bullet \mapsto \mathrm{RHom}(L_\bullet, k) : \mathrm{D}^-(\mathrm{Ab}_k) \to \mathrm{D}^+(\mathrm{Ab}_k),$$

inducing a perfect duality within the category of "perfect" objects in $\mathrm{D}(\mathrm{Ab}_k)$, namely objects which are isomorphic to those which may be described by complexes in Ab_k having a bounded span of degrees, and all of whose components are projective of finite type. But this autoduality of course transforms *chain* complexes into *cochain* complexes – we have to shift these in order to get chain complexes again. This suggests that maybe we'll hit upon an actual autoduality, provided we go over from $\mathrm{Hot}_0(k)$ to a suitable "stabilized" category, the suspension category say (deduced from the initial one by introducing formally a quasi-inverse for the suspension functor). The homology functor $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ extends to a functor from the suspension category to the corresponding category for $\mathrm{Hotab}_0(k)$, say $\mathrm{D}^-(\mathrm{Ab}_k)$. The $\mathrm{L}\pi_\bullet$ functor, though, is lost on our way – [p. 529]
I am afraid I am confusing the kind of duality I am after here, with a rather different type of duality, kin to Poincaré duality, and discovered I believe by J. H. C. Whitehead, in the context of spaces having the homotopy type of a finite complex. For such a complex, the idea (as far as I remember) is to embed X into a large-dimensional sphere, and to take the complement $X^\vee$ of an open tubular neighborhood of X. Up to suspension, the homotopy type of $X^\vee$ does not depend on the choices made and (shifting back by n) we thus get a canonical object in the

suspension category, depending contravariantly on X. The functor $X \mapsto X^\vee$ (if I got it right) is an autoduality of the relevant full subcategory of the suspension category, and the (reduced) homology functor $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ maps this subcategory into perfect complexes, and commutes to autodualities. The whole story seems tailored towards a study of duality relations for the functor $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ exclusively – without any reference to the homotopy invariants π_i. I don't know if one can devise a similar story for $\mathrm{L}\pi_\bullet$, by stabilizing with respect to the loop space functor (or is this just nonsense?). At any rate, there doesn't seem any autoduality in view, exchanging homology and homotopy invariants...

E) Conservativity properties: The functors $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ and $\mathrm{L}\pi_\bullet$ are both conservative, i.e., a map in $\mathrm{Hot}_0(k)$ which by either of these functors becomes an isomorphism, is an isomorphism. (Here of course the 0-connectedness assumption for the homotopy types we are working with is essential, as far as the functor $\mathrm{L}\pi_\bullet$ is concerned at any rate.)

F) Base change properties: All four functors, and the adjunction and inversion maps (7) and (9), are compatible with ring extension $k \to k'$, it being understood that such a ring homomorphism defines functors

$$\mathrm{Hotab}_0(k) \to \mathrm{Hotab}_0(k'), \quad \mathrm{Hot}_0(k) \to \mathrm{Hot}_0(k') \tag{12}$$

compatible with the fibration and cofibration structures.

In opposite direction, there should be, too, a "restriction" functor (less important, though, I feel), I didn't try to find out what its formal properties should be with respect to the formalism developed here.

G) Comparison with ordinary homotopy types: We got functors

$$\mathrm{Hotab}_0(k) \to \mathrm{Hotab} = \mathrm{D}_\bullet(\mathrm{Ab}), \quad \mathrm{Hot}_0(k) \to \mathrm{Hot}_0, \tag{13}$$

(the first being interpreted as "forget k" functor, the second as a sections
[p. 530] functor on semisimplicial schematic models). This pair of functors is compatible with the functors $\widetilde{K}$, but (even if $k = \mathbb{Z}$) definitely *not* with their left adjoints $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$, namely with homology. Despite this fact, we hope if $k = \mathbb{Z}$ that the functors (13) are compatible with the functors $\widetilde{S}$ (assuming that $\widetilde{S}$ can actually be constructed in the discrete set-up too), so that spheres are transformed into spheres. Compatibility with $\mathrm{L}\pi_\bullet$ doesn't have a meaning strictly speaking, as this functor is not defined on ordinary homotopy types, however, the functors π_i are defined in both contexts, and the sections functor should commute to these (for arbitrary k). Thus, the only serious incompatibility trouble concerns the homology functor $\mathrm{LH}_\bullet$ and should not arise, however, for sphere-like objects $S(H,n)$. I forgot to state from the outset that the two functors (13) are expected to respect fibration and cofibration sequence structures, of course.

13.10.

Digression on Baues' cofibration and fibration categories, and on weak equivalences alone as the basic data for a model category.

125 During the last eight days, I have been busy with a number of things, which left little leisure for mathematical ponderings. I got a letter from H. J. Baues, who had seen my notes, which induced him to send me his preprint "On the homotopy classification problem" (Chapters I to V + chapter Ext). I spent two evenings looking through parts of these, where Baues carries as far as possible (namely quite far indeed) the homotopy formalism in the context of his so-called "cofibration (or fibration) categories", using as his leading thread his ideas on "obstruction theory". He makes the point that he tried by his basic notion to pinpoint the weakest axiomatic set-up, sufficient however for developing all the major familiar (and even some not at all familiar!) notions, operations and statements of usual homotopy theory. In his letter, he suggested that maybe in any "universe" where homotopy constructions make sense, one or the other of his two mutually dual set-ups should be around. Such suggestion was of course quite interesting for my present reflections, as I do have the hope indeed that there exists a "universe" of schematic homotopy types, which may be described in terms of the models (namely semisimplicial unipotent bundles) I have been using so far, or at any rate by closely related kinds of models. More specifically, I do hope for the two kinds of operations to make sense, namely "integration" (including homotopy cofibers and suspensions) and "cointegration" (including homotopy fibers and loop objects), which in Baues' set-up should correspond on the category of models to a structure of a cofibration category and a fibration category, respectively. As the kind of models I have been working with don't allow for an amalgamated sum (or "push-out") construction, except in the trivial case when one of the two arrows to be amalgamated is an isomorphism, it is clear that we cannot hope to get with these models a cofibration category. There may be, however, a fibration category structure, and I started playing around a little with a possible notion of Kan fibration – without coming to a definite conclusion yet, however. I feel I shouldn't dwell much longer still, for the time being, on getting off the ground a theory of schematic homotopy types, maybe I'll come back upon it later, after next chapter, when (hopefully) I'll be a little more "in the game" of Kan type conditions, closed model category structures and Baues' "halved" variants of these.

[I'm not sure what this refers to; perhaps Baues (1985)...]

[p. 531]

At any rate, whether or not a homotopy cofiber construction can be carried through for schematic homotopy types, it seems rather clear to me that Baues' suggestion or expectation, about the set-up of cofibration and fibration categories he ended up with, is not quite justified. Already by the time (in the late sixties) when I first heard from Quillen about his approach (and the same applied to Baues' ones), when applied say to semisimplicial objects in some category A as "models", is pretty strongly relying on the existence of "enough" projectives (or dually, of enough injectives, if working with cosemisimplicial objects) in A. When A is a topos say, then the category $\mathbf{Hom}(\Delta^{\mathrm{op}}, A)$ of semisimplicial objects of A doesn't carry, it seems (except in very special cases, say of a totally disconnected topos), a reasonable structure of a Quillen model category, nor even (I would think) of a cofibration or a fibration category in the sense of Baues; however, I am pretty sure that the derived category (with respect to the notion of weak equivalence introduced by Illusie) has important geometric meaning and is indeed a "universe for homotopy types" – and the first steps in developing such theory have been taken by Illusie already in Chapter I of his thesis. Quite similarly, if A is an abelian category, Verdier's derived category $\mathrm{D}(A)$, deduced from the category $\mathrm{C}(A)$ of complexes in A by localizing with respect to the set W of quasi-isomorphisms, does allow for a homotopy formalism (as does any "triangulated category" in the sense of Verdier), however, it doesn't seem that this formalism may be deduced from a structure of a cofibration category or a fibration category on $\mathrm{C}(A)$. The same I guess holds for the subcategories $\mathrm{C}^-(A)$ and $\mathrm{C}^+(A)$, giving rise to $\mathrm{D}^-(A)$ and $\mathrm{D}^+(A)$, which are more important still than $\mathrm{D}(A)$ in the everyday cohomology formalism of "spaces" of all kinds (namely, essentially, of ringed topoi). When A admits enough projectives or enough injectives, respectively, these categories (as pointed out by Quillen) are associated to closed model structures on $\mathrm{C}^-(A)$ and $\mathrm{C}^+(A)$, respectively, but (it seems) not otherwise.

[AG wanted to insert a reference, but didn't, probably to Illusie (1971, 1972)...]

[p. 532]

It strikes me that nobody apparently so far has tried to develop homotopy theory, starting as basic data with a category of models M together with a set W of weak equivalences in M, satisfying suitable assumptions, without giving moreover a notion of "fibration" or of "cofibrations" in M. Various examples, including the all-important case (from my point of view) of (Cat), suggest that the choice of the notion of fibration or cofibration isn't really so imperative, that it is to a certain extent arbitrary, different choices (compatible with the basic data W) leading to essentially "the same" homotopy theory. This is due to the fact that they lead to the same notions of "integration" and "cointegration" of homotopy types, which depends indeed only on W and which, in my eyes, are *the* two main operations of homotopy theory (compare section 69). They seem to me the key for defining such a thing as a specific "homotopy theory", independently of any particular choice of a category of models (+ extra structure on it, and notably, weak equivalences) used for describing it. The precise technical notion achieving this is of course the notion of a "derivator" – and I do hope that it shouldn't be too awfully hard to construct, for instance, the "derivator of schematic homotopy types", and maybe even characterize it axiomatically (as well as the derivator of usual homotopy types), without having to make any mention of models for such characterization. One may say that, after the one major step in the foundations of homological algebra, consisting in introducing the derived category of an abelian category (and systematically working with derived categories for stating the main facts about cohomology of all kind of "spaces", namely topoi, such as usual topological spaces, schemes and the like...), the work on foundations more or less stopped short, while the next step to take was to come to a grasp of the full structure involved in derived categories, namely the structure of a derivator. And it turns out that as well the step of taking derived categories (namely localizations of suitable model categories), as the next, namely introducing the corresponding structure of a derivator, make good sense also in non-abelian set-ups, namely for doing "homotopy theory", thus viewed as a non-commutative version of homology theory. [p. 533]

Dissymmetry of $\mathrm{L}\widetilde{\mathrm{H}}_*$ versus $\mathrm{L}\pi_*$ (no filtration dual to Postnikov dévissage).

126 Let's come back, though, to schematic homotopy types. Last thing we looked at was the four basic functors between "schematic homotopy types" and "linear schematic homotopy types" over a given ground ring k, making up the two basic categories

$$\mathrm{Hotab}_0(k) \quad \text{and} \quad \mathrm{Hot}_0(k).$$

We have been dwelling somewhat on the remarkable formal symmetry to be expected for these relations. It is tempting, then, to try and dualize any kind of basic notion or construction which makes sense in terms of the basic data, namely the four functors and the adjunction and inversion maps relating them. Maybe the very first thing which forces attention is the Postnikov dévissage of an object of $\mathrm{Hot}_0(k)$, which had been (together with abelianization) the very starting point for our approach towards schematic homotopy types. One main ingredient of this dévissage is the "tower" of the Cartan-Serre type functors

$$X \mapsto X_n \tag{1}$$

and maps

$$X \to X_n \tag{2}$$

(with n a natural integer), where X_n is deduced from X by "killing its π_i's for $i > n$". We may call an object Y of $\mathrm{Hot}_0(k)$ *n-co-connected* (a notion in a way symmetric to n-connectedness) if $\pi_i(X) = 0$ for $i > n$, and denote by

$$\mathrm{Hot}_0(k)_{/n} \tag{3}$$

the full subcategory of $\mathrm{Hot}_0(k)$ made up with these objects, which is therefore the inverse image by the functor

$$\mathrm{L}\pi_i : \mathrm{Hot}_0(k) \to \mathrm{Hotab}_0(k)$$

of the corresponding full subcategory

$$\mathrm{Hotab}_0(k)_{\leq n} \tag{4}$$

of $\mathrm{Hotab}_0(k)$. The most natural way for defining X_n in terms of X (in an axiomatic set-up with the four functors as basic data), together with the canonical "fibration" (2), is by describing (2) as the "universal" map (in $\mathrm{Hot}_0(k)$) of X into an n-co-connected object. In other words, we are surmising that the inclusion functor

$$\mathrm{Hot}_0(k)_{\leq n} \hookrightarrow \mathrm{Hot}_0(k)$$

[p. 534] admits a left adjoint, and the latter is denoted by $X \mapsto X_n$. This description gives rise at once to the familiar "tower" structure for variable n

(5) $$X_{n+1} \to X_n \to \cdots \to X_1 \to X_0 (= e).$$

Intuitively, the maps in (5) are viewed as being (surjective) fibrations between (connected) "spaces", X being viewed as a kind of inverse limit of the X_n's. Dually, we would expect to get a sequence of inclusions

(*) $${}_0X \hookrightarrow {}_1X \hookrightarrow \cdots \hookrightarrow {}_nX \hookrightarrow {}_{n+1}X \hookrightarrow \cdots,$$

with X appearing as a kind of direct limit. In the set-up of ordinary homotopy types, modelized by semisimplicial sets, one will think at once of the filtration by skeleta – which however isn't quite the right thing surely, because if we dualize the familiar Cartan-Serre requirement on (2) (namely that it induces an isomorphism for π_i for $i \le n$), we see that the "inclusion"

$${}_nX \hookrightarrow X$$

should induce an isomorphism for H_i for $i \le n$, which isn't quite true for the skeletal filtration (it is OK for $i \le n-1$ only); the condition

$$\mathrm{H}_i({}_nX) = 0 \quad \text{for } i > n$$

is OK, though, for this filtration. So the next idea would be to modify a little the n-skeleton ${}_nX$, to straighten things out. I played around some along these lines, and after some initial optimism, came to the feeling that there does not exist (in the discrete nor in the schematic set-up) such an increasing canonical filtration of a homotopy type. I didn't make any formal statement and proof for this (in the set-up of ordinary homotopy types, say), however, in the process of playing around in became soon clear that in various ways, *there is some essential dissymmetry* between the seemingly "dual" situations, when trying to get the two types of "filtrations" of the object X. One dissymmetry occurs already in the very definition of the subcategory (3), and of the corresponding "dual" subcategory

(6) $${}_{(\le n)}\mathrm{Hot}_0(k)$$

of objects satisfying

$$\mathrm{H}_i(Y) = 0 \quad \text{for } i > n.$$

Namely, both subcategories (3) and (6) are defined in terms of the
[p. 535] *same* subcategory of $\mathrm{Hotab}_0(k)$, namely (4), as the inverse image of the latter by either $\mathrm{L}\pi_\bullet$, or $\mathrm{LH}_\bullet$. Now, the point is that the properties of this subcategory, with respect to the inner structure of $\mathrm{Hotab}_0(k)$ (involving the "left" and the "right" homotopy operations, notably the suspension and the loop functors, respectively) are by no means autosymmetric. The main dissymmetry, it would seem, turns out in this, that (4) is stable under the loop functor, and by no means under the suspension

functor. This is the reason why, even if we assume that (in analogy to what happens for the categories (3)) the inclusion functors from the categories (5) into $\mathrm{Hot}_0(k)$ do admit the relevant (namely right) adjoints (which I greatly doubt anyhow. . .), and thus give rise for any object X to an increasing filtration (*), the Postnikov-type relationships between X_n and X_{n+1} cannot be quite dualized to a similar relationship between ${}_nX$ and ${}_{n+1}X$. We may think next, of course, of defining an increasing sequence of subcategories (5) of $\mathrm{Hot}_0(k)$ in terms of $\mathrm{LH}_\bullet$ and a corresponding sequence of subcategories of $\mathrm{Hotab}_0(k)$, different from the categories (4), and stable under suspensions. But there doesn't seem to be anything reasonable around along these lines.

To sum up, it doesn't seem one should overemphasize the somewhat startling symmetry which appeared in section 124 between "homotopy" embodied in $\mathrm{L}\pi_\bullet$, and "homology" embodied in $\mathrm{LH}_\bullet$ – in some essential respects, it would seem that the corresponding two functors do have non-mutually symmetry properties. I guess I have to apologize for having taken that long for coming to a conclusion which, presumably, must be felt as a kind of self-evidence by all homotopy people!

Schematic homotopy types and Illusie's derived category for a topos.

127 Before leaving (for the time being) the topic of schematic homotopy types and schematization, I would like still to add a few comments, about various possibilities for working with different kinds of models for defining schematic homotopy types. My point here is not in replacing the basic test-category we are working with, here $\triangle$, by some other (say the category © of standard hemispheres) – this choice I feel should be more or less irrelevant, so we may as well keep $\triangle$. Thus, we are going to work with semisimplicial objects, and the main question then is (for a given ground ring k) to say precisely what kind of objects we are allowing (or imposing!) as components for our complexes. At any rate, they should be "objects over k", and the most encompassing choice for such objects seems to be sheaves on the category of all schemes over k (or equivalently, of all affine schemes over k), for a suitable topology such as the fpqc topology (compare section 111, pages 446–447). Apart from the technico-logical nuisance of this not being a $\mathcal{U}$-site, where $\mathcal{U}$ is our basic universe (cf. p. 492), which we'll ignore here (as it isn't really too serious a difficulty), when working the semisimplicial sheaves, the embryo of foundational work of Illusie's on the derived category of a topos via semisimplicial sheaves becomes available. Thus, for a map of semisimplicial sheaves over k, [p. 536]

[Illusie (1971), chapter 1]

(1) $$X_* \to Y_*,$$

we know already what it means to be a quasi-isomorphism (= weak equivalence). It seems likely that, even when drastically restricting the sheaves allowed as components for our semisimplicial models, the notion of quasi-isomorphism relevant for these models should be the same as Illusie's for the encompassing topos. This is one point we have been neglecting so far. I am not going here to recall Illusie's definition (in terms of the homotopy sheaves $\boldsymbol{\pi}_i(X_*, s)$ where s is a section of X_*),

[in the typescript, the "π" is underlined. . .]

or translate it into cohomological terms. One point I want to make here, is that it is by no means automatic that if (1) is a weak equivalence, the same holds for the corresponding map of semisimplicial sets

(2) $$X_*(k) \to Y_*(k).$$

Thus, there is no more-or-less tautological "sections" functor from Illusie's derived category to the category (Hot) of usual homotopy types. (There surely *is* a canonical functor, though, in the case of a general topos T and Illusie's corresponding derived category, namely what we would like to call the *"cointegration over T" functor*, which should come out of the formalism of stacks we haven't begun to develop yet.) Presumably, however, when working with semisimplicial models whose components are restricted to be unipotent or something pretty close to these (see below for examples), whenever (1) is a weak equivalence, the same will hold for (2). As the choice of objects (let's call them simply the "bundles") we are allowing should clearly be stable under ring extension, it will then follow that more generally, for any algebra k' over k, the corresponding map

(2') $$X_*(k') \to Y_*(k')$$

[p. 537] is a weak equivalence, too. Thus, a "schematic homotopy type over k" should define a functor $\mathrm{Alg}_{/k} \to \mathrm{Hot}$.

For any choice of a "section"

(3) $$s \in X_0(k)$$

of X_*, Illusie's constructions yield sheaves

(4) $$\pi_i(X_*, s) \quad (i \geq 0),$$

where π_0 is a sheaf of sets, π_1 a sheaf of groups, acting on the higher π_i's which are abelian sheaves. We will be mainly interested of course in the case when π_0 is the final sheaf (we'll say that X_* is *relatively 0-connected over k*), and moreover π_1 is abelian and its action on the higher π_i is trivial (let's say in this case that the relative homotopy type defined by the semisimplicial sheaf X_* is *pseudo-abelian*). In this case, up to canonical isomorphisms, the abelian sheaves $\pi_i(X_*, s)$ do not depend on the choice of s, and (provided the pseudoabelian notion is defined locally) they make sense, independently even of the existence of a section s. Our hope is, by suitably restricting our notion of "bundle", to get on the abelian sheaves π_i for $i \geq 2$, or even on all π_i $(i \geq 1)$, a natural structure of an $\mathcal{O}_k$-module, and one moreover which in many "good" cases turns it into a quasi-coherent $\mathcal{O}_k$-module. Also, the natural maps

(5) $$\pi_i(X_*(k)) \to \pi_i(X_*)(k)$$

should be isomorphisms, i.e., taking π_i and π_i should commute to taking sections (which will imply indeed that if (1) is a quasi-isomorphism, then so is (2)). Thus, a first basic question here is to find a suitable notion of a "bundle" over k, in such a way that for semisimplicial bundles satisfying some mild extra assumption (such as $X_0 = X_1 = e$) implying that X_* is pseudoabelian, the maps (5) should be isomorphisms. I expect this to be true for the notion we have been working with so far, namely for "unipotent bundles" (cf. section 111), but I haven't made any attempt yet to prove this. But even granting this property for a given notion of "bundles", the module structure on the π_i's at this point remains still a mystery, as long as we don't tie them in with the Lie functor (compare section 118)...

Remark. Maybe, in the set-up of a general topos T and taking sections on the latter, the maps (5) are isomorphisms, whenever the homotopy sheaves π_i satisfy the relations [p. 538]

(6) $$\mathrm{H}^i(T, \pi_j(X_*)) = 0 \quad \text{for } i > 0,$$

as a consequence, maybe, of some spectral sequence whose abutment is the graded homotopy of the cointegration of X_* over T. If so, then (5) are quasi-isomorphisms whenever the sheaves π_i can be endowed with the structure of a quasi-coherent $\mathcal{O}_k$-module.

128 Next requirement about the notion of "bundles" we are going to work with, is about existence of a "linearization functor"; associating to every bundle X another bundle, namely its "linearization" $L(X)$, endowed moreover with the structure of an $\mathcal{O}_k$-module. *We insist that for a given bundle X, $L(X)$ should be at any rate a bundle too*. More specifically, if $U(k)$ denotes the category of all bundles over k, we should also introduce a corresponding "k-linear" category, or more accurately an $\mathcal{O}_k$-linear category, $U_{\mathrm{ab}}(k)$, whose objects should be objects of $U(k)$ endowed with the extra structure of an $\mathcal{O}_k$-module, possibly subject to some restrictions – thus, we'll get a forgetful functor

Looking for the right notion of "bundles"; V-bundles versus W-bundles.

(7) $$K : U_{\mathrm{ab}}(k) \to U(k).$$

When we take $U(k)$ to be unipotent bundles in the sense of section 111, the evident choice for $U_{\mathrm{ab}}(k)$ is to take the category of quasicoherent $\mathcal{O}_k$-modules, equivalent to the category of k-modules, Ab_k. Dually, we may take $U(k)$ to be the category of sheaves over k isomorphic to the underlying sheaf of sets of a vector bundle $V(M)$ over k associated to a k-module M, by the requirement

(8) $$V(M)(k') = \mathrm{Hom}_{(k\text{-Mod})}(M, k').$$

The evident corresponding choice for $U_{\mathrm{ab}}(k)$ is to take the category of all vector bundles over k, which is equivalent to the category $\mathrm{Ab}_k^{\mathrm{op}}$ *opposite* to the category of k-modules, as the functor $M \mapsto V(M)$ is contravariant in M. Maybe we should distinguish between these two choices of bundles by different notations, namely

(9) $$U_W(k) \text{ and } U_V(k),$$

where the subscripts W and V are meant to suggest the standard descriptions of objects via (the underlying sheaves of sets of the $\mathcal{O}_k$-modules)

$$W(M) \text{ and } V(M)$$

respectively, where (we recall)

$$W(M)(k') = M \otimes_k k'. \tag{10}$$

[p. 539] Reverting to a general notion of "bundles" $U(k)$, we define now a linearization functor

$$L : U(k) \to U_{\mathrm{ab}}(k) \tag{11}$$

to be a functor left adjoint to the forgetful functor (7), i.e., giving rise to an adjunction isomorphism

$$\mathrm{Hom}_{U(k)}(X, K(L)) \simeq \mathrm{Hom}_{U_{\mathrm{ab}}(k)}(L(X), L), \tag{12}$$

where X is in $U(k)$, L in $U_{\mathrm{ab}}(k)$. Passing over to semisimplicial objects and the corresponding derived categories, the functors K and L in (7) and (11) should give rise to the functors $\widetilde{K}$ and $\mathrm{LH}_\bullet$ of section 124.

When we take $U(k) = U_W(k)$, there is a drawback, though, because (as we saw in section 115) the functor K *does not* admit a left adjoint, only a *proadjoint*, associating to an object X a *proobject* $L(X)$ of $U_{\mathrm{ab}}(k)$. As pointed out in section 124 (p. 522), if we want a nice pair (K, L) of adjoint functors, this forces us to work with pro-unipotent bundles instead of just unipotent ones, thus getting out of the haven of sheaves over k and into the somewhat dubious sea of prosheaves and semisimplicial prosheaves, which have not been provided for in Illusie's foundational ponderings! If we do stick to the W-approach, this promises us a fair amount of extra sweat, putting in proobjects everywhere, not too enticing a prospect, is it?

It would seem that we are better off with the V-approach, in which case the bundles (more accurately, V-bundles) we are working with are actual schemes, indeed affine schemes over k, as we get

$$V(M) \simeq \mathrm{Spec}(\mathrm{Sym}_k(M)). \tag{13}$$

Now, let more generally X be any scheme, and let's look at maps from X into any vector bundle $V(M)$, we get

$$\begin{aligned} \mathrm{Hom}(X, V(M)) &\overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{O}_X}(p^*(\widetilde{M}), \mathcal{O}_X) \\ &\overset{\text{adjunction}}{\simeq} \mathrm{Hom}_{\mathcal{O}_S}(\widetilde{M}, p_*(\mathcal{O}_X)), \end{aligned} \tag{*}$$

where

$$p : X \to S = \mathrm{Spec}\, k$$

is the structural map of X, and $\widetilde{M}$ the restriction of $W(M)$ to the usual Zariski site of $S = \mathrm{Spec}(k)$. It is well-known that under a rather mild restriction on X (namely X quasi-compact and quasi-separated), always satisfied when X is affine, p_* takes quasi-coherent sheaves on X (for the usual small Zariski site) into quasi-coherent sheaves on $S = \operatorname{Spec} k$; thus, if A is the k-module (a k-algebra as a matter of fact) such that [p. 540]

$$\widetilde{A} \simeq p_*(\mathcal{O}_X), \quad \text{i.e., } A = \Gamma(S, p_*(\mathcal{O}_X)) \simeq \Gamma(X, \mathcal{O}_X),$$

the last member of (*) is

$$\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{A}) \simeq \operatorname{Hom}_k(M, A) \simeq \operatorname{Hom}_{\mathcal{O}_k}(V(A), V(M)),$$

and finally we get

$$\operatorname{Hom}(X, V(M)) \simeq \operatorname{Hom}_{\mathcal{O}_k}(V(A), V(M)). \tag{14}$$

This shows that the forgetful functor from the category $U_{\mathrm{ab}V}$ of all vector bundles over k to the category of all k-schemes which are quasi-compact and quasi-separated, admits a left adjoint L_V, where

$$L_V(X) = V(A), \quad \text{where } A = \Gamma(S, p_{X*}(\mathcal{O}_X)) \simeq \Gamma(X, \mathcal{O}_X). \tag{15}$$

In case X is affine, A is just the affine ring of X, which is a k-algebra, from which we retain only (in formula (15)) the k-module structure.

Thus, as far as the notion of k-linearization goes, V-bundles behave in a considerably nicer way than W-bundles, without any need to go over to proobjects. Thus, it may be preferable to work with V-bundles rather than W-bundles. We may wonder at this point why not admit, then, as "bundles" any k-scheme X which is quasi-compact and quasi-separated, or at any rate any affine k-scheme, as these can be quite conveniently described in terms of k-algebras. Thus, semisimplicial affine schemes over k just correspond to *co*-semisimplicial commutative algebras over k, and likewise for maps – and linearization just corresponds to forgetting the algebra structure in this co-semisimplicial object, and retain the structure of a co-semisimplicial k-module, which corresponds dually to a semisimplicial vector bundle. This is a perfectly simple relationship – why bother about restricting the "bundles" from arbitrary affine k-schemes to those which are isomorphic to a vector bundle?

We should remember here our initial motivation, which was, in case when $k = \mathbb{Z}$, to get a category of "schematic" homotopy types as close as possible to the usual one. One plausible way of achieving this is by restricting the notion of a bundle the more we can, so as to get still the possibility of "schematization" for a very sizable bunch of ordinary homotopy types. Postnikov dévissage then suggested to work with so-called "unipotent bundles", and it was almost a matter of chance 50/50 that we took first the choice of using W-bundles, rather than [p. 541] V-bundles which are the dual choice, giving rise in some respects to a simpler algebraic formalism (and to a less satisfactory one in some others...). In both cases, an instinct of "economy" is leading us. It isn't always clear that instinct isn't misleading at times – after all, it would be nice too to have a so-called "schematic" homotopy type (and hence a usual one) associated to rather general types of semisimplicial schemes, say. But here already, if we want to get a usual homotopy type just by taking sections, we've seen that this isn't so automatic, that this is tied up with the expectation that the maps (5) should be isomorphisms, which presumably will not be true unless we make rather drastic extra assumptions on the "bundles" we are working with.

Thus, a first main test whether the choice of W-bundles or of V-bundles is a workable one, is to see whether this condition on (5) is satisfied, possibly under a suitable extra assumption, such as $X_0 = e$ or $X_0 = X_1 = e$. The next test, presumably a deeper one, is whether these choices allow for a description of the homotopy sheaves $\pi_i(X_*)$ in terms of the Lie functor, under the natural flatness assumption on the components of X_* (compare section 118). The only clue so far for such a relationship comes from Postnikov dévissage, and this relationship isn't proved either, except in the more or less tautological "Postnikov case", and without naturality. In a way, it appears as a rather strange kind of relationship, one which implies that the homotopy type of a semisimplicial bundle is very strongly dominated (almost determined, one might say) by its formal completion along a section – and even by the first-order infinitesimal neighborhood already. Instinct again tells us that reducing, as it were, a scheme to a tangent space at one of its points might make sens when the scheme is isomorphic to affine n-space, and that it is surely nonsense for general affine schemes (such as an elliptic curve minus a point say!). To say it differently, we feel that such a thing may be reasonable only when the given schemes X_n may be thought of as "homotopically trivial" in some sense or other, which for an algebraic curve, say, over an algebraically closed field k is surely *not* the case, unless precisely the curve is isomorphic to the affine line.

[p. 542] 14.10.

129 *Quasi-coherent homological quasi-isomorphisms, versus weak equivalences.*

I have not been clear enough in yesterday's notes, when introducing the linearization functor (11)

$$L : U(k) \to U_{\mathrm{ab}}(k)$$

for a suitable notion of "bundles" and "linear bundles", that it is by no means automatic that such a functor (even when its existence is granted) will induce a linearization functor

$$\mathrm{LH}_\bullet : \mathrm{Hot}_0(k) \to \mathrm{Hotab}_0(k) \tag{16}$$

for the corresponding derived categories. In other words, it is by no means clear (and we haven't even tried yet to prove in the U_W set-up of "unipotent bundles") that if

$$X_* \to Y_* \tag{17}$$

is a weak equivalence of semisimplicial objects of $U(k)$, that the corresponding map

$$L(X_*) \to L(Y_*) \tag{17'}$$

of semisimplicial objects in $U_{\mathrm{ab}}(k)$ will again be a weak equivalence. At any rate, for this statement even to make sense, we'll have to make clear what notion of weak equivalence we are taking for maps between semisimplicial objects of $U_{\mathrm{ab}}(k)$, so as to get a localized category $\mathrm{Hotab}_0(k)$. The first obvious choice, of course, that comes to mind is to take the notion of weak equivalence for the corresponding semisimplicial sheaves of sets, which presumably is going to be the right one. Thus, the existence of a "total homology" or "linearization" functor (16) may be viewed as the "schematic" analog of W. H. C. Whitehead's theorem in the discrete set-up, which we had been puzzling about already in section 92 (in connection with replacing the test category $\triangle$ by a more or less arbitrary small category). In the case of semisimplicial W-bundles or V-bundles, under some extra assumption, presumably, such as $X_0 = e$ or $X_0 = X_1 = e$ on the semisimplicial objects we are working with, I do expect that the linearization functor transforms weak equivalences into weak equivalences, and that the converse holds true, too. This seems to me to be the third main test (besides the two considered on previous page 541) about a notion of "bundle" being suited for developing a theory of schematic homotopy types.

This suggests that when working with more general semisimplicial sheaves over k, such as semisimplicial schemes, say, it may be useful to [p. 543] introduce a notion of "quasi-coherent homological quasi-isomorphism" between such objects, as a map (17) such that the corresponding map (17') should be a quasi-isomorphism, which presumably may be interpreted also in cohomological terms, as usual. As just noticed, it is doubtful that this is implied by (17) being a weak equivalence, and even if it should be implied, it looks considerably weaker in a way – just as in the set-up of usual homotopy types, a homology equivalence is a considerably weaker notion than homotopy equivalence, unless we make a 1-connectedness assumption. We could reinforce, of course, this notion of (quasi-coherent) "homological" or "cohomological" quasi-isomorphism by taking to the cohomological version of it, and instead of quasi-coherent coefficients *coming from the base* $S = \mathrm{Spec}(k)$, admit equally the analog of "*twisted coefficients*", which would amount here to taking as coefficients quasi-coherent sheaves F_* on Y_*, such that for any structural morphism

$$\varphi : Y_m \to Y_n$$

associated to a map in $\triangle$, the corresponding map of quasi-coherent sheaves on Y_m

$$\varphi^*(F_n) \to F_m$$

should be an isomorphism. (These "twisted coefficients" should correspond to quasi-coherent sheaves on $S = \mathrm{Spec}(k)$, *on which the sheaf of groups* $\pi_1(Y_*)$ *operates*, in case at least when Y_* is relatively 0-connected and endowed with a section, so that $\pi_1(Y_*)$ makes sense.) My point here is that it may be interesting to take the derived category of a suitable category of semisimplicial schemes (submitted to some very mild conditions, such as quasi-compactness and quasi-separation, say) with respect to this notion of q.c.h. quasi-isomorphism – with the hope that the set of

maps with respect to which we are now localizing is wide enough, so as to get *the same* derived category as when working only with "bundles", namely getting just schematic homotopy types. This would give a very strong link between more or less arbitrary semisimplicial schemes over k (and stronger still when $k = \mathbb{Z}$), and ordinary homotopy types, of a much subtler nature than the known link with ordinary pro-homotopy types via étale cohomology with discrete coefficients. But maybe the daydreaming is getting here so much out of reach or maybe simply crazy,
[p. 544] that I better stop along this line!

130 *A case for non-connected bundles.*

In a more down-to-earth line, and reverting to the "unipotent" approach still (in either W- or V-version for "unipotent bundles"), I would still like to point out one rather mild extension of the set-up as contemplated initially. The suggestion here is to admit as components for our semisimplicial models not merely sheaves of sets which are "unipotent bundles", but equally direct sums of such. After making such extension, the linearization functor L ((11) p. 539) still makes sense, provided $U_{\mathrm{ab}}(k)$ is stable under infinite sums (otherwise, we'd have to restrict to a finite number of connected components for our "bundles"). Presumably, once we get into this, we will have to admit "twisted" finite direct sums as well, in order to have basic notions compatible with descent – never mind such technicalities at the present stage of reflections! Thus, applying componentwise (i.e., to each component X_n) the π_0-functor ("connected components"), a semisimplicial "bundle" X_* in the wider sense gives rise to an associated usual semisimplicial set, let's call it $\pi_{0/k}(X_*)$, together with a map

$$X_* \to \pi_{0/k}(X_*), \tag{18}$$

where the second-hand side is interpreted as a semisimplicial constant object over k. (For simplicity, we have assumed here $\mathrm{Spec}(k)$ to be 0-connected, and that the direct sums involved in the X_n's are not twisted...) Intuitively, we may interpret (18) as defining X_* as a ("strict", namely componentwise connected) *schematic* homotopy type, lying "over" the *discrete* (or "constant") homotopy type $\pi_{0/k}(X_*) = E_*$. The latter introduces homotopy invariants of its own, which strictly speaking shouldn't be viewed as being of a "schematic" nature. Thus, when our exclusive emphasis is on studying the "strict" schematic homotopy types, we'll restrict our models X_* by demanding that the associated discrete homotopy type $\pi_{0/k}(X_*)$ should be *aspheric*, i.e., isomorphic in Hot to a one-point space. Under this restriction, presumably, working with those slightly more general models should give (up to equivalence) the same derived category $\mathrm{Hot}_0(k)$, as when working with (connected) unipotent bundles. Allowing connected components may prove useful for giving a little more "elbow freedom" in working with models, as it allows for instance anodyne operations such as taking direct sums. Thus, all constant semisimplicial objects will be allowed, or at any rate those which correspond to aspheric semisimplicial sets – which includes
[p. 545] notably the objects of $\triangle^\wedge$ represented by the standard simplices $\triangle_n$;

and it is useful indeed to be able to have these among our models.

My motivation for suggesting to allow for connected components for our bundles, comes from an attempt to perform the "integration" operation for an indexes family of schematic homotopy types, under suitable assumptions – one among these being that the indexing small category I should be aspheric (in order to meet the above asphericity requirement on E_*, for the semisimplicial bundle obtained). When just paraphrasing the construction of integration for ordinary homotopy types (as indicated in section 69 – we're going to come back upon this in a later chapter), we just cannot help but run into a bunch of connected components, for the components of the semisimplicial sheaf expressing the integrated type. The general idea here, of course, is to perform the basic homotopy operations (essentially, integration and cointegration operations) *in the context of semisimplicial sheaves*, and then look and see whether (or when) this doesn't take us out of the realm semisimplicial "bundles". I am sorry I didn't work out any clear-cut result along these lines yet, and I am going to leave it at that for the time being.

One serious drawback, however, when allowing for connected components of the components X_n or our semisimplicial bundles, is that we can hardly expect anymore that the homotopy sheaves of X_* may be expressed in terms of tangent sheaves, as contemplated in section 118. At any rate, this extension of our category of "models" would seem a reasonable one only if we are able to show that we get the same derived category (up to equivalence) as when working with the more restricted models, using as components X_n only (connected) unipotent bundles. I didn't make any attempt either to try and prove such a thing.

16.9.

131

Tentative description of the spherical functor $\widetilde{S}$, and "infinitesimal" extension of the basic notion of "bundles".

Yesterday again, I didn't do any mathematics – instead, I have been writing a ten pages typed report on the preparation and use of kimchi, the traditional Korean basic food of fermented vegetables, which I have been practicing now for over six years. Very often friends ask me for instructions for preparing kimchi, and a few times already I promised to put it down in writing, which is done now. Besides this, I wrote to Larry Breen to tell him a few words about my present ponderings as he is the one person I would think of for whom my rambling reflections on schematization and on schematic homotopy types may make sense. [p. 546]

Definitely, my suggestion in the last notes, to work with non-connected "bundles", isn't much more than the reflection of my inability so far to make a breakthrough and get the "left" homotopy constructions in terms of semisimplicial (connected) unipotent bundles alone. Besides the serious drawback already pointed out at the end of section 130, another one occurred to me – namely that with the suggested extension, one is losing track of the extra condition $X_0 = e$ (or even $X_0 = X_1 = e$) on our models, which is often remaining implicit in the notes, and which, however, is an important restriction, actually needed for the kind of things we want to do. For one thing, when we drop it, and even when we otherwise restrict to components X_n which are standard affine spaces $E_k^{d_n}$, say, there is no hope of getting a Lie-type description for the homotopy sheaves $\pi_i(X_*)$ – for instance, the Lie-type invariants we get by using different sections of X_0 over $S = \mathrm{Spec}(k)$ are by no means related (as the π_i should) by a transitive system of canonical isomorphisms; this is easily seen already when taking the "trivial" semisimplicial bundle described by $X_n = X_0$ for all n (the constant functor with value X_0 from $\triangle^{\mathrm{op}}$ to $U(k)$)! Of course, even when allowing connected components for the "bundles" X_n, we may formally still throw in the condition $X_0 = e$ – but this is cheating and no use at all, if we remember that the main motivation for allowing connected components was in order to be able (in suitable cases) to perform an integration operation of schematic homotopy types. But when following the standard construction, for the resulting X_* the component X_0 as well as all others will have lots of connected components, and hence the condition $X_0 = e$ will not hold true. The same remark applies also to the "constant" semisimplicial bundles (namely with components which are "constant" schemes over S) defined by the standard simplices $\triangle_n$. To sum up, while it *is* certainly quite useful to view semisimplicial "bundles", used for describing the "schematic homotopy types" we are after, as particular cases of more general semisimplicial sheaves on the fpqc site of $S = \mathrm{Spec}(k)$, we will probably have to be quite careful in keeping the "bundles" we are working with restricted enough, and not confuse our models for schematic homotopy types (allowing for a nice description of the basic $\mathrm{L}\pi_\bullet$ and $\mathrm{LH}_\bullet$ invariants) with more general semisimplicial sheaves which may enter the picture in various ways.

[p. 547] I would like, however, to suggest still another extension of the notion of a "bundle", which maybe will prove something better than just a random way out of embarrassment! It has to do with an attempt to come to a description of one among the four "basic functors" of section 124, namely the "spherical" functor

$$\widetilde{S} : \mathrm{Hotab}_0(k) \to \mathrm{Hot}_0(k), \tag{1}$$

which for the time being is remaining hypothetical, due notably to my inability so far to carry out the suspension operation for schematic homotopy types. There are two basic formal properties giving us some clues about this functor, namely, it should be left adjoint to the "total homotopy" or "Lie" functor $\mathrm{L}\pi_\bullet$, and it should be right inverse to the (reduced) total homology functor $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ (cf. pages 523–525). Let's work for the sake of definiteness, for describing $\mathrm{Hot}_0(k)$, with semisimplicial V-bundles X_* satisfying the extra assumption $X_0 = e$ (and presumably a little more), corresponding therefore to co-semisimplicial algebra A^* satisfying $A^0 = k$, such that the components A^n be isomorphic to symmetric algebras

$$A^n \simeq \mathrm{Sym}_k(M^n), \tag{2}$$

where we'll assume the modules M^n (or equivalently, the algebras A^n) *flat* over k. The condition $A^0 = k$ implies that the algebras A^n are

k-augmented, and if $\mathfrak{m}^{(n)}$ is the augmentation ideal, we get

$$\mathfrak{m}^{(n)}/(\mathfrak{m}^{(n)})^2 \simeq M^n, \tag{3}$$

thus, the modules M^n may be viewed as the components of a co-semisimplicial k-module M^*, giving rise to a semisimplicial vector bundle $V(M^*)$ over k, and we may write

$$\mathrm{L}\pi_\bullet(X_*) \simeq V(M^*), \tag{4}$$

where the second member is viewed as a chain complex (rather than as a semisimplicial module) in the usual way. Consider now an object L_* in $\mathrm{Hotab}_0(k)$, it is described in terms of a co-semisimplicial k-module L^* by the formula

$$L_* = V(L^*), \tag{5}$$

and we get now

$$\mathrm{Hom}_{\mathcal{O}_k}(L_*, \mathrm{L}\pi_\bullet(X_*)) \simeq \mathrm{Hom}_{\mathcal{O}_k}(V(L^*), V(M^*)) \simeq \mathrm{Hom}_{\mathcal{O}_k}(M^*, L^*). \tag{*}$$

Now, if M is any k-module, let

$$D(M) = \mathrm{Sym}_k(M)(1) \simeq k \oplus M \tag{6}$$

be the corresponding augmented k-algebra admitting M as a square-zero augmentation ideal, and let [p. 548]

$$I(M) = \mathrm{Spec}(D(M)) \tag{7}$$

its spectrum, which is a first-order infinitesimal neighborhood of $S = \mathrm{Spec}(k)$. For variable M, $D(M)$ depends covariantly, $I(M)$ contravariantly on M. Thus,

$$D(L^*) \tag{8}$$

is a co-semisimplicial algebra, depending covariantly on L^*, hence contravariantly on $L_* = V(L^*)$, and accordingly

$$I(L^*) = \mathrm{Spec}(D(L^*)) \tag{9}$$

(where in both members I and Spec are applied again componentwise) is a semisimplicial affine scheme, indeed a first-order infinitesimal one, depending contravariantly on L^*, hence covariantly on L_*. Now, the last member of (*) may be interpreted non-linearly as

$$\mathrm{Hom}_{\text{co-ss }k\text{-alg}}(A^*, D(L^*)) \simeq \mathrm{Hom}(I(L^*), X_*), \tag{**}$$

where in the second term the Hom means homomorphisms of semisimplicial schemes. Let's now write (with an obvious afterthought!)

$$\widetilde{S}(L_*) = I(L^*) \quad \text{whenever } L_* = V(L^*), \tag{10}$$

the sequence of isomorphisms (*) and (**) may be summed up by

$$\mathrm{Hom}(L_*, \mathrm{L}\pi_\bullet(X_*)) \simeq \mathrm{Hom}(\widetilde{S}(L_*), X_*), \tag{11}$$

where the first Hom means maps of semisimplicial $\mathcal{O}_k$-modules, whereas the second is a Hom of semisimplicial schemes. It looks very much like the adjunction formula (p. 524, (7b)) we are after, with however two big grains of salt. The smaller one is that this formula doesn't take place in derived categories, but rather in the categories of would-be models. The considerably bigger grain of salt is that $\widetilde{S}(L_*)$ isn't at all in our model category, its components are purely infinitesimal, first-order schemes, and far from being vector bundles! Essentially, what we have been using for getting the (admittedly quite tautological) adjunction formula (11) is that the Lie functor on schemes-with-section over S is representable in an obvious way, namely by the scheme-with-section

$$I(k) = \mathrm{Spec}(k[T]/(T^2)),$$

which is a first-order infinitesimal scheme-with-section over k.

[p. 549] Our tentative $\widetilde{S}$ functor in (10) has been constructed in the most evident way, in order to satisfy an adjunction formula (11), valid on the level of semisimplicial objects (and carrying over, hopefully, to the similar adjunction formula for suitable derived categories). Next question is then, what about the inversion formula

$$\mathrm{L}\widetilde{\mathrm{H}}_\bullet(\widetilde{S}(L_*)) \simeq L_*? \tag{12}$$

The question makes sense, as $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ is defined for any semisimplicial affine scheme-with-section over k, or equivalently for any co-semisimplicial augmented k-algebra A^* over k, by taking the augmentation ideal $\mathfrak{m}^*$ in the latter and retaining only its linear (co-semisimplicial) structure. Keeping this in mind, formula (12) comes out indeed a tautology again!

The tentative description we just got is indeed of a most seducing simplicity, as seducing indeed as the description of homotopy in terms of the Lie functor, and closely related to the latter. It gives as a particular case an exceedingly simple description of the sought-for "spheres over k" $S(k,n)$. But it is clear that this description is liable to makes sense only at the price of suitably extending the notion of a "bundle" we are working with, in a rather different direction, I would say, from adding (or allowing) connected components, as suggested in the previous section. Maybe we might view it, though, as a kindred, but somewhat subtler extension of our initial bundles, namely that we are now allowing, not a discrete non-trivial "set" or k-scheme of connected components, but rather, an infinitesimal one. More specifically, the suggestion which comes to mind here, is to call now "bundle" over k any scheme X over k admitting a subscheme

$$X_* \subset X$$

in such a way that X_0 should be a V-bundle (namely isomorphic to a vector bundle) over k, and X should be an infinitesimal neighborhood of X_0, i.e., X_0 should be definable by a quasi-coherent ideal on X which is nilpotent. Equivalently, in terms of the affine ring A of X, we are demanding that A should admit a nilpotent ideal J (which is of course not part of its structure), such that A/J should be isomorphic to a symmetric algebra over k (with respect to some k-module M). Possibly, we may have moreover to impose further flatness restrictions.

When working with this extended notion of bundles, there is no problem for describing for the corresponding semisimplicial models the [p. 550] three functors $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$, $\widetilde{K}$, $\widetilde{S}$. Indeed, as we just recalled, the first of the three functors is well-defined and has an evident description for all semisimplicial affine schemes over k. As for $\widetilde{K}$ and $\widetilde{S}$, they are obtained in terms of a variable co-semisimplicial k-module L^* (representing the semisimplicial vector bundle $L_* = V(L^*)$) by applying componentwise the functor Sym_k and the first-order truncation $\mathrm{Sym}_k(-)(1)$, respectively – one may hardly imagine something simpler! This brings to my attention that in terms of this description, we get a canonical functorial map

$$\widetilde{S}(L_*) \hookrightarrow \widetilde{K}(L_*) \tag{13}$$

when working with the semisimplicial models, and hence presumably a corresponding map for the functors between the relevant derived categories $\mathrm{Hotab}_0(k)$ and $\mathrm{Hot}_0(k)$. Working either in the model or in the derived categories, this map, as a matter of fact, may be deduced from the basic formulaire of section 124, where it had by then escaped my attention. Indeed, $\widetilde{S}$ is a left adjoint of $\mathrm{L}\pi_\bullet$, and $\widetilde{K}$ a right adjoint of $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$, to give such a map (13) is equivalent with giving either one of two maps

$$\begin{cases} \text{a)} \quad L_* \to \mathrm{L}\pi_\bullet(\widetilde{K}(L_*)) \\ \text{b)} \quad \mathrm{L}\widetilde{\mathrm{H}}_\bullet(\widetilde{S}(L_*)) \to L_* \end{cases}, \tag{14}$$

and the formulaire provides for two such maps, namely the "inversion isomorphisms" ((9), p. 525). Thus, there is an extra property which was forgotten in the formulaire, namely that the two maps (13) associated to the two inversion isomorphisms should be the same. A nicer way, then, to state the formulaire is to consider the map (13) as a basic datum, and say that the two maps in (14) deduced from it by the adjunction property should be isomorphisms. The situation is reminiscent of the two ways by which we could obtain the Hurewicz map (p. 525, C)) – presumably, the basic data for the formulaire of section 124 should be the functors $\widetilde{S}$ and $\widetilde{K}$ and the map (13) between them, with the property that the relevant adjoint functors $\mathrm{L}\pi_\bullet$ and $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ exist, and that the corresponding maps in (14) should be isomorphisms, which then will allow to define a *unique* Hurewicz map $\mathrm{L}\pi_\bullet \to \mathrm{L}\widetilde{\mathrm{H}}_\bullet$.

As long as we are sticking to the purely formal aspect, and even when working in the larger context of semisimplicial affine schemes over k satisfying merely $X_0 = e$, or more generally still, dropping the last restriction and taking "k-pointed" semisimplicial affine schemes instead, the whole "four functors formalism" (including even $\mathrm{L}\pi_\bullet$) as contemplated in section 124 (and with the extra feature (13) above [p. 551] as just notice) goes over very smoothly, in an essentially tautological way. As recalled on p. 547, the $\mathrm{L}\pi_\bullet$-functor, when interpreted on the co-semisimplicial side of the dualizing functor, appears as a quotient of the $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$-functor, the latter identified to the functor obtained by taking augmentation ideals of co-semisimplicial algebras – the quotient being obtained by dividing out by the squares of the latter ideals. Dually, we get the Hurewicz map for semisimplicial vector bundles, which is always an inclusion. Again, imagine something simpler! The only trouble (but an extremely serious one indeed!) is that in this general set-up, the relation of the so-called $\mathrm{L}\pi_\bullet$-functor to homotopy groups-or-sheaves becomes a very dim one. Definitely, the only firm hope here is that the relationship between the two is OK (as contemplated in section 118) whenever the components X_n are actual flat vector bundles, satisfying moreover $X_0 = e$ (at the very least) – plus possibly even some extra Kan type conditions (sorry for the vagueness of even this one "firm hope"!). If we take already the "next best" set of assumptions, namely essentially that the X_n be flat "bundles" in the sense above (not necessarily vector bundles, though), then the hoped-for relationship again seems to vanish. The first case of interest, of course, is the case when X_* is of the form $\widetilde{S}(L_*)$, which includes (if our $\widetilde{S}$ functor is "the right one" indeed) the n-spheres over k. We get in this case (namely when X_* is a first-order neighborhood of the marked section) the trivial, and really stupid relation

$$\mathrm{L}\pi_\bullet(\widetilde{S}(L_*)) \simeq L_* \quad (!!!),$$

which translates into: the homotopy groups of a sphere, computed in the most naive "Lie" way, are canonically isomorphic to its homology groups! Not much of a success...

132 *A crazy tentative wrong-quadrant (bi)complex for the homotopy groups of a sphere.*

This makes it very clear that, while the functors $\widetilde{K}$, $\widetilde{S}$, $\mathrm{L}\widetilde{\mathrm{H}}_\bullet$ in our new context of semisimplicial "bundles" make perfectly good sense as they are, the $\mathrm{L}\pi_\bullet$ functor computed naively (taking tangent spaces) definitely doesn't, except when actually working with flat (hence, essentially "smooth") vector bundles as components of our semisimplicial models. This, after all, shouldn't be too much of a surprise, if we remember the way differentials and tangent spaces fit into a sweeping homology or cohomology formalism. It has become quite familiar to people "in the know" that taking the sheaf of 1-differentials, say, or its dual, or a sheaf [p. 552] of 1-differentials or a tangent sheaf along a section, behaves as "the" good object in terms of homological algebra and obstruction theory in various geometric situations, *only* in the case when the relative scheme (X say) we are working with is *smooth* over the base scheme S – which in the present case amounts to saying (when $X = X_n$ is a component of a semisimplicial "bundle") that X is indeed a flat vector bundle over S. In more general cases, the work of André-Quillen-Illusie tells us that the relevant object which replaces $\Omega^1_{X/S}$ or its dual $\mathrm{T}_{X/S}$ is the relative tangent or cotangent *complex* $\mathrm{L}^{X/S}_\bullet$ or $\mathrm{L}^\bullet_{X/S}$, the second being the dual of the other

[André (1967, 1974), Illusie (1971), and Quillen (1970); see also Berthelot, Grothendieck, and Illusie (SGA 6)...]

(15) $$\mathrm{L}^\bullet_{X/S} = \mathrm{R}\mathbf{Hom}_{\mathcal{O}_X}(\mathrm{L}^{X/S}_\bullet, \mathcal{O}_X),$$

these objects being viewed, respectively, as objects in the derived categories $\mathrm{D}_\bullet(\mathcal{O}_X)$ and $\mathrm{D}^\bullet(\mathcal{O}_X)$ (deduced from chain and cochain complexes of $\mathcal{O}_X$-modules). When X is endowed with a section over S, the naive differentials and codifferentials along this section should in the same way be replaced by the co-Lie and Lie complexes

$$\ell_\bullet(X/S,s) \quad \text{and} \quad \ell^\bullet(X/S,s) \simeq \mathrm{R}\mathbf{Hom}(\ell_\bullet(X/S,s),\mathcal{O}_S), \tag{16}$$

where s is the given section, obtained from the previous complexes by taking its inverse images $\mathrm{L}s^*$ by s. As a matter of fact, the chain complex $\mathrm{L}_\bullet^{X/S}$ can be realized canonically, up to unique isomorphism, via a semisimplicial module

$$\mathrm{L}_*^{X/S}$$

on X, whose components are free $\mathcal{O}_X$-modules. Accordingly, we get (16) in terms of a well-defined semisimplicial $\mathcal{O}_S$-module,

$$\ell_*(X/S,s) \tag{17}$$

whose components are free – and as $S = \mathrm{Spec}(k)$, we may interpret this more simply as a semisimplicial k-module with free components. When we apply this to the components X_n of a semisimplicial bundle X_*, we get however the co-Lie invariants; in order to get the relevant Lie invariants we'll have to take the duals

$$\ell^*(X/S,s) = \mathrm{Hom}_k(\ell_*(X/S,s),k), \tag{17'}$$

where X is any one among the X_n's, and s its marked section. Thus, the "corrected" description of $\mathrm{L}\pi_\bullet$, by using the André-Quillen-Illusie version of the "Lie-functor along a section", would seem to be

$$\mathrm{L}\pi_\bullet(X_*) = \ell^*(X_*/k,e_*) \quad (?), \tag{18}$$

where now the second member appears as a mixed complex of k-modules [p. 553]

$$(n,p) \mapsto \ell^p(X_n/k,e_n) : \triangle^{\mathrm{op}} \times \triangle \to \mathrm{Ab}_k,$$

contravariant with respect to the index n, covariant with respect to p. Translating this via Kan-Dold-Puppe, we get a bicomplex of k-modules, which we'll write in cohomological notation (with the two partial differential operators of degree $+1$)

$$(C^{n,p}) = (C^{n,p}(X_*))$$

situated in the "quadrant"

$$n \le 0, p \ge 0.$$

As we finally want an object of the derived category $\mathrm{D}(\mathrm{Ab}_k)$ of the category of k-modules, and even an object in the subcategory $\mathrm{D}_\bullet(\mathrm{Ab}_k)$, the evident thing that seems to be done now is to take the associated simple complex, which hopefully may prove to be the "correct" expression of the looked-for $\mathrm{L}\pi_\bullet$ – and this (if any) should be the precise meaning of (18).

The associations for getting (18) are very tempting, indeed, the expression we got makes us feel a little uneasy, though. The main point is that the quadrant where our bicomplex lies in is one of the two "awkward" ones, which implies that a) for a given total degree, there are an infinity of summands occurring (and one has to be careful, therefore, if these should be "summands" indeed, or rather "factors", namely if we should take an infinite direct sum, or an infinite product instead); and b) the total complex will have components of any degree both positive and negative, and it isn't clear at all that it should be (as an object of $\mathrm{D}(\mathrm{Ab}_k)$) of the nature of a chain complex, namely that its cohomology modules vanish for (total) degree $d > 0$. If it should turn out that this is not so (I didn't yet check any particular case), this would imply for the least that (18) should be corrected, by taking the relevant truncation of the second-hand side.

Working with the $\mathrm{L}^*_{X/S}$ and $\ell^*(X/S,s)$ invariants brings in a slightly awkward feature of its own which we have been silent about, namely (except under suitable finiteness conditions) it brings in non-quasi-coherent $\mathcal{O}_X$ or $\mathcal{O}_S$-modules. This may encourage us to dualize (18), which will amount to working with the co-semisimplicial algebra A^* expressing X_*, and taking componentwise the reduced (via augmentations) André-
[p. 554] Quillen complexes (rather than their duals). At any rate, the would-be expression of "total co-homotopy" of X_* we'll get this way isn't so much more appealing than (18) – it lies still in one of the wrong quadrants, which definitely makes us feel uncomfortable.

In principle, the tentative formula (18), when applied say to an object such as

$$S(n,k) \overset{\mathrm{def}}{=} \widetilde{S}(k[n]),$$

gives a rather explicit (but for the time being highly hypothetical!) expression of the homotopy modules of the n-sphere over k, which in case $k = \mathbb{Z}$ are hoped to be just the homotopy groups of the ordinary n-sphere. To test whether this makes at all sense, we'll have to understand first the structure of the André-Quillen "Lie complex" of an algebra (6) of the type $D(M)$, for variable k-module M. I haven't started looking into this yet, and I doubt I am going to do it presently.

At any rate, whether or not the formula (18) we ended up with is essentially correct, in order (among other things) to get a method for computing the total homotopy $\mathrm{L}\pi_\bullet$ for a semisimplicial bundle X_* that isn't a flat vector bundle, we'll have to find out some more or less explicit means of replacing X_* by some X'_* whose components *are* flat vector bundles, and which is isomorphic to X_* in the relevant derived category $\mathrm{Hot}_0(k)$. When k isn't a field, the question arises already even when the components X_* *are* vector bundles, when these are not flat. The first idea that comes to mind, from the André-Quillen theory precisely, in terms of the co-semisimplicial algebra A^* expressing X_*, is to take "projective resolutions" of the various components A^n by polynomial algebras. Again we end up with a mixed complex, this time a complex of augmented algebras depending on two indices n, p, covariant in n and contravariant in p, or the reverse if we replace those algebras A_p^n by their spectra X_n^p. In any case, we again end up in a "wrong quadrant"!

The hesitating question that comes to mind now is whether it is at all feasible to work with a category of models which isn't a category of *semisimplicial* bundles say, but one of such *mixed wrong-quadrant* bundles; namely, use these as "models" for getting hold of a reasonable derived category of "schematic homotopy types"? I never heard of anything such yet, and I confess that at this point my (anyhow rather poor!) formal intuition of the situation breaks down completely – maybe the suggestion is complete nonsense, for some wholly trivial reason! Maybe Larry Breen could tell me at once – or someone else who has more feeling than I for semisimplicial and cosemisimplicial models?

Part VII

Linearization of homotopy types (2)

22.10. [p. 555]

Birth of Suleyman.

133 Again nearly a week has passed by without writing any notes – the tasks and surprises of life took up almost entirely my attention and my energy. It were days again rich in manifold events – most auspicious one surely being the birth, three days ago, of a little boy, Suleyman, by my daughter. The birth took place at ten in the evening, in the house of a common friend, in a nearby village where my daughter had been awaiting the event in quietness. It came while everyone in the house was in bed, the nearly five years old girl sleeping next to her mother giving birth. The girl awoke just after the boy had come out, and then ran to tell Y (their hostess) she got a little brother. When I came half an hour later, the little girl was radiant with joy and wonder, while telling me in whispers, sitting next to her mother and to her newborn brother, what had just happened. As usual with grownups, I listened only distractedly, anxious as I was to be useful the best I could. There wasn't too much by then I could do, though, as the mother knew well what was to be done and how to do it herself – presently, tie the chord twice, and then cut it in-between. "Now you are on your own, boy!" she told him with a smile when it was done. A little later she and Y helped the baby to a warm bath – and still later, while the girl was asleep next to her brother and Y in her room, the mother took a warm bath herself in the same basin, to help finishing with the labors. When her mother arrived a couple of hours later, everybody in the house was asleep except me in the room underneath, awaiting her arrival while taking care of some fallen fruit which had been gathered that very day.

This birth I feel has been beneficial, a blessing I might say, in a number of ways. The very first one I strongly perceived, was about Suleyman's sister, who had so strongly participated in his birth. This girl has been marked by conflict, and her being is in a state of division, surely as strongly as any other child of her age. However, after this experience, if ever her time should come to bear a child and give birth, she will do so with joy and with confidence, with no secret fear or refusal interfering with her labors and with the act of giving birth – this most extraordinary act of all, maybe, which a human being is allowed to accomplish; this unique privilege of woman over man, this blessing, carried by so many like a burden and like a curse... And there are other blessing too in this birth which I perceive more or less clearly, and surely others still which escape my conscious attention. [p. 556]

I suspect that potentially, in every single instance anew, the act of giving birth, and the sudden arrival and presence of a newborn, are a blessing, carry tremendous power. In many cases, however, the greatest part (if not all) of this beneficial power is lost, dispersed through the action of crispation and fear (including the compulsory medical "mise en scène", from which it gets over harder to get rid in our well-to-do countries...). A great deal could be said on this matter in general, while my own leanings at present would be, rather, to ponder about the manifold personal aspects and meaning of the particular, manifold event I have just been involved in. But to yield fully now to this leaning of mine would mean to stop short with the mathematical investigation and with these notes. The drive carrying this investigation is very much alive, though, and I have been feeling it was time getting back to work – to this kind of work, I mean.

This brings me to somewhat more mundane matters – such as the beginning of school and teaching duties. This year I am in charge of preparing the "concours d'agrégation", something which, I was afraid, would be rather dull. Rather surprisingly, the first session of common work on a "problème d'agrègue" wasn't dull at all – the problem looked interesting, and two of the three students who turned up so far were interested indeed, and the atmosphere relaxed and friendly. It looks as though I was going again to learn some mathematics through teaching, or at any rate to apply things the way I have known and understood them (in a somewhat "highbrow" way, maybe) to the more down-to-earth vision going with a particular curriculum (here the "programme d'agrègue") and corresponding virtuosity tests – something rather far, of course, from my own relation to mathematical substance! Besides these arpeggios, we started a microseminar with three participants besides me, on the Teichmüller groupoids; I expect one of the participants is going to take a really active interest in the stuff. I have been feeling somewhat reluctant to start this seminar while still involved with the homotopy story, which is going to keep me busy easily for the next six months still, maybe even longer. Sure enough, the little I told us about some of the structures and operations to be investigated, while progressively gaining view of them again after a long oblivion – and as discovering them again hesitatingly, while pulling them out of the mist by bribes and bits – this little was enough to revive the special fascination of these structures, and all that goes with them. It seems to me there hasn't been a single thing in mathematics, including motives, which has exerted such a fascination upon me. It will be hard, I'm afraid, to carry on a seminar on such stuff, and go on and carry to their (hopefully happy!) end those ponderings and notes on homotopical algebra, so-called – [p. 557]

[these are the civil service competitive examinations for high school (lycée) teachers...]

[unreadable margin note]

and a somewhat crazy one too at times, I am afraid! We'll see what comes out of all this! The night after the seminar session at any rate, and already during the two hours drive home, the Teichmüller stuff was brewing anew in my head. Maybe it would have gone on for days, but next day already several things came up demanding special attention, last not least being the birth of Suleyman. . .

26.10. [the typescript says "26.8." but it must be a misprint!]

Back to linearization in the modelizer (Cat)*: another handful of questions around Kan-Dold-Puppe.*

134 It is time to come back to the "review" of linearization of (ordinary) homotopy types, and the homology and cohomology formalism in the context of the modelizer (Cat), which has been pushed aside now for over two months, for the sake of that endless digression on schematization of homotopy types. The strong tie between these two strains of reflection has been the equal importance in both of the linearization process. Technically speaking though, linearization, as finally handled in the previous chapter (via the so-called "integrators"), looks pretty much different from the similar operation in the schematic set-up, due (at least partly) to the choice we made of models for expressing schematic homotopy types, namely taking semisimplicial scheme-like objects; which means, notably, working with Δ rather than more general test categories, and relying heavily on the Kan-Dold-Puppe relationship. In the discrete set-up, it turned out (somewhat unexpectedly) that the latter can be replaced, when working with more general categories A than Δ (which need not even be test categories or anything of the kind), by the $\mathrm{L}p_!^{\mathrm{ab}}$ operation (when $p : A \to e$ is the map from A to the final object e in (Cat)), computable in terms of a choice of an "integrator" for A. More specifically, recalling that

$$p_!^{\mathrm{ab}} : A^\wedge_{\mathrm{ab}} \to (\mathrm{Ab}) \tag{1}$$

is defined as the left adjoint of the inverse image functor

$$p^* : (\mathrm{Ab}) \to A^\wedge_{\mathrm{ab}},$$

associating to every abelian group the corresponding "context" abelian presheaf on A, and [p. 558]

$$\mathrm{LH}_\bullet = \mathrm{L}p_!^{\mathrm{ab}} : \mathrm{D}^-(A^\wedge_{\mathrm{ab}}) \to \mathrm{D}^-(\mathrm{Ab}), \quad \text{inducing } \mathrm{D}_\bullet(A^\wedge_{\mathrm{ab}}) \to \mathrm{D}_\bullet(\mathrm{Ab}) \tag{2}$$

is its left derived functor (computed using projective resolution of complexes bounded above in $A^\wedge_{\mathrm{ab}}$), composing (2) with the tautological inclusion functor

$$A^\wedge_{\mathrm{ab}} \hookrightarrow \mathrm{D}_\bullet(A^\wedge_{\mathrm{ab}}) \hookrightarrow \mathrm{D}^-(A^\wedge_{\mathrm{ab}}),$$

we get a canonical functor

$$\mathrm{L}p_!^{\mathrm{ab}} \upharpoonright A^\wedge_{\mathrm{ab}} : A^\wedge_{\mathrm{ab}} \to \mathrm{D}_\bullet(\mathrm{Ab}), \tag{3}$$

which may be expressed in terms of an integrator $L_\bullet^B$ for $B = A^{\mathrm{op}}$, i.e., a projective resolution of the constant presheaf $\mathbb{Z}_B$ in $B^\wedge_{\mathrm{ab}}$, as the composition

$$A^\wedge_{\mathrm{ab}} \to \mathrm{Ch}_\bullet(\mathrm{Ab}) \to \mathrm{D}_\bullet(\mathrm{Ab}), \tag{4}$$

where the first arrow is

$$F \mapsto F *_{\mathbb{Z}} L_\bullet^B : A^\wedge_{\mathrm{ab}} \to \mathrm{Ch}_\bullet(\mathrm{Ab}), \tag{5}$$

and the second is the canonical localization functor. The functors (2) and (3) may be viewed as the (total) homology functors of A, with coefficients in complexes of abelian presheaves, resp. in abelian presheaves simply. When we focus attention on the latter, we may introduce in $A^\wedge_{\mathrm{ab}}$ the set of arrows which become isomorphisms under the total homology functor (3), let's call them "*abelian weak equivalences*" in $A^\wedge_{\mathrm{ab}}$, not to be confused with the notion of quasi-isomorphism for a map between complexes in $A^\wedge_{\mathrm{ab}}$. Let's denote by

$$\mathrm{Hotab}_A = (W_A^{\mathrm{ab}})^{-1} A^\wedge_{\mathrm{ab}} \tag{6}$$

the corresponding localized category of $A^\wedge_{\mathrm{ab}}$, where W_A^{ab} denotes the set of abelian weak equivalences in $A^\wedge_{\mathrm{ab}}$. Thus, the choice of an integrator $L_\bullet^B$ for A (i.e., a cointegrator for B) gives rise to a commutative diagram of functors

$$\begin{tikzcd} A^\wedge_{\mathrm{ab}} \arrow[r] \arrow[d] & \mathrm{Ch}_\bullet(\mathrm{Ab}) \arrow[d] \\ \mathrm{Hotab}_A \arrow[r] & \mathrm{D}_\bullet(\mathrm{Ab}) \overset{\mathrm{def}}{=} \mathrm{Hotab} \end{tikzcd} , \tag{7}$$

where the lower horizontal arrow is defined via (3) (independently of the choice of $L_\bullet^B$), the vertical arrows being the localization functors. Beware that even when $A = \triangle$ and $L_\bullet^B$ is the usual, "standard" integrator for $\triangle$, the upper horizontal arrow in (7) is *not* the Kan-Dold-Puppe equivalence of categories, it has to be followed still by the "normalization" operation. Thus, *we certainly should not expect in any case the functor* (5) *to be an equivalence – however, we suspect that when A is a test category* (maybe even a weak test category would do it), *then the lower horizontal arrow in* (7) [p. 559]

$$\mathrm{Hotab}_A \to \mathrm{D}_\bullet(\mathrm{Ab}) = \mathrm{Hotab} \tag{8}$$

is an equivalence of categories. Whenever true, for a given category A, this statement looks like a reasonable substitute (on the level of the relevant derived categories) for the Kan-Dold-Puppe theorem, known in the two cases $\triangle$ and $\bigcirc$.

There are however still two important extra features in the case $A = \triangle$, which deserve to be understood in the case of more general A. One is that a map $u : F \to G$ in $A^\wedge_{\mathrm{ab}}$ is in W_A^{ab} (i.e., is an "abelian weak equivalence") iff it is in W_A, i.e., iff it is a weak equivalence when forgetting the abelian structures. In terms of a final object e in A (when such object exists indeed), viewing the categories $A_{/F}$ (for F in $A^\wedge_{\mathrm{ab}}$) pointed by the zero map $e \to F$, which yields the necessary base-point e_F for defining homotopy invariants $\pi_i(F)$ for $i \geq 0$, the relationship just

it is enough that A be 1-connected instead of having a final object considered between abelian weak equivalence and weak equivalence, will follow of course whenever we have functorial isomorphisms

$$\pi_i(F) \overset{\text{def}}{=} \pi_i(A_{/F}, e_F) \simeq \mathrm{H}_i(A, F), \tag{9}$$

which are known to exist indeed in the case $A = \mathbb{\Delta}$. It should be noted that in section 92, when starting (in a somewhat casual way) with some reflections on "abelianization", we introduced a category (denoted by Hotab_A) by localizing $A^\wedge_{\mathrm{ab}}$ with respect to the maps which are weak equivalences (when forgetting the abelian structures), whereas it has now become clear that, in case the latter should not coincide with the "abelian weak equivalences" defined in terms of linearization or homology, it is the category (6) definitely which is the right one. Still, the question of defining isomorphisms (9) when A has a final object, and whether the equality

$$W_A^{\mathrm{ab}} = \mathrm{forg}_A^{-1}(W_A) \tag{10}$$

holds, where

$$\mathrm{forg}_A : A^\wedge_{\mathrm{ab}} \to A^\wedge$$

is the "forgetful functor", should be settled for general A.

The other "extra feature" is about the relationship of W_A^{ab} with the notion of homotopism. In case $A = \mathbb{\Delta}$, a map in $A^\wedge_{\mathrm{ab}}$ is a weak equivalence (or equivalently, an abelian weak equivalence) iff it is a homotopism when forgetting the abelian structures – this follows from the well-known fact that semisimplicial groups are Kan complexes. There is of course also a notion of homotopism in the stronger abelian sense – a particular case of the notion of a homotopism between semisimplicial objects in an arbitrary category (here in (Ab)). The Kan-Dold-Puppe theory implies that if F and G in $\mathbb{\Delta}^\wedge_{\mathrm{ab}}$ have as values projective $\mathbb{Z}$-modules, then a map $F \to G$ in $\mathbb{\Delta}^\wedge_{\mathrm{ab}}$ is a weak equivalence iff it is an "abelian" homotopism; and likewise, two maps $u, v : F \rightrightarrows G$ are equal in $\mathrm{Hotab}_{\mathbb{\Delta}}$ iff they are homotopic (in the strict, *abelian* sense of the word). The corresponding statements still make sense and are true, when replacing $\mathbb{Z}$ by any ground ring k, working in $\mathbb{\Delta}^\wedge_k$ rather than $\mathbb{\Delta}^\wedge_{\mathbb{Z}} = \mathbb{\Delta}^\wedge_{\mathrm{ab}}$ – and more generally still, when working in [p. 560]

$$\mathbb{\Delta}^\wedge_{\mathcal{M}} = \mathbf{Hom}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{M}),$$

where $\mathcal{M}$ is any abelian category. Now, replacing $\mathbb{\Delta}$ by an arbitrary object A in (Cat), these statements still make sense, it would seem, provided we got on $A^\wedge$ a suitable "homotopy structure", more specifically, a suitable "homotopy interval structure" (in the sense of section 51). In section 97 (p. 355) we reviewed the three standard homotopy interval structures which may be introduced on a category $A^\wedge$, and their relationships – the main impression remaining was that, in case A is a *contractor* (cf. section 95), those three structures coincide, and may be defined also in terms of a *contractibility structure* on $A^\wedge$ (section 51), the latter giving rise (via the general construction of section 79) to the usual notion of weak equivalence W_A in $A^\wedge$. Thus, we may hope that the feature just mentioned for $\triangle^\wedge$ may be valid too for $A^\wedge$, whenever A is a contractor. In case A is only a test category, then the most reasonable homotopy interval structure on $A^\wedge$, I would think, which possibly may still yield the desired "extra feature", is the one defined in terms of the set W_A of weak equivalences in $A^\wedge$ as in section 54 (namely h_{W_A}).

135 *Proof of "integrators are abelianizators" (block against homology receding?).*

[p. 561]

[I guess AG thought this was still Part VI. Since there was never a second volume of PS, we refer to Cisinski (2006) for a discussion of closed model structures on presheaf topoi, now known as *Cisinski model structures*. . .]

This handful of questions (some of which we met with before) is mainly a way of coming into touch again with the abelianization story, which has been becoming somewhat remote during the previous two months. I am not sure I am going to make any attempt, now or later, to come to an answer. The last one, anyhow, seems closely related to the formalism of closed model structure on categories $A^\wedge$, and the proper place for dealing with it would seem to be rather next chapter VII, where such structures are going to be studied. What I would like to do, however, here-and-now, is to establish at last the long promised relationship "integrators are abelianizators", which we have kept turning around and postponing ever since section 99, when those integrators were finally introduced, mainly for this purpose (of furnishing us with "abelianizators").

First of all, I should be more outspoken than I have been before, in defining the "*abelianization functor*" (or "*absolute Whitehead functor*"):

$$\mathbf{Wh} : (\mathrm{Hot}) \overset{\mathrm{def}}{=} \mathcal{W}^{-1}(\mathrm{Cat}) \to \mathrm{Hotab} \overset{\mathrm{def}}{=} \mathrm{D}_\bullet(\mathrm{Ab}), \tag{1}$$

without any use of the semisimplicial machinery which, at the beginning of our reflections, had rather obscured the picture (section 92). Defining such a functor amounts to defining a functor

$$\mathrm{LH}_\bullet : (\mathrm{Cat}) \to \mathrm{D}_\bullet(\mathrm{Ab}), \tag{2}$$

or "total homology functor", which should take weak equivalences into isomorphisms. For an object A in (Cat), we define

$$\mathrm{LH}_\bullet(A) = \mathrm{LH}_\bullet(A, \mathbb{Z}_A) = \mathrm{L}p_{A!}^{\mathrm{ab}}(\mathbb{Z}_A), \tag{3}$$

where $\mathbb{Z}_A$ is the constant abelian presheaf on A with value $\mathbb{Z}$, and

$$p_A : A \to e$$

is the map to the final object of (Cat). We have to define the functorial dependence on A. More generally, for pairs

$$(A, F), \quad \text{with } A \text{ in (Cat)}, F \text{ in } A^\wedge_{\mathrm{ab}},$$

the expression

$$\mathrm{LH}_\bullet(A, F) \quad \text{in } \mathrm{D}_\bullet(\mathrm{Ab})$$

is functorial with respect to the pair (A, F), where a map

$$(A, F) \to (A', F')$$

is defined to be a pair (f,u), where

(4) $$f : A \to A', \quad u : F \to f^*(F'),$$

the composition of maps being the obvious one. To see that such a pair defines a map

(5) $$\mathrm{LH}_\bullet(f,u) : \mathrm{LH}_\bullet(A,F) \to \mathrm{LH}_\bullet(A',F'),$$

we use [p. 562]

$$p_A = p_{A'} \circ f,$$

hence

(*) $$(p_A)_!^{\mathrm{ab}} \simeq (p_{A'})_!^{\mathrm{ab}} \circ f_!^{\mathrm{ab}}, \quad \text{and} \quad \mathrm{L}(p_A)_!^{\mathrm{ab}} \simeq \mathrm{L}(p_{A'})_!^{\mathrm{ab}} \circ \mathrm{L}f_!^{\mathrm{ab}},$$

taking into account that $f_!^{\mathrm{ab}}$ maps projectives to projectives. Hence, we get

$$\mathrm{LH}_\bullet(A,F) \overset{\mathrm{def}}{=} \mathrm{L}(p_A)_!^{\mathrm{ab}}(F) \simeq \mathrm{L}(p_{A'})_!^{\mathrm{ab}}(\mathrm{L}f_!^{\mathrm{ab}}(F)) \overset{\mathrm{def}}{=} \mathrm{LH}_\bullet(A',\mathrm{L}f_!^{\mathrm{ab}}(F)).$$

This, in order to get (5), we need only define a map in $\mathrm{D}^-(A'^{\wedge}_{\mathrm{ab}})$

(6) $$\mathrm{L}f_!^{\mathrm{ab}}(F) \to F',$$

which will be obtained as the composition

(6') $$\mathrm{L}f_!^{\mathrm{ab}}(F) \to f_!^{\mathrm{ab}}(F) \to F',$$

where the first map in (6') is the canonical augmentation maps towards the H_0 object, and where the second corresponds to u in (4) by adjunction. This defines the map (5), and compatibility with compositions should be a tautology. Hence the functor (2). To get (1), we still have to check that when

$$f : A \to A'$$

is a weak equivalence, then

$$\mathrm{LH}_\bullet(f) : \mathrm{LH}_\bullet(A) \to \mathrm{LH}_\bullet(A')$$

is an isomorphism in $\mathrm{D}_\bullet(\mathrm{Ab})$. It amounts to the same to check that for any object $K^\bullet$ in $\mathrm{D}^+(\mathrm{Ab})$, the corresponding map between the RHom's with values in $K^\bullet$ is an isomorphism in $\mathrm{D}^+(\mathrm{Ab})$. But the latter map can be identified with the map for *cohomology*

$$\mathrm{RH}^\bullet(A',K^\bullet_{A'}) \to \mathrm{RH}^\bullet(A,K^\bullet_A),$$

with coefficients in the constant complex of presheaves defined by $K^\bullet$ on A' and on A, which is an isomorphism, by the very definition of weak equivalences in (Cat) via cohomology.

Now, the statement "an integrator is an abelianizator" may be rephrased rather evidently, without any reference to a given integrator, as merely

the commutativity, up to canonical isomorphism, of the following dia-
[p. 563] gram for a given A in (Cat):

(7)
$$\begin{tikzcd} A\hat{} \arrow[r, "\varphi_A"] \arrow[d, "\mathrm{Wh}_A"'] & (\mathrm{Hot}) \arrow[d, "\mathbf{Wh}"] \\ A\hat{}_{\mathrm{ab}} \arrow[r, "\mathrm{LH}_\bullet(A,-)"'] & (\mathrm{Hotab}) \overset{\mathrm{def}}{=} \mathrm{D}_\bullet(\mathrm{Ab}), \end{tikzcd}$$

or equivalently, of the corresponding diagram where the categories $A\hat{}$, $A\hat{}_{\mathrm{ab}}$ are replaced by the relevant localizations:

(7')
$$\begin{tikzcd} \mathrm{Hot}_A \arrow[r] \arrow[d] & \mathrm{Hot} \arrow[d] \\ \mathrm{Hotab}_A \arrow[r] & \mathrm{Hotab} \ . \end{tikzcd}$$

The left vertical arrow in (7) is of course the trivial abelianization functor in $A\hat{}$:

$$\mathrm{Wh}_A : A\hat{} \to A\hat{}_{\mathrm{ab}}, \quad X \mapsto \mathbb{Z}^{(X)} = \big(a \mapsto \mathbb{Z}^{(X(a))}\big).$$

Going back to the definitions, the commutativity of (7) up to isomorphism, means that for X in $A\hat{}$, there is a canonical isomorphism

(8)
$$\mathrm{LH}_\bullet(A_{/X}, \mathbb{Z}) \simeq \mathrm{LH}_\bullet(A, \mathbb{Z}^{(X)}),$$

functorial with respect to X. To define (8), let

$$f : A' \overset{\mathrm{def}}{=} A_{/X} \to A$$

be the canonical functor, then we get (using (*) of the previous page, with the roles of A and A' reversed)

$$\mathrm{LH}_\bullet(A', \mathbb{Z}_{A'}) \simeq \mathrm{LH}_\bullet(A, \mathrm{L}f_!^{\mathrm{ab}}(\mathbb{Z}_{A'})),$$

and the relation (8) follows from the more precise relation

(9)
$$\mathrm{L}f_!^{\mathrm{ab}}(\mathbb{Z}_{A'}) \xrightarrow{\sim} \mathbb{Z}^{(X)}.$$

To get (9), we remark that we have the tautological relation

(9')
$$f_!^{\mathrm{ab}}(\mathbb{Z}_{A'}) \simeq \mathbb{Z}^{(X)},$$

hence the map (9). To prove it is an isomorphism amounts to proving

$$\mathrm{L}_i f_!^{\mathrm{ab}}(\mathbb{Z}_{A'}) = 0 \quad \text{for } i > 0,$$

but we have indeed

(10)
$$\mathrm{L}_i f_!^{\mathrm{ab}} = 0 \quad \text{for } i > 0, \text{ i.e., } f_!^{\mathrm{ab}} \text{ is } \textit{exact},$$

[p. 564] not only right exact, a rather special feature, valid for a localization functor like $f : A_{/X} \to A$, namely for a functor which is fibering (in the sense of the theory of "fibered categories") and has discrete fibers. It comes from the explicit description of $f_!^{\mathrm{ab}}$, as

$$f_!^{\mathrm{ab}}(F) = \Big(a \mapsto \bigoplus_{u \in X(a)} F(a)_u\Big), \tag{11}$$

where, for a group object F in $A^\wedge{}_{/X} \simeq (A_{/X})^\wedge$, and u in $X(a)$ (defining therefore an object of $A_{/X}$) $F(a)_u$ = fiber of $F(a)$ at $u \in X(a)$ is the corresponding abelian group. I'll leave the proof of (11) to the reader, it should be more or less tautological.

Once the whole proof is written down, it looks so simple that I feel rather stupid and can't quite understand why I have turned around it for so long, rather than writing it down right away more than three months ago! The reason surely is that I have been accustomed so strongly to expressing everything via cohomology rather than homology, that there has been something like a block against doing work homologically, when it is homology which is involved. This block has remained even after I took the trouble of telling myself quite outspokenly (in section 100) that homology was just as important and meaningful as cohomology, and more specifically still (in section 103) that the proof I had in mind first, via Quillen's result about $A \simeq A^{\mathrm{op}}$ in (Hot) and via cohomology, was an "awkward", an "upside-down", one. The review on abelianization I went into just after made things rather worse in a sense, as there I took great pains to make the point that homology and cohomology were just one and the same thing (so why bother about homology!). Still, I *did* develop some typically homologically flavored formalism with the $*_k$ operation, and I hope that at the end that block of mine is going to recede...

27.10. [p. 565]

136 I just spent a couple of hours, after reading the notes of last night, trying to get a better feeling of the basic homology operation in the context of the basic localizer (Cat), namely taking the left derived functor $\mathrm{L}f_!^{\mathrm{ab}}$ of the "unusual" direct image functor for abelian presheaves, associated to a map

Preliminary perplexities about a full-fledged "six operations duality formalism" within (Cat).

$$f : A \to B$$

in (Cat). This led me to read again the notes of section 92, when I unsuspectingly started on an "afterthought, later gradually turning into a systematic reflection on abelianization. With a distance of nearly four months, what strikes me most now in these notes is awkwardness of the approach followed at start, when yielding to the reflex of laziness of describing abelianization of homotopy types via the semisimplicial grindmill. The uneasiness in these notes is obvious throughout – I kind of knew "au fond" that dragging in the particular test category $\triangle$ was rather silly. In section 100 only, does it get clear that the best description for abelianization, with the modelizer (Cat), is via the unusual direct image $p_!^{\mathrm{ab}}$ corresponding to the projection

["au fond" – at the bottom – deep down]

$$p = p_A : A \to e$$

of the "model" A in (Cat) to the final object (formulæ (11) and (12) page 359), by applying $p_!^{\mathrm{ab}}$ to a "cointegrator" $L_\bullet^A$ for A, namely to a projective resolution of $\mathbb{Z}_A$ (where A is written B by the way, as I had been led before to replace a given A in (Cat) by its "dual" or opposite $B = A^{\mathrm{op}}$, bound as I was for interpreting "integrators" for A in terms of "cointegrators" for B...); and in the next section the step is finally taken (against the "block"!) to write

$$p_!^{\mathrm{ab}}(L_\bullet^B) = \mathrm{L}p_!^{\mathrm{ab}}(\mathbb{Z}_B),$$

which inserts abelianization into the familiar formalism of (left) derived functors of standard functors. The reasonable next thing to do was of course what I finally did only yesterday, namely check the commutativity of the diagram (7) of p. 563, namely compatibility of this notion of abelianization (or homology) with Whitehead's abelianization, when working with models coming from $A\hat{}$, A any object in (Cat). This by the way, when applied to the case $A = \mathbb{\Delta}$, gives at once the equivalence of the intrinsic definition of abelianization, with the semisimplicial one we started with – provided we remark that in this case, $\mathrm{L}p_!^{\mathrm{ab}}$ (rather, its
[p. 566] restriction to $\mathbb{\Delta}\hat{}_{\mathrm{ab}}$) may be equally interpreted as the Kan-Dold-Puppe functor (more accurately, the composition of the latter with the localization functor $\mathrm{Ch}_\bullet(\mathrm{Ab}) \to \mathrm{D}_\bullet(\mathrm{Ab}) = \mathrm{Hotab}$). The equivalence of the two definitions of abelianization is mentioned on the same p. 369, somewhat as a chore I didn't really feel then to dive into. Besides the "block" against homology, the picture was being obscured, too, by the "computational" idea I kept in mind, of expressing $\mathrm{L}p_!^{\mathrm{ab}}(F)$ for F in $A\hat{}_{\mathrm{ab}}$ in terms of an "integrator" for A, i.e., as $F *_{\mathbb{Z}} L_\bullet^B$ – whereas I should have known best myself that for establishing formal properties relating various derived functors, the particular approaches used for "computing" them more or less elegantly are wholly irrelevant...

One teaching I am getting out of all this, is that when expressing abelianization, or presumably any other kind of notion or operation of significance for homotopy types, one should be careful, for any modelizer one chooses to work in, to dig out the description which fits smoothly those particular models. Clearly, when working with semisimplicial models, the description via tautological abelianization $\mathrm{Wh}_{\mathbb{\Delta}}$ and using Kan-Dold-Puppe is the best. When working within the modelizer (Cat), though, making the detour through $\mathbb{\Delta}\hat{}$ is awkward and makes us just miss the relevant facts. Once we got this, it gets clear, too, what to do when $\mathbb{\Delta}$ is replaced by any other object A in (Cat) when taking models in $A\hat{}$ (never minding even whether $A\hat{}$ is indeed a "modelizer"): namely, apply Wh_A, and take total homology!

This brings to my mind another example, very similar indeed. Some time after I got across Thomason's nice paper, showing that (Cat) is a closed model category (see comments in section 87), I got from Tim Porter another reprint of Thomason's, where he gives the description of
[Thomason (1979)] "homotopy colimits" (or "integration", as I call it) within the modelizer (Cat), in terms of the total category associated to a fibered category (compare section 69). There he grinds through a tedious, highly technical proof, whereas the direct proof when describing "integration" as the

left adjoint functor to the tautological "inverse image" functor, is more or less tautological, too. The reason for this awkwardness is again that, rather than being content to work with the models as they are, Thomason refers to the Bousfield-Kan description of colimits in the semisimplicial set-up which he takes as his definition for colimits. I suspect that (Cat) is the one modelizer most ideally suited for expressing the "integration" operation, and that the Bousfield-Kan description is just the obvious, not-quite-as-simple one which can be deduced from the former, using the relevant two functors between (Cat) and $\mathbb{\Delta}^\wedge$ which allow to pass from one type of models to another. (Sooner or later I should check in Bousfield-Kan's book whether this is so or not. . .) Thomason, however, did the opposite, and it is quite natural that he had to pay for it by a fair amount of sweat! (Reference of the paper: Homotopy colimits in the category of small categories, Math. Proc. Cambridge Philos. Soc. (1979), 85, p. 91–109.) [Bousfield and Kan (1972)] [p. 567]

The proof written down yesterday for compatibility of abelianization of homotopy types with the Whitehead abelianization functor within a category $A^\wedge$, still goes through when replacing abelianization by k-linearization, with respect to an arbitrary ground ring k (not even commutative). It wasn't really worth while, though, to introduce a ring k, as the general result should follow at once from the case $k = \mathbb{Z}$, by the formula

$$\text{(1)} \qquad \mathrm{L}p_!^k(k_A) \simeq \mathrm{L}p_!^{\mathbb{Z}}(\mathbb{Z}_A) \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}} k,$$

where in the left-hand side we are taking the left derived functor for the functor

$$p_!^k : A_k^\wedge \to (\mathrm{Ab}_k)$$

generalizing $p_!^{\mathrm{ab}} = p_!^{\mathbb{Z}}$ (with $p : A \to e$ as above), and where in the right-hand side we are using the ring extension functor

$$\overset{\mathrm{L}}{\otimes}_{\mathbb{Z}} : \mathrm{D}_\bullet(\mathrm{Ab}) \to \mathrm{D}_\bullet(\mathrm{Ab}_k)$$

for the relevant derived categories. This reminds me of the need of developing a more or less exhaustive formulaire around the basic operations

$$\text{(2)} \qquad \mathrm{L}f_!, \quad f^*, \quad \mathrm{R}f_*,$$

including the familiar one for the two latter, valid more generally for maps between ringed topoi, and including also a "*projection formula*" generalizing (1). Such a formula will be no surprise, surely, to a reader familiar with a duality context (such as étale cohomology, or "coherent" cohomology of noetherian schemes, say), where a formalism of the "four variances" $f_!$, f^*, f_*, $f^!$ and the "two internal operations" $\overset{\mathrm{L}}{\otimes}$ and R**Hom** can be developed – it would seem that the formal properties of the triple (2), together with the two internal operations just referred to, are very close (for the least) to those of the slightly richer one in duality set-ups, including equally an "unusual inverse image" $f^!$, right adjoint to $\mathrm{R}f_*$

[p. 568] (denoted sometimes simply by f_*). This similarity is a matter of course, as far as the two last among the functors (2), together with the two internal operations, are concerned, as in both contexts (homology and cohomology formalism within (Cat) on the one hand, and the "sweeping duality formalism" on the other) the formulaire concerning these four operations

$$f^*, \quad \mathrm{R}f_*, \quad \overset{\mathrm{L}}{\otimes}, \quad \mathrm{R}\mathbf{Hom}$$

is no more, no less than just the relevant formulaire in the context of arbitrary (commutatively) ringed topoi, and maps between such. The common notation $f_!$ (occurring in $\mathrm{L}f_!$ in the (Cat) context, in $\mathrm{R}f_!$ in the "duality" context) is a very suggestive one, for the least, and I am rather confident that most reflexes (concerning formal behavior of $f_!$ with respect to the other operations) coming from one context, should be OK too in the other. Whether this is just a mere formal analogy, or whether there is a deeper relationship between the two kinds of contexts, I am at a loss at present to say. It doesn't seem at all unlikely to me that among arbitrary maps in (Cat), one can single out some, by some kind of "finite type" condition, for which a functor

$$(*) \qquad f^! : \mathrm{D}^+(B_k^\wedge) \to \mathrm{D}^+(A_k^\wedge)$$

can be defined (where $f : A \to B$), right adjoint to the familiar $\mathrm{R}f_*$ functor, so that (2) can be completed to a sequence of *four* functors

$$(3) \qquad \mathrm{L}f_!, \quad f^*, \quad \mathrm{R}f_*, \quad f^!,$$

forming a sequence of mutually adjoint functors between derived categories, in the usual sense (the functor immediately to the right of another being its right adjoint). I faintly remember that Verdier worked out such a formalism within the context of discrete or profinite groups, or both, in a Bourbaki talk he gave, this being inspired by the similar work he did within the context of usual topological spaces. In the latter, the finiteness condition required for a map

$$f : X \to Y$$

of topological spaces to give rise to a functor $f^!$ (between, say, the derived categories of the categories of abelian sheaves on Y and X) is mainly that X should be locally embeddable in a product $Y \times \mathbb{R}^d$ – a rather natural condition indeed! As we are using (Cat) as a kind of algebraic paradigm for the category of topological spaces, this last example for "sweeping duality" makes it rather plausible that something of the same kind should exist indeed in (Cat) – and likewise for the
[p. 569] context of groups (which we may view as just particular cases of models in (Cat)).

I am sorry I was a bit confused, when describing $f^!$ as a right adjoint to $\mathrm{R}f_*$ – I was thinking of the analogy with a map of schemes, or spaces, which is not only "of finite type" in a suitable sense, but moreover *proper* – in which case, in those duality contexts, $\mathrm{R}f_*$ is canonically isomorphic with the functor denoted by $f_!$ or $\mathrm{R}f_!$. Otherwise, the ("non-trivial") "duality theorem" will assert, rather, that the pair

$$\mathrm{R}f_!, \quad f^!$$

is a pair of adjoint functors, just as is the pair

$$f^*, \quad \mathrm{R}f_* \quad \text{(or simply } f_*\text{)}.$$

But even when f is assumed to be proper, the sequence (3) isn't a sequence of adjoint functors in the standard duality contexts, namely $\mathrm{R}f_!$ is by no means left adjoint to f^*, i.e., f^* isn't isomorphic to $f^!$ (except in extremely special cases, practically I would think only étale maps are OK, which in the context of (Cat) would correspond to maps in (Cat) isomorphic to a map $A_{/X} \to A$ for X in $A^\wedge$, namely maps which are fibering with discrete fibers). This *does* make an important discrepancy indeed, between the two kinds of contexts – and increases my perplexity, as to whether or not one should expect a "four variance duality formalism" to make sense in (Cat). If so, presumably the $f_!$ or $\mathrm{R}f_!$ it will involve (perhaps via a suitable notion of "proper" maps in (Cat), as already referred to earlier (section 70)) will be different after all from the $\mathrm{L}f_!$ we have been working with lately, embodying homology properties. But so does $\mathrm{R}f_!$ too, in a rather strong sense, via the "duality theorem"!

In the various duality contexts, a basic part is played by the three particular classes of maps: proper maps, smooth maps, and immersions, and factorizations of maps into an immersion followed by either a proper, or a smooth map. In the context of Cat, there is a very natural way indeed to define the three classes of maps, as we'll see in the next chapter, presumably – so natural indeed, that it is hard to believe that there may be any other reasonable choice! One very striking feature (already mentioned in section 70) is that the two first notions are "dual" to each other in the rather tautological sense, namely that a map $f : A \to B$ in (Cat) is proper iff the corresponding map $f^{\mathrm{op}} : A^{\mathrm{op}} \to B^{\mathrm{op}}$ is smooth (whereas the notion of an immersion is autodual). How does this fit [p. 570] with the expectation of developing a "four variance" duality formalism within (Cat)? It rather heightens perplexity at first sight! Proper maps include cofibrations (in the sense of category theory, not in Kan-Quillen's sense!); dually, smooth maps include fibrations. Consequently, maps which are moth smooth and proper include bifibrations, and hence are not too uncommon. Now, how strong a restriction is it for a map $f : A \to B$ in (Cat) to factor into

$$f = p \circ i,$$

where i is an immersion $A \to A'$ (namely, a functor identifying A to a full subcategory of A', whose essential image includes, with any two objects x, y, any other z which is "in between": $x \to z \to y$), and p is both proper and smooth (a bifibration, say)?

I start feeling like a battle horse scenting gunpowder again – still, I don't think I'll run into it. Surely, there is something to be cleared up, and perhaps once again a beautiful duality formalism with the six operations and all will emerge out of darkness – but this time I will not do the pulling. Maybe someone else will – if he isn't discouraged beforehand, because the big-shots all seem kind of blasé with "big duality", derived categories and all that. As for my present understanding, I feel that the question isn't really about homotopy models, or about foundations of homotopy and cohomology formalism – at any rate, that I definitely don't need this kind of stuff, for the program I have been out for. I shouldn't refrain, of course, to pause on the way every now and then and have a look at the landscape, however remote or misty – but I am not going to forget I am bound for a journey, and that the journey should not be an unending one. . .

27.10.

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Looking for the relevant notions of properness and smoothness for maps in (Cat)*. Case of ordered sets as a paradigm for cohomology theory of conically stratified spaces.*

[p. 571]

It occurred to me that I have been a little rash yesterday, when asserting that the notions of "smoothness" and "properness" for maps in (Cat) which I hit upon last Spring is the only "reasonable" one. Initially, I referred to these notions by the names "cohomologically smooth", "cohomologically proper", as a measure of caution – they were defined by properties of commutation of base change to formation of the Leray sheaves $\mathrm{R}^i f_*$ (i.e., essentially, to "cointegration"), which were familiar to me for smooth resp. proper maps in the context of schemes, or ordinary topological spaces. These cohomological counterparts of smoothness and properness fit very neatly into the homology and cohomology formalism, and I played around enough with them, last Spring as well as more than twenty years ago when developing étale cohomology of schemes, for there being no doubt left in my mind that these notions are relevant indeed. However, I was rather rash yesterday, while forgetting that these cohomological versions of smoothness and properness are considerably weaker than the usual notions. Thus, in the context of schemes over a ground field, the product of any two schemes is cohomologically smooth over its factors – or equivalently, any scheme over a field k is cohomologically smooth over k! Similarly, in the context of (Cat), any object in (Cat), namely any small category, is both cohomologically smooth and proper over the final object e (as it is trivially "bifibered" over e). On the other hand, it isn't reasonable, of course, to expect any kind of Poincaré-like duality to hold for the cohomology (with twisted coefficients, say) of an arbitrary object A in (Cat). To be more specific, it is easy to see that in many cases, the functor

$$\mathrm{R}f_* : \mathrm{D}^+(A^\wedge_{\mathrm{ab}}) \to \mathrm{D}^+(\mathrm{Ab})$$

does not admit a right adjoint (which we would call $f^!$). For instance, when A is discrete, then $f^!$ exists (and may then be identified with f^*) iff A is moreover *finite* – a rather natural condition indeed, when we keep in mind the topological significance of the usual notion of properness! This immediately brings to mind some further properties besides base change properties, which go with the intuitions around properness –

for instance, we would expect for proper f, the functors f_* and $\mathrm{R}f_*$ to commute to filtering direct limits, and the same expectation goes with the assumption that $\mathrm{R}f_*$ should admit a right adjoint. This exactness property is not satisfied, of course, when A is discrete infinite. We now may think (still in case of target category equal to e) to impose the drastic condition that the category A is finite. Such restriction however looks in some respects too weak, in others too strong. Thus, it will include categories defined by finite groups, which goes against the rather natural expectation that properness + smoothness, or any kind of Poincaré duality, should go with *finite cohomological dimension*. On the other hand, there are beautiful infinite groups (such as the fundamental group of a compact surface, or of any other compact variety that is a $K(\pi,1)$ space...) which satisfy Poincaré duality.

These reflections make it quite clear that the notions of properness and of smoothness for maps in (Cat), relevant for a duality formalism, have still to be worked out. Two basic requirements to be kept in mind are the following: 1) for a proper map $f : A \to B$, and any ring of coefficients k, the functor [p. 572]

$$\mathrm{R}f_* : \mathrm{D}^+(A_k^\wedge) \to \mathrm{D}^+(B_k^\wedge)$$

should admit a right adjoint $f^!$, and 2) for a smooth map f factored as $f = g \circ i$, with g proper and i an "open immersion", the composition $f^! \overset{\text{def}}{=} i^* \circ g^!$ should be expressible as

$$f^! : K^\bullet \mapsto f^*(K) \overset{\mathrm{L}}{\otimes}_k T_f(k)[d_f], \tag{5}$$

where $T_f(k)$ is a presheaf of k-modules on A locally isomorphic to the constant presheaf k_A ($T_f(k)$ may be called the *orientation sheaf* for f, with coefficients in k), and d_f is a natural integer (which may be called the *relative dimension* of A over B, or of f). (For simplicity, I assume in 2) that A is connected, otherwise d_f should be viewed as a function on the set of connected components of A.) This again should give the correct relationship, for f as above, between the (for the time being hypothetical) $\mathrm{R}f_!$ ($\overset{\text{def}}{=} \mathrm{R}g_* \circ \mathrm{L}f_!$) and our anodyne $\mathrm{L}f_!$, for an argument $L_\bullet$ in $\mathrm{D}^\bullet(A_k^\wedge)$ say:

$$\mathrm{R}f_!(L_\bullet) \simeq \mathrm{L}f_!(L_\bullet \otimes T_f^{-1})[-d_f], \tag{6}$$

where the left-hand side is just $\mathrm{R}f_*(L_\bullet)$, if we assume moreover f to be proper.

This precise relationship between the two possible versions of an $f_!$ operation between derived categories, namely $\mathrm{L}f_!$ embodying homology, defined for any map f in (Cat), and $\mathrm{R}f_!$ embodying "cohomology with proper supports", defined for a map that may be factored as $g \circ i$ with g "proper" and i an (open, if we like) immersion, relation valid if f is moreover assumed to be "smooth", greatly relaxes yesterday's perplexity, coming from a partial confusion in my mind between the operations $\mathrm{R}f_!$ and $\mathrm{L}f_!$. (Beware the notation $\mathrm{R}f_!$ is an abuse, as it doesn't mean at all anything like the right derived functor of the functor $f_!$!) At the same time, I feel a lot less dubious now about the existence of a "six operations" duality formalism in the (Cat) context – I am pretty much convinced, now, that such a formalism exists indeed. The main specific work ahead is to get hold of the relevant notions of proper and smooth [p. 573] maps. The demands we have on these, besides the relevant base change properties, are so precise, one feels, that they may almost be taken as a definition! Maybe even the "almost" could be dropped – namely that a comprehensive axiomatic set-up for the duality formalism could be worked out, in a way applicable to the known instances as well as to the presently still unknown one of (Cat), by going a little further still than Deligne's exposition in SGA 5 (where the notions of "smooth" and "proper" maps were supposed to be given beforehand, satisfying suitable properties). [Deligne (SGA $4\frac{1}{2}$, Dualité)] Before diving into such axiomatization game, one should get a better feeling, though, through a fair number of examples (not all with e as the target category moreover), of how the proper, the smooth and the proper-and-smooth maps in (Cat) actually look like. Here, presumably, Verdier's work in the context of discrete infinite groups should give useful clues.

Other important clues should come from the opposite side so to say – namely ordered sets. Such a set I, besides defining in the usual way a small category and hence a topos, may equally be viewed as a topological space, more accurately, the topos it defines may be viewed as being associated to a topological space, admitting I as its underlying set (cf. section 22, p. 18). This topological space is noetherian iff the ordered set I is – for instance if I is finite. In such a case, an old algebraic geometer like me will feel in known territory, which maybe is a delusion, however – at any rate, I doubt the duality formalism for topological spaces (using factorizations of maps $X \to Y$ via embeddings in spaces $Y \times \mathbb{R}^d$) makes much sense for such non-separated spaces. However, as we saw in section 22, when I satisfies some mild "local finiteness" requirement (for instance when I is finite), we may associate to it a *geometrical realization* $|I|$, which is a locally compact space (a compact one indeed if I is finite) endowed with a "conical subdivision" (index by the opposite ordered set I^{op}), hence canonically triangulated via the "barycentric subdivision". The homotopy type of this space is canonically isomorphic to the homotopy type of I, viewed as a "model" in (Cat). What is more important here, is that a (pre)sheaf of sets (say) on the category I may be interpreted as being essentially the same as a sheaf of sets on the geometric realization $|I|$ *which is locally constant* (*and hence constant*) *on each of the* "open" *strata or "cones" of* I. This description then carries over to sheaves of k-modules. The "clue" I had in mind is that *within the context of locally finite ordered sets, the looked-for "six operations duality formalism" should be no more, no less than the accurate reflection of the same formalism within the context of* (*locally compact*) [p. 574] *topological spaces*, as worked out by Verdier in one of his Bourbaki talks – it being understood that when applying the latter formalism to spaces endowed with conical stratifications, maps between these which are compatible with the stratifications (in a suitable sense which should still be pinned down), and to sheaves of modules which are compatible with the stratifications in the sense above, these will give rise (via the "six operations") to sheaves satisfying the same compatibility. (NB when speaking of "sheaves", I really mean complexes of sheaves $K^\bullet$, and the compatibility condition should be understood as a condition for the ordinary sheaves of modules $\mathrm{H}^i(K^\bullet)$.) This remark should allow to work out quite explicitly, in purely algebraic (or "combinatorial") terms, the "six operations" in the context of locally finite ordered sets, at any rate.

This interpretation suggests that an ordered set I should be viewed as a "proper" object of (Cat) iff I is finite. In the same vein, whenever I is "locally proper", namely locally finite, and moreover its topological realization $|I|$ is a "smooth" topological space in the usual sense, namely is a topological *variety* (for which it is enough that for every x in I, the topological realization

(7) $$|I_{>x}|$$

of the set of elements y such that $y > x$ should be a sphere), we would consider I as a "smooth" object in (Cat). If A is any object in (Cat) and I is an ordered set which is finite resp. "smooth" in the sense above, we will surely expect $A \times I \to A$ to be "proper" resp. "smooth" for the duality formalism we wish to develop in (Cat).

It should be kept in mind that for the algebraic interpretation above to hold, for sheaves on a conically stratified space $|I|$ in terms of an ordered set I, we had to take on the indexing set I for the strata the order relation *opposite* to the inclusion relation between (closed) strata – otherwise, the correct interpretation of sheaves constant on the open strata is via *covariant* functors $I \to$ (Sets), i.e., *presheaves on* I^{op} (not I). At any rate, as there is a canonical homeomorphism

(8) $$|I| \simeq |I^{\mathrm{op}}|$$

respecting the canonical barycentric subdivisions of both sides, notions for I (such as properness, or smoothness) which are expressed as intrinsic properties of the corresponding topological space $|I|$ (independently of its subdivision) are autodual – they hold for I iff they do for I^{op}. [p. 575]
This is in sharp contrast with the more naive notions of "cohomological" properness and smoothness via base change operations, which are interchanged by duality (which was part of yesterday's perplexities, now straightening out...).

I feel I should be a little more outspoken about the relevant notion of "proper maps" between ordered sets, which should be the algebraic expression of the geometric notion alluded to above, of a map between conically stratified spaces being "compatible with the conical stratifications". In order for the corresponding direct image functor for sheaves to take sheaves compatible with the stratification above to sheaves of same type below, we'll have to insist that the image by f of a strata above should be a whole strata below. Now, it is clear that any map

(9) $$f : I \to J$$

between ordered sets takes flags into flags, and hence induces a map

$$|f| : |I| \to |J| \tag{10}$$

between the geometric realizations, compatible with the barycentric triangulations and thus taking simplices into simplices. But even when I and J are finite (hence "proper"), the latter map does not always satisfy the condition above. Thus, when I is reduced to just one point, hence (9) is defined by the image $j \in J$ of the latter, the corresponding map (10) maps the unique point of $|I|$ to the barycenter of the stratum C_j in $|J|$, which is a stratum of $|J|$ iff C_j is a minimal stratum, i.e., j is a *maximal* element in J. A natural algebraic condition to impose upon f, in order to ensure that $|f|$ satisfy the simple geometric condition above, is that for any x in I, and any y' in J such that

$$y' > y \stackrel{\text{def}}{=} f(x),$$

there should exist an x' in I satisfying

$$x' > x \quad \text{and} \quad f(x') = y'. \tag{11}$$

This condition strongly reminds us of the valuative criterion for properness in the context of schemes, where the relation $y' \geq y$ or $y \to y'$, say, should be interpreted as meaning that y' is a *specialization* of y. However, in the valuative criterion for properness (for a map of preschemes of finite type over a noetherian prescheme say), if one wants actual
[p. 576] properness indeed (including separation of f, not just that f is universally closed), one has to insist that the x' above should be unique: every specialization y' of $y = f(x)$ lifts to a *unique* specialization x' of x. If we applied this literally in the present context, this would translate into the condition that $|f|$ should map *injectively* each closed stratum C_x of $|I|$ – which would exclude such basic maps as the projection $I \to e$ to just one point!

One may wonder why trouble about the analogy with algebraic geometry and any extra condition on f besides the one we got. The point, however, is that we would like the map

$$C_x = |I_{\geq x}| \to C_y = |I_{\geq y}| \tag{12}$$

analogy: aspheric fibers! [I *think* that's what the footnote says; it's hard to read…]

between corresponding strata induced by $|f|$ to be "cohomologically trivial" in a suitable sense, not only surjective – in analogy, say, with the usual notion of maps between triangulated spaces; if we don't have some condition of this type, we will have no control over the structure of the direct image and the higher direct images $\mathrm{R}^i|f|_*$ of an "admissible" sheaf upstairs. I didn't really analyze the situation carefully, in terms of what we are after here in the context of those stratified geometric realizations. I feel pretty sure, though, that there the general notion of "cohomological properness" which I worked out last Spring fits in just right, to give the correct answer. The criterion I obtained (necessary and sufficient for the relevant compatibility property of $\mathrm{R}^i f_*$ with arbitrary base change $J' \to J$ in (Cat)) reads as follows: let

$$I(f,x,x') = \{x' \in I \mid x' > x \text{ and } f(x') = y'\} \tag{13}$$

be the subset of I satisfying the conditions (11) above. Instead of demanding only that this set be non-empty, or going as far as demanding that it should be reduced to just one point, we'll demand that this set should be *aspheric*. (When we are concerned with cohomology with commutative coefficients only, presumably it should be enough to demand only that this set be *acyclic* – which should be enough for the sake of a mere duality formalism...) In more geometric terms, this should mean, I guess, that the inverse image, by the map (12) above between closed strata, of any closed stratum below, should be aspheric. I doubt this condition holds under the mere assumption that the sets (13) be non-empty, i.e., (12) be surjective – I didn't sit down, though, to try and make an example.

Thus, we see that when working with the notion of properness of maps, even for such simple gadgets as finite ordered sets, which should be viewed as "proper" (or "compact") objects by themselves, this notion is far from being a wholly trivial one – for instance, it does not hold true that any map between such "proper" objects is again "proper". This now seems to me just a mathematical "fact of life", which we may not disregard when working with finite ordered sets, say, in view of expressing in algebraic terms some standard operations in the cohomology theory of sheaves on topological spaces, endowed with suitable conical stratifications. The fault, surely, is not with the notion of conical stratification itself, which may be felt by some as being ad hoc, awkward and what not. I know the notion is just right – but at any rate, even when working with strata which are perfect topological cells (so that nobody could possibly object to them), exactly the same facts of life are there – not every map between such cellular decompositions, mapping cells *onto* cells, will fit into a "combinatorial" description when it comes to describing the standard operations of the cohomology of sheaves, for sheaves compatible with the stratifications... [p. 577]

To sum up, it seems to me that definitely, there is a very rich experimental material available already at present, in order to come to a feeling of what duality is like in the context of (Cat), and for developing some of the basic intuitions needed for working out, hopefully, "the" full-fledged duality theory which should hold in (Cat). Before leaving this question, I would like to point out still one other property connected with the intuitions around "properness" – more generally, around maps "of finite type" in a suitable sense, which sometimes may translate into: factorizable as a composition $f = g \circ i$, with g proper and i an immersion. This is about stability of "constructibility" or "finiteness" conditions for (complexes of) sheaves of k-modules, with respect to the standard operations $\mathrm{R}f_*$, $\mathrm{R}f_!$, $f^!$ (stability by f^* being a tautology in any case). Such stability of course, whenever it holds, is an important feature, for instance for making "virtual" calculations in suitable "Grothendieck groups" (where Euler-Poincaré type invariants may be defined). It should be recalled, however, that the six fundamental operations in the duality formulaire, as well as nearly all of the formulaire itself, make sense (and formulæ hold true) without any finiteness conditions on the complexes of sheaves we work with, except just boundedness conditions on the degrees of those complexes.

[p. 578] **Remarks.** 1) One of the (rather few) instances in the duality formulaire where finiteness conditions are clearly needed, is the so-called "biduality theorem", when taking R**Hom**'s with values in a so-called "dualizing complex", which here should be an object

$$R^\bullet \quad \text{in } \mathrm{D}^{\mathrm{b}}(A_k^\wedge)$$

(for given A in (Cat) and given ring of coefficients k, for instance $k = \mathbb{Z}$). The question arises here, for any given A and k, a) whether there exists a dualizing complex $R^\bullet$ (which, as usual, will be unique up to dimension shift and "twist" by an invertible sheaf of k-modules on A), and b) can such dualizing complex be obtained as

$$R^\bullet = p^!(k_e),$$

where $p : A \to e$ is the projection to the final object of (Cat)? It should be easy enough, once the basic duality formalism is written down for ordered sets, as contemplated above, to show that when A is a finite ordered set (or only locally finite of finite combinatorial dimension), then $p^!(e_k)$ is indeed a dualizing complex. To refer to something more "en vogue" at present than those poor dualizing complexes, it is clear, from what I heard from Illusie and Mebkhout about the "complexe d'intersection" for stratified spaces, that this complex can be described also in the context of locally finite ordered sets (if I got the stepwise construction of this complex right). As the construction here corresponds to stratifications where the strata are by no means even-dimensional, I am not too sure, though, if the complex obtained this way is really relevant – it isn't a topological invariant of the topological space at any rate, independently of its stratification – a bad point indeed. Too bad!

2) Here is a rather evident example showing that the asphericity condition on (13) is not automatic. Take I with a smallest element x (hence $C_x = |I|$), and $J = \mathbb{\Delta}_1 = (0 \to 1)$. To give a map $I \to J$ amount to give $I_0 = f^{-1}(0) \subset I$, which is any open subset of I (i.e., such that $a \in I_0$, $b \leq a$ implies $b \in I_0$). If we take $I_0 = \{x\}$, then properness of f is equivalent with $I_1 = I \setminus I_0 = I \setminus \{x\}$ being aspheric, while the weaker condition contemplated first (the sets (13) non-empty) means only that $I_1 \neq \emptyset$. Now, I_1 may of course be taken to be any (finite say) ordered set – the construction made amounting to taking the cone over the map $|I_1| \to e$. In this examples, all fibers of $|f| : |I| \to |J|$ are (canonically)
[p. 579] homeomorphic to I_1, except the fiber at the "barycenter" 1, which is reduced to a point – visibly not a very "cellular" behavior when $|I_1|$ isn't aspheric! At any rate, as the sheaves $\mathrm{R}^i|f|_*(F)$ (for F a constant sheaf above, say) may be computed fiberwise, we see that if I_1 isn't aspheric, these sheaves (which are constant on $|J| \setminus \{1\}$) are *not* going to be constant on $|J| \setminus \{0\}$, as they should if we want an algebraic paradigm of operations like $\mathrm{R}|f|_*$ in terms of sheaves on finite ordered sets.

In this example we could take $|I|$ to be a perfect n-cell ($n \geq 1$), hence I_1 is an $(n-1)$-sphere, whereas $|J|$ isn't really a (combinatorial) 1-cell, as its boundary has just *one* point 0, instead of two. The 1-cell structure corresponds to the ordered set (*opposite* to the ordered set formed by the two endpoints and the dimension 1 stratum)

$$J = \begin{tikzcd}[column sep=small] & y \arrow[dl] \arrow[dr] & \\ 0 & & 1 \end{tikzcd}.$$

For any ordered set I, to give a map $f : I \to J$ amounts to giving the two subsets

$$I_0 = f^{-1}(0), \quad I_1 = f^{-1}(1),$$

subject to the only condition of being open and disjoint. In terms of strata of $|I|$, this means that we give two sets I_0, I_1 of strata, containing with any stratum any smaller one, and having no stratum in common. The condition that the sets (13) should be non-empty says that any point in I which is neither in I_0 nor I_1 admits majorants in both – or geometrically, any stratum which isn't in I_0 nor in I_1 meets both $|I_0|$ and $|I_1|$, i.e., contains strata which are in I_0 and strata which are in I_1. Even when $|I|$ is a combinatorial 2-cell, i.e., a polygonal disc, this condition does not imply asphericity of the sets (13) (not even 0-connectedness). To see this, we take the set (13) where x is the dimension 2 stratum, i.e., the smallest element of I, mapped to y (the smallest in J), and y' either 0 or 1. The reader will easily figure out on a drawing the structure of the map $|f|$, as I just did myself: the fibers at the endpoints of the segment $|J|$ are, as given, discrete with cardinal m, the fiber at a point different from the endpoints and from their barycenter are disjoint sums of m segments (hence homotopic to the former fibers), whereas the fiber at the barycenter is the union of m segments meeting in their common middle, hence is contractible. Thus, the $\mathrm{R}^0|f|_*$ of a constant sheaf on the disc $|I|$ is by no means constant on the open, dimension one stratum of $|J|$. [p. 580]

These examples bring to my mind that for any map $f : I \to J$ between finite ordered sets, possibly submitted to the mild restriction that the sets (13) should be non-empty, the homotopy types of those ordered sets (13) (for the order relation induced by I) should be exactly the homotopy types of the fibers of the maps (12) between strata. Thus, asphericity of these ordered sets should express no more, no less than the contractibility of those fibers. This latter condition is exactly what is needed in order to ensure stability by $\mathrm{R}|f|_*$ of the notion of complexes of sheaves compatible with the stratifications.

28.10.

138 *Niceties and oddities:* R$f_!$ *commutes to colocalization, not localization.*

Yesterday and the day before, I got involved in that unforeseen digression around the foreboding of a "six operations duality formalism" in (Cat), and suitable notions for smoothness and (more important still) of properness for a map in (Cat). This digression wholly convinced me that the usual duality formalism should hold in (Cat) too. Working this out in full detail should be a most pleasant task indeed, and presumably the best, or even the only way for gaining complete mastery of the cohomology formalism within (Cat) or, what more or less amounts to the same, for topoi which admit sufficiently many projective objects. In the previous two sections, I referred to such duality formalism as one concerned with sheaves of k-modules, for any fixed ring k – but from the example of étale duality for schemes, say, it is likely that instead of fixing a ring k, we may as well take objects A in (Cat) endowed with an arbitrary sheaf of rings $\mathcal{O}_A$ (which we'll only have to suppose commutative when concerned with the two internal operations $\overset{\mathrm{L}}{\otimes}$ and R**Hom**), and taking maps of such ringed objects as the basic maps. In the present context, the usual "six operations" in duality theory will be enriched, however, by still another one, namely L$f_!$, defined for any map f between such ringed objects (not to be confused with the R$f_!$ operation, defined only under suitable finiteness assumptions, such as "properness", for the underlying map in (Cat)), whose relationship to the other operations should be understood and added to the standard duality formulaire.
[p. 581] One such formula, namely the precise relation between L$f_!$ and R$f_!$ for a smooth f, was given yesterday (p. 572 (6)).

The day before, I was out for trying to get a better understanding of the R$f_!$ operation (including the case of non-constant sheaves of rings for the modules we work with). This operation still remains unfamiliar to me, very unlike my old friend Rf_* – there is a number of things about it which are not quite clear yet in my mind, even when just taking the functor $f_!$ between modules, before taking a left derived functor. For instance, for general rings of operators $\mathcal{O}_A$ and $\mathcal{O}_B$, when f is a ringed map

(1) $$f : (A, \mathcal{O}_A) \to (B, \mathcal{O}_B),$$

it doesn't seem that formation of $f_!$ commutes to restriction of rings of operators (to the constant rings $\mathbb{Z}_A$ and $\mathbb{Z}_B$, say), namely that for a given $\mathcal{O}_A$-module F, $f_!(F)$ may be interpreted as just $f_{0!}^{\mathrm{ab}}(F)$ with suitable operations of $\mathcal{O}_B$ on the latter, where

(2) $$f_0 : A \to B$$

is the map in (Cat) underlying f. When reducing to a suitable "universal" case, this may be expressed by saying that for given map f_0 and given abelian sheaf F on A, if we define

$$\mathcal{O}_A = \mathbf{End}_{\mathbb{Z}}(F), \quad \mathcal{O}_B = f_*(\mathcal{O}_A),$$

there doesn't seem to be a natural operation of $\mathcal{O}_B$ upon $f_{0!}^{\mathrm{ab}}(F)$; all we can say, it seems, is that the ring of global sections of $\mathcal{O}_B$ operates on $f_{0!}^{\mathrm{ab}}(F)$. More generally, reverting to the general case of a map f of ringed objects in (Cat), the ring of global sections of $\mathcal{O}_B$ operates on $f_{0!}^{\mathrm{ab}}(F)$. When $\mathcal{O}_B$ is a constant sheaf of rings k_B, this implies of course that k operates on this abelian sheaf on B, from which will follow by an obvious argument that with this structure of a sheaf of k-modules, $f_{0!}^{\mathrm{ab}}(F)$ may indeed be identified with $f_!(F)$.

Going over to $\mathrm{L}f_!$ which we would like to express via $\mathrm{L}f_{0!}^{\mathrm{ab}}$, the situation is worse, as we still have to check (granting $\mathcal{O}_B$ to be constant) that for F a projective $\mathcal{O}_A$-module, we got

$$\mathrm{L}_i f_{0!}^{\mathrm{ab}}(F) = 0 \quad \text{for } i > 0. \tag{3}$$

This isn't always true, even when B is the final object in (Cat), and A has a final object, hence $\mathcal{O}_A$ is a projective module over itself, and (3) reads [p. 582]

$$\mathrm{H}_i(A, \mathcal{O}_A) = 0 \quad \text{for } i > 0,$$

which isn't always true. (NB if it were for any commutative ring $\mathcal{O}_A$ on A, it would be too for any abelian sheaf M on A, as we see by taking $\mathcal{O}_A = \mathbb{Z}_A \otimes M$, hence A would be homological dimension 0, a drastic restriction, indeed, even when A has a final object.)

When however $\mathcal{O}_A$ is equally a constant sheaf of rings, say $\mathcal{O}_A = k'_A$, then the relation (3) holds for any projective module on A. We need only check it for

$$F = \mathcal{O}_A^{(a)} = k'^{(a)},$$

with a in A, then (3) follows from

$$\mathrm{L}_i f_{0!}^{\mathrm{ab}}(M^{(a)}) = 0 \quad \text{for } i > 0,\ M \text{ in (Ab)}, \tag{4}$$

To check (4), we take a projective resolution of $M^{(a)}$, by using a projective resolution $L_\bullet$ of M in (Ab) and taking $L_\bullet^{(a)}$ (using the fact that the functor

$$L \mapsto L^{(a)} : (\mathrm{Ab}) \to A^\wedge_{\mathrm{ab}}$$

is exact). We then get

$$\mathrm{L}f_{0!}(M^{(a)}) = f_{0!}(L_\bullet^{(a)}) = L_\bullet^{(b)} \simeq M^{(b)},$$

where $b = f_0(a)$, and where the last equality stems from exactness of $L \mapsto L^{(b)}$. (NB the relation (4) generalizes a standard acyclicity criterion in the homology theory of discrete groups...)

Thus, we get finally that in case both rings $\mathcal{O}_A$ and $\mathcal{O}_B$ are constant, that formation of $\mathrm{L}f_!$ commutes to restriction of operator rings (provided the rings to which we are restricting are constant too – say they are just the absolute $\mathbb{Z}_A$ and $\mathbb{Z}_B$). Presumably, a little more care should show the similar result for locally constant rings.

Reverting to the case (1) of a general map between ringed objects in (Cat), our inability, for a given $\mathcal{O}_A$-module F on A, to define an operation of $\mathcal{O}_B$ upon $f_{0!}^{\mathrm{ab}}$ (while there is an operation of the ring $\Gamma(B, \mathcal{O}_B)$ upon it), is tied up with this difficulty, that formation of $f_{0!}^{\mathrm{ab}}$, and a fortiori of $f_!$ for arbitrary rings $\mathcal{O}_A$ and $\mathcal{O}_B$, does not commute to "localization" (as [p. 583] f_* and $\mathrm{R}f_*$ does), namely to base change of the type

$$B_{/b} \to B, \tag{5}$$

where b is a given object in B, and $B_{/b}$ designates as usual the category of all "objects over b" in B, i.e., of all arrows in B with target b. This gives rise to the cartesian square in (Cat)

$$\begin{array}{ccc} A_{/b} & \longrightarrow & A \\ \downarrow & & \downarrow \\ B_{/b} & \longrightarrow & B \end{array}, \tag{6}$$

where $A_{/b}$ is the category of all pairs

$$(a, u) \text{ with } u : f_0(a) \to b,$$

which may be identified equally with the category $A_{/f_0^*(b)}$. The commutation property we have in mind is a tautology for f_*, and it follows for $\mathrm{R}f_*$, because the inverse image by $A_{/b} \to A$ of an injective module on A is an injective module on $A_{/b}$. The latter fact is true, more generally, for any "localization map", of the type $A_{/X} \to A$, with X in $A^\wedge$, i.e., any map which is fibering with discrete fibers. Thus, the commutation property for $\mathrm{R}f_*$ is valid more generally for any base change of the type

$$B_{/Y} \to B,$$

with Y in $B^\wedge$ (not necessarily in B). More generally still, it can be shown to hold for any map

$$B' \to B$$

which is fibering (not necessarily with discrete fibers).

On the other side of the mirror, when taking $f_!$ and its left derived functor, already the former definitely does *not* commute to base change of the type (5), i.e., to "localization on the base" – something a little hard to get accustomed to! The base changes which will do here, are those of the dual type

$$_{b\backslash}B \to B, \tag{5'}$$

we may call them maps of "colocalization" on the base B. It gives rise to a cartesian diagram in (Cat) dual to (6)

$$\begin{array}{ccc} _{b\backslash}A & \longrightarrow & A \\ \downarrow & & \downarrow \\ _{b\backslash}B & \longrightarrow & B \end{array}, \tag{6'}$$

[p. 584] where now $_{b\backslash}A$ is the category of pairs

$$(a,u) \text{ with } u : b \to f_0(a).$$

We'll have to assume now that both shaves of rings $\mathcal{O}_A$, $\mathcal{O}_B$ are constant, and correspond to the same ring k. Thus, modules on A and B are just contravariant functors from these categories to the category Ab_k of k-modules, and accordingly, $f_!$ (left adjoint to the composition functor f^*) may be computed by a well-known formula, involving direct limits on the categories ${}_{b\backslash}A$:

$$f_!(F)(b) \simeq \varinjlim_{{}_{b\backslash}A} F(a), \tag{7}$$

where the limit in the second member is relative to the composition

$${}_{b\backslash}A \to A \xrightarrow{F} \mathrm{Ab}_k.$$

[It seems to me that we're forgetting that F is contravariant. Should be easily fixed by inserting $^{\mathrm{op}}$'s, though...]

(NB This formula is dual to the formula for f_*, the right adjoint of f^*, closer to intuition – to mine at any rate – because the $\varprojlim$ in

$$f_*(F)(b) \simeq \varprojlim_{A_{/b}} F(a)$$

may be "visualized" as the set of "sections" of F over $A_{/b}$.) From this formula (7) follows at once commutation of $f_!$ with colocalization. To get the corresponding result for $\mathrm{L}f_!$, we have only to use the fact that the relevant inverse image functor (corresponding to ${}_{b\backslash}A \to A$) transforms projective modules into projective modules. As the map ${}_{b\backslash}B \to B$ is a *cofibering functor with discrete fibers*, so is the map

$$h : A' = {}_{b\backslash}A \to A$$

deduced by base change. Now, for any such functor $h : A' \to A$ between small categories, the inverse image functor

$$h^* : A^\wedge \to A'^\wedge, \quad \text{or} \quad h_k^* : A_k^\wedge \to A_k'^\wedge$$

carries indeed projectives into projectives. This statement is dual formally to the corresponding statement for injectives, valid when we make on h the dual assumption of being *fibering with discrete fibers* – in the latter case the (well-known) proof comes out formally from the fact that the left adjoint functor $h_!$ or $h_!^k$ carries monomorphisms into monomorphisms – a fact that we used in section 135 in the form $\mathrm{L}h_!^k = h_!^k$ (in case $k = \mathbb{Z}$). In the present case, the proof is essentially the dual one – as a matter of fact, as there are enough projectives, the statement about h^* or h_k^* taking projectives into projectives is equivalent with the right adjoint h_* or h_*^k (for sheaves of sets, resp. sheaves of k-modules) transforming epimorphisms into epimorphisms, which in the case of k-modules can be written equally under the equivalent form [p. 585]

$$\mathrm{R}^i h_* = 0 \quad \text{for } i > 0. \tag{8}$$

Now, this exactness property for h_*, in the case when h is cofibering with discrete fibers, is I guess well known (it is well-known to me at any rate), and comes from the specific computation of $h_*(F)$ for F in $A'^{\wedge}$, valid whenever h is *cofibering* (with arbitrary fibers)

$$(9) \qquad h_*(F)(x) \simeq \Gamma(A'_x, F \upharpoonright A'_X) \quad \text{for } x \text{ in } A,$$

where A'_X is the fiber of A' over x (a category not to be confused with $A'_{/x}$, the two being closely related, however...). In case the fibers of h are discrete, the right-hand side of (9) may be written as a product, hence the formula

$$(10) \qquad h_*(F)(x) \simeq \prod_{x' \text{ in } A'_X} F(x'),$$

(which may be viewed as the dual of the formula (11) p. 564). As in the category of sets (and hence also in Ab_k) a product of epimorphisms is again an epimorphism, the result we want follows indeed.

Remarks. 1) The results just given, as well as their proofs, concerning inverse images of injectives or projectives, are valid not only in the case of a common constant sheaf of rings on A and A', but more generally for any sheaf of rings $\mathcal{O}_A$ on A, when taking on A' the "induced" sheaf of rings

$$(11) \qquad \mathcal{O}_{A'} = h^*(\mathcal{O}_A).$$

However, it doesn't seem that the result about commutation of $\mathrm{L}f_!$ to colocalization is valid under the corresponding assumption $\mathcal{O}_A = f^*(\mathcal{O}_B)$, without assuming moreover $\mathcal{O}_B$ to be constant (hence $\mathcal{O}_A$ too), because already for the functor $f_!$ itself for modules it doesn't seem that commutation will hold.

2) I should correct as silly mistake I made at the very beginning of this section, when rashly stating that the functor $\mathrm{L}f_!$ may be defined for *any* map (1) between ringed objects in (Cat). I was thinking of the fact that for any ringed object $(A, \mathcal{O}_A)$ in (Cat), there are indeed enough
[p. 586] projectives in the category of $\mathcal{O}_A$-modules – thus, the modules

$$(12) \qquad \mathcal{O}_A^{(a)}, \quad \text{for } a \text{ in } A,$$

are clearly projective, and there are "sufficiently many". Thus, any additive functor from $\mathrm{Mod}(\mathcal{O}_A)$ to an abelian category admits a total left derived functor. However, it is not always true, for a map (1) of small ringed categories, that the corresponding inverse image functor for modules

$$G \mapsto f^*(G) = f_0{}^*(G) \otimes_{f_0{}^*(\mathcal{O}_B)} \mathcal{O}_A$$

admits a left adjoint $f_!$, or what amounts to the same, that this functor (which is right exact) is left exact and commutes to small products. It isn't even true, necessarily, when we assume A and B to be the final category! Left exactness of f^* just means flatness of $\mathcal{O}_A$ over $\mathcal{O}_B$, i.e., that for any a in A, $\mathcal{O}_A(a)$ is flat as a module over $\mathcal{O}_B(b)$, where $b = f(a)$. As for commutation to small products, it amounts (together with the first condition) to the still more exacting condition that $\mathcal{O}_A(a)$ should be a *projective module of finite type over* $\mathcal{O}_B(b)$, for any a in A. This condition is so close to the condition $\mathcal{O}_A = f_0{}^*(\mathcal{O}_B)$ (already considered in (11) above), that for a bird's eye view as we are aiming at here, we may as well assume this slightly stronger condition! Anyhow, as noticed before, to get commutation of $f_!$ with restriction of scalars and with colocalization, even this assumption isn't enough, apparently, and we'll have to assume moreover that $\mathcal{O}_B$ (hence also $\mathcal{O}_A$) is constant, or for the least, locally constant.

This brings us back to the case when a fixed ring k is given, and when we are working with categories of presheaves of k-modules – a situation studied at length in chapter V. In case the target category B is the final category, hence $\mathrm{L}f_!$ is just (absolute) total homology of A, this then may be computed nicely, using an "integrator" for A, namely a projective resolution of $k_{A^{\mathrm{op}}}$ in $(A^{\mathrm{op}})^{\wedge}_k$. This is more or less where we ended up by the end of chapter V, when developing a nicely autodual homology-cohomology set-up, replacing the category of k-modules Ab_k by a more-or-less arbitrary abelian category. I was about to go on and carry through a similar treatment in the "relative" case, namely for an arbitrary map f in (Cat) (but then I got caught unsuspectingly by that unending digression on schematic homotopy types, finally making up a whole chapter by itself). Maybe it is still worthwhile to come back to this without necessarily grinding through a complete formulaire for the [p. 587]
five main operations we got so far (namely $\mathrm{L}f_!$, f^*, $\mathrm{R}f_*$, $\overset{\mathrm{L}}{\otimes}$, $\mathrm{R}\mathbf{Hom}$). Not later than two pages ago or so, we were faced again with two visible dual statements, one about $\mathrm{L}f_!$, the other about $\mathrm{R}f_*$ – and feeling silly not to be able to merely deduce one from the other!

29.10.

Retrospective on ponderings on abelianization, and on coalgebra structures in (Cat).

139 Last night I still pondered a little about the $\mathrm{L}f_!$ operation, and did some more reading in the notes of chapter V of about three months ago, which had been getting a little distant in my mind. In those notes a great deal of emphasis goes with the notions of integrators and cointegrators – as a matter of fact, that whole chapter sprung from an attempt to understand the meaning of certain "standard complexes" associated with standard test categories such as $\triangle$, which then led us to the notions of an integrator (via the intermediate one of an "abelianizator"). This in turn brought us to the $*_k$ formalism, expressing most conveniently the relationship between categories such as $A^{\wedge}_k$ and $(A^{\mathrm{op}})^{\wedge}_k$ (k any commutative ring), and its various avatars. I was so pleased with this formalism and its "computational" flavor, that in my enthusiasm I subsumed under it the dual treatment of homology and cohomology for an object A of (Cat) (with coefficients in a complex of $\mathcal{M}$-valued presheaves, $\mathcal{M}$ being an abelian category satisfying some mild conditions), in section 108, as a particular case of the total derived functors $F_\bullet \overset{\mathrm{L}}{*}_k L'_\bullet$ and $\mathrm{RHom}_k(L_\bullet, K^\bullet)$

(with values in the derived category of $\mathcal{M}$). The main point here was using *projective resolutions* of the argument $L_\bullet$ in $B_k^\wedge$ or $A_k^\wedge$ (where $B = A^{\mathrm{op}}$), which in the most important case was just the constant sheaf of rings k_B or k_A – rather than resolve the argument $F_\bullet$ (projectively) or $K^\bullet$ (injectively), namely the coefficients for homology or cohomology. It was the enthusiasm of the adept of a game he just discovered – I was going to try it out for the next step, namely relative homology and cohomology $\mathrm{L}f_!$ and $\mathrm{R}f_*$, with f any map in (Cat) – but then I got caught by the more fascinating schematization game. Coming back now upon the rather routine matter of looking up a comprehensive mutually dual treatment of $\mathrm{L}f_!$ and $\mathrm{R}f_*$, it doesn't seem that the formalism of integrators and cointegrators is going to be of much help. To be more specific, in order
[p. 588] to compute (or simply define) $\mathrm{L}f_!(F_\bullet)$ or $\mathrm{R}f_*(K^\bullet)$, for a general map

$$f : A \to B$$

in (Cat), and $F_\bullet \in \mathrm{D}^-(A_k^\wedge)$, $K^\bullet \in \mathrm{D}^+(A_k^\wedge)$, I do not see any means of bypassing projective resolutions of $L_\bullet$, injective ones of $K^\bullet$. Taking the more familiar case of $\mathrm{R}f_*(K^\bullet)$, one natural idea of course would be to take a projective resolution $L_\bullet^A$ of k_A (i.e., a cointegrator for A), and write tentatively

$$\mathrm{R}f_*(K^\bullet) \overset{?}{\simeq} f_*(\mathbf{Hom}_k^{\bullet\bullet}(L_\bullet^A, K^\bullet). \tag{1}$$

The (misleading) reflex inducing us to write down this formula, is that this formula looks as if it were to boil down to the similar (correct) formula for the maps $A_{/b} \to e$, when taking the localizations on B,

$$B_{/b} \to B.$$

For this intuition to be correct, it should be true that the restriction (or "localization") of $L_\bullet^A$ to $A_{/b}$ (which is of course a resolution of the constant sheaf k on $A_{/b}$) is indeed a cointegrator on $A_{/b}$, namely that its components are still projective. This, however, we suspect, will hold true only under very special assumptions – as in general, it is inverse image by *colocalization* (not by localization) that takes projectives into projectives. Thus, I don't expect a relation (1) to hold, except under most exacting conditions on f and $K^\bullet$, which I didn't try to pin down. To take an example, assume A has a final object, hence k_A is projective and we may take $L_\bullet^A = k_A$, then (1) reads (when $K^\bullet = K$ is reduced to degree zero)

$$\mathrm{R}f_*(K) \simeq f_*(K), \quad \text{i.e., } \mathrm{R}^i f_*(K) = 0 \text{ for } i > 0,$$

which need not hold true even if K is constant ($= k_A$ say). For instance, we may start with an arbitrary object A_0 of (Cat) and add a final object e_1 to get A (intuitively, it is the "cone" over A_0), which is mapped into the cone over e, $B = \Delta_1 = (0 \to 1)$, in the obvious way. Then for any sheaf K on A, with restriction K_0 to A_0, we get

$$\mathrm{R}^i f_*(K)_0 \text{ (fiber at 0) } = \mathrm{H}^i(A_0, K_0),$$

which needs not be 0 for $i > 0$.

There is a big blunder at the end of section 101, where the formulæ (12), (12'), (13) (p. 377) (supposed to be "essentially trivial"), are false for essentially the same kind of reason. The formulæ ran into the typewriter as a matter of course, as they looked just the same as familiar ones from the standard duality formulaire (with $\mathrm{L}f_!$, f^* replaced by $\mathrm{R}f_!$, $f^!$). The first one reads, in case as above when A has a final object ε [p. 589]

$$\mathrm{R}f_*(f^*(K)) \overset{?}{\simeq} \mathbf{Hom}_k(f_!(k_A), K), \tag{2}$$

But we have

$$f_!(k_A) = f_!(k^{(\varepsilon)}) = k^{(b)}, \quad \text{where } b = f_!(\varepsilon) = f(\varepsilon),$$

and if f takes final object into final object, we thus get $f_!(k_A) = k_B$, and the right-hand side of (2) is just K, and hence (2) implies

$$\mathrm{R}^i f_*(f^*(K)) = 0 \quad \text{for } i > 0,$$

which, however, needs not be true even for $K = k_B$, as we saw with the previous example.

To come back to the general $\mathrm{L}f_!$, $\mathrm{R}f_*$ formalism, it seems it can't be helped, we'll have to do the usual silly thing and just resolve the argument involved, projectively in one case, injectively in the other. Even in the "absolute case" when $B = e$, we couldn't help it, either taking such resolutions, when it comes to working with the internal operations $\mathrm{R}\mathbf{Hom}_k(L_\bullet, K^\bullet)$ or $F_\bullet \overset{\mathrm{L}}{\otimes}_k L'_\bullet$, as we saw already in section 108 (which should have tuned down a little my committedness to integrators, but it didn't!).

Once this got clear, in order to turn the homological algebra mill, all we still need is a handy criterion for existence of "enough" projectives or injectives in categories of the type

$$A^\wedge_{\mathcal{M}} = \mathbf{Hom}(A^{\mathrm{op}}, \mathcal{M}),$$

with $\mathcal{M}$ an abelian category. In section 109 we got such a criterion (prop. 4 p. 433) – the simplest one can imagine: it is sufficient that (besides the stability under direct or inverse limits, needed anyhow in order for the functor $f_!$ resp. f_* from $A^\wedge_{\mathcal{M}}$ to $B^\wedge_{\mathcal{M}}$ to exist) that in $\mathcal{M}$ there should be enough projectives resp. injectives! This gives what is needed, surely, in order to grind through a mutually dual treatment of relative homology $\mathrm{L}f_!$ and cohomology $\mathrm{R}f_*$. I have the feeling that the little work ahead, for defining the basic operations and working out the relevant formulaire (including "cap products"), is a matter of mere routine, and I don't really expect any surprise may come up. Therefore, I don't feel like grinding through this, and rather will feel free to use whenever needed the most evident formulæ, as the adjunction formulæ between the three functors [p. 590]

$$\mathrm{L}f_!, \quad f^*, \quad \mathrm{R}f_*,$$

transitivity isomorphisms for a composition of maps, possibly also various projection formulæ – trying however to be careful with base change questions, notably for $\mathrm{L}f_!$, and not repeat the same blunders!

In retrospect, the main role for me of the reflections of chapter V on abelianization has been to become a little more familiar with the *homology* formalism in (Cat), namely essentially with the $\mathrm{L}f_!$ operation, which has been more or less a white spot in my former experience, centered rather upon cohomology. An interesting, still somewhat routine byproduct of these ponderings has been the careful formulation of the duality relationship around the pairings between sheaves and cosheaves, namely the operations $*_k$ and $\circledast_k$. Granting this, the game with integrators and cointegrators boils down to the standard reflex, of taking projective resolutions of the "unit" sheaf or cosheaf, k_A or $k_{A^{\mathrm{op}}}$ – which are the most obvious objects at hand of all, in our coefficient categories.

The one idea which seems to me of wider scope and significance in this whole reflection on abelianization, is the "further step in linearization", whereby the models in (Cat) are replaced by their k-additive envelope, endowed moreover with their natural *diagonal map* (section 109). The psychological effect of this discovery has been an immediate one – it triggered at once the reflection on schematization of chapter VI. This reflection, apparently, took me into a rather different direction from those "k-coalgebra structures in (Cat)", which had reawakened and made more acute the feeling that homotopy types should make sense "over any ground ring". Coming back now to the homology and coho- [p. 591] mology formalism within (Cat), it remains a very striking fact indeed for me, that as far as I can see at present, all basic operations in the (commutative) homology and cohomology formalism in (Cat), and their basic properties and interrelations, should make sense for these linearized objects, which therefore may be viewed as more perfect carriers still than small categories for embodying the relevant formalism. To what extent this feeling is indeed justified, cannot of course be decided beforehand – only experience can tell. For instance, does the "six operations duality formalism" contemplated lately for (Cat) (sections 136, 137), which has still to be worked out, carry over to this wider, linearized set-up? This will become clearer when the relevant notions in (Cat) are understood, so that it will become a meaningful question whether for a map in (Cat), the property of being proper, or smooth, or an immersion, may be read off in terms of properties of the corresponding map of coalgebra structures in (Cat). And what about subtler types of cohomology operations, such as the Steenrod operations (which, I remember, may be defined in the context of cohomology with coefficients in general sheaves of $\mathbb{F}_p$-modules)?

When concerned with homology and cohomology formalism in (Cat), involving general sheaves of coefficients, not merely locally constant ones, we are leaving, strictly speaking, the waters of a reflection on "homotopy models". The objects of (Cat) now are no longer viewed as mere models for homotopy types, but rather, each one as defining a topos, with the manifold riches it carries; a richness similar to the one of a topological space, almost all of which is being stripped off when looking at the mere homotopy type – including even such basic properties as dimension, compactness, smoothness, cardinality and the like. When passing from an object in (Cat) to the corresponding coalgebra structure in (Cat), much of this richness is preserved – maybe everything, indeed, which can be expressed in terms of *commutative* sheaves of coefficients. As for the realm of non-commutative cohomology formalism (which is supposed to be the main theme of these notes, with overall title "Pursuing Stacks"!), it doesn't seem, not at first sight at any rate, that much of this could be read off the enveloping coalgebra $P = \mathrm{Add}_k(A)$ of a given object A in (Cat) – except of course in the case when A can be recovered in terms of P, maybe as the category of "exponential" pairs (x, u), where x is in P and

$$u : \delta(x) \simeq x \otimes_k x$$

an isomorphism.

4.11. [p. 592]

140 Life keeps pushing open the doors of that well-tempered hothouse of my mathematical reflections, as a fresh wind and often an impetuous one, sweeping off the serene quietness of abstraction, – a breath rich with the manifold fragrance of the world we live in. This is the world of conflict, weaving around each birth and each death and around the lovers' play alike – it is the world we all have been born into without our choosing. I used to see it as a stage – the stage set for our acting. Our freedom (rarely used indeed!) includes choosing the role we are playing, possibly changing roles – but not choosing or changing the stage. It doesn't seem the stage ever changes during the history of mankind – only the decors kept changing. More and more, however, over the last years, I have been feeling this world I am living in, the world of conflict, somehow as a *meal* – a meal of inexhaustible richness. Maybe the ultimate fruit and meaning of all my acting on that stage, is that parts of that meal, of that richness, be actually eaten, digested, assimilated – that they become part of the flesh and bones of my own being. Maybe the ultimate purpose of conflict, so deeply rooted in every human being, is to be the raw material, to be eaten and digested and changed into understanding about conflict. Not a collective "understanding" (I doubt there is such a thing!), written down in textbooks or sacred books or whatever, nor even something expressed or expressible in words necessarily – but the kind of immediate knowledge, rather, the walker has about walking, the swimmer about swimming, or the suckling about milk and mother's breasts. My business is to be a learner, not a teacher – namely to allow this process to take place in my being, letting the world of conflict, of suffering and of joy, of violence and of tenderness, enter and be digested and become knowledge about myself.

The meal and the guest.

I am not out, though, to write a "journal intime" or meditation notes, so I guess I better get back to the thread of mathematical reflection where I left it, rather than write allusively about the events of these last days, telling me about life and about myself through one of my children.

12.11.

Maybe at times I like to give the impression, to myself and hence others, that I am the easy learner of things of life, wholly relaxed, "cool" and all that – just keen for learning, for eating the meal and welcome smilingly whatever comes with its message, frustration and sorrow and destructiveness and the softer dishes alike. This of course is just humbug, an image d'Épinal which at whiles I'll kid myself into believing I am like. Truth is that I am a hard learner, maybe as hard and reluctant as anyone. At any rate, the inbuilt mechanisms causing rejection of the dishes unpalatable to my wholly conditioned, wholly ego-controlled taste, are as much present in me now as they have ever been in my life, and as much as in anyone else I know. This interferes a lot with the learning – it causes a tremendous amount of friction and energy dispersion (wholly unlike what happens when I am learning mathematics, say, namely discovering things about any kind of substance which my own person and ego is not part of...). If there has been anything new appearing in my life, it is surely not the end of this process of dispersion, or the end of inertia, closely related to dispersion. It is something, rather, which causes learning to take place all the same – be it the hard way, as it often happens, very much like the troubled digestion of one who took a substantial meal reluctantly or in a state of nervousness, of crispation. Once one is through with the digestion, though, the food one ate is transformed into flesh and muscles, blood and bones and the like, just as good and genuine as if the meal had been taken relaxedly and with eager appetite, as it deserved. What really counts for the process of assimilation to be able to take place, is that in a certain sense the food, palatable or not, be *accepted* – not vomited, or just kept in the bowels like a foreign body, sometimes for decennaries. The remarkable fact I come to know through experience, is that even after having been kept thus inertly for a lifetime, a process of digestion and assimilation may still come into being and transform the obtrusive stuff into living substance.

[p. 593]

[the Épinal prints being proverbial for a naive depiction, showing only good aspects...]

[decennary = decade]

During the last week I have been sick for a few days – a grippe I might say, but surely a case of troubled digestion too. It seems though I am through now – till the next case at any rate! I daresay life has been generous with me for these last three months, while I haven't even taken the trouble to stop with the mathematical nonsense for any more than a week or two. This week, too, I still did some mathematical scratchwork, still along the lines of abelianization, which keeps showing a lot richer than suspected.

[grippe = the flu]

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